Monotonicity-based methods for inverse coefficient problems

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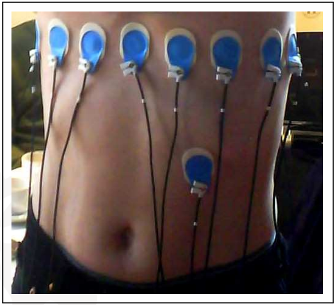
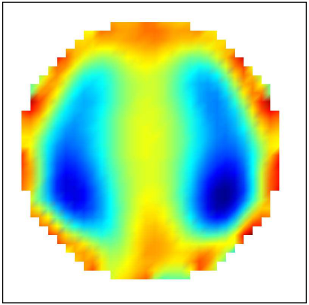
April 14, 2016

Electrical impedance tomography (EIT)





- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject.

Mathematical Model

Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega$$

$\Omega \subset \mathbb{R}^n$: imaged body, $n \geq 2$
$\sigma(x)$: conductivity
$u(x)$: electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_\nu u(x)|_{\partial\Omega}$: applied electric current
$u(x)|_{\partial\Omega}$: measured boundary voltage (potential)

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves } (1)\} \text{ ?}$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma): L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Partial/local data

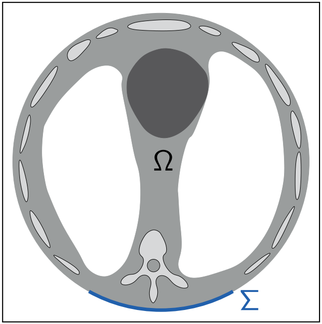
Measurements on open part of boundary $\Sigma \subseteq \partial\Omega$
($\partial\Omega \setminus \Sigma$ is kept insulated.)

Recover σ from

$$\Lambda(\sigma): \; L^2_\diamond(\Sigma) \to L^2_\diamond(\Sigma), \quad g \mapsto u|_\Sigma,$$

where u solves $\nabla \cdot (\sigma \nabla u) = 0$ with

$$\sigma \partial_\nu u|_\Sigma = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$



Uniqueness results

- Measurements on complete boundary (full data):

  Calderón (1980), Druskin (1982+85), Kohn/Vogelius (1984+85), Sylvester/Uhlmann (1987), Nachman (1996), Astala/Päivärinta (2006)

- Measurements on part of the boundary (local data):

  *Bukhgeim/Uhlmann (2002), Knudsen (2006), Isakov (2007), Kenig/Sjöstrand/Uhlmann (2007), **H**. (2008), Imanuvilov/Uhlmann/Yamamoto (2009+10), Kenig/Salo (2012+13)*

Rough summary of known results:

- L^∞ coefficients are uniquely determined from full data in 2D.
- In all cases, piecew.-anal. coefficients are uniquely determined.
- Sophisticated research on uniqueness for $\approx C^2$-coefficients

  (based on CGO-solutions for Schrödinger eq. $-\Delta u + qu = 0$, $q = \frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}}$).

Inversion of $\sigma \mapsto \Lambda(\sigma) = \Lambda_{\text{meas}}$?

Generic solvers for non-linear inverse problems:

- Linearize and regularize:

  $$\Lambda_{\text{meas}} = \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).$$

  σ_0: Initial guess or reference state (e.g. exhaled state)

  $\rightsquigarrow$ Linear inverse problem for σ

  (Solve using linear regularization method, repeat for Newton-type algorithm.)

- Regularize and linearize:

  E.g., minimize non-linear Tikhonov functional

  $$\|\Lambda_{\text{meas}} - \Lambda(\sigma)\|^2 + \alpha \|\sigma - \sigma_0\|^2 \to \text{min!}$$

Advantages of generic solvers:

- Very flexible, additional data/unknowns easily incorporated
- Problem-specific regularization can be applied

  (e.g., total variation penalization, stochastic priors, etc.)

Inversion of $\sigma \mapsto \Lambda(\sigma) = \Lambda_{\text{meas}}$?

Problems with generic solvers

- High computational cost

  (Evaluations of $\Lambda(\cdot)$ and $\Lambda'(\cdot)$ require PDE solutions)

- Convergence unclear

  (Validity of TCC/Scherzer-condition is a long-standing open problem for EIT.)

  - Convergence against true solution for exact meas. Λ_{meas}?

    (in the limit of infinite computation time)
  - Convergence against true solution for noisy meas. $\Lambda_{\text{meas}}^{\delta}$?

    (in the limit of $\delta \to 0$ and infinite computation time)
  - Global convergence? Resolution estimates for realistic noise?

D-bar method

- Convergent 2D-implementation for $\sigma \in C^2$ and full bndry data

  (Knudsen, Lassas, Mueller, Siltanen 2008)

Linearization and shape reconstruction

Theorem (**H.**/Seo, SIMA 2010)

Let κ, σ, σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Longrightarrow \quad \mathrm{supp}_\Sigma \kappa = \mathrm{supp}_\Sigma(\sigma - \sigma_0)$$

supp_Σ: outer support ($=$ supp $+$ parts unreachable from Σ)

$\rightsquigarrow$ Linearized EIT equation contains correct shape information

Next slides: Idea of proof using monotonicity & localized potentials.

Monotonicity

For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\sigma_0 \leq \sigma_1 \quad \Longrightarrow \quad \Lambda(\sigma_0) \geq \Lambda(\sigma_1)$$

This follows from

$$\int_\Omega (\sigma_1 - \sigma_0) |\nabla u_0|^2 \geq \int_\Sigma g \left(\Lambda(\sigma_0) - \Lambda(\sigma_1)\right) g \geq \int_\Omega \frac{\sigma_0}{\sigma_1} (\sigma_1 - \sigma_0) |\nabla u_0|^2$$

for all solutions u_0 of

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_\nu u_0|_{\partial\Omega} = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$

(e.g., Kang/Seo/Sheen 1997, Ikehata 1998)

Localized potentials

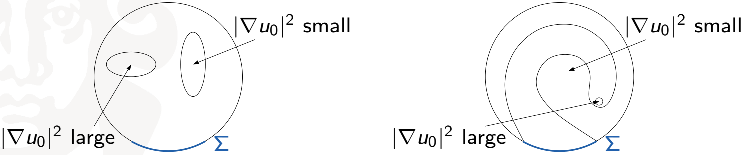
Theorem *(**H.**, IPI 2008)*

Let σ_0 fulfill unique continuation principle (UCP),

$$\overline{D_1} \cap \overline{D_2} = \emptyset, \quad \text{and} \quad \Omega \setminus (\overline{D}_1 \cup \overline{D}_2) \text{ be connected with } \Sigma.$$

Then there exist solutions $u_0^{(k)}$, $k \in \mathbb{N}$ with

$$\int_{D_1} \left|\nabla u_0^{(k)}\right|^2 \mathrm{d}x \to \infty \quad \text{and} \quad \int_{D_2} \left|\nabla u_0^{(k)}\right|^2 \mathrm{d}x \to 0.$$

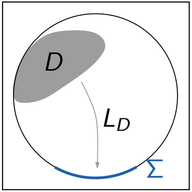


Proof 1/3

Virtual measurements:

$$L_D: \ L^2(D)^n \to L^2_\diamond(\Sigma), \quad F \mapsto u|_\Sigma, \text{ with}$$

$$\int_\Omega \sigma_0 \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega F \cdot \nabla v \, \mathrm{d}x \quad \forall v \in H^1_\diamond(D).$$



By (UCP): If $\overline{D_1} \cap \overline{D_2} = \emptyset$ and $\Omega \setminus (\overline{D}_1 \cup \overline{D}_2)$ is connected with Σ, then

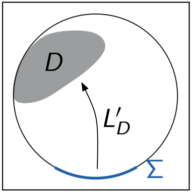
$$\mathcal{R}(L_{D_1}) \cap \mathcal{R}(L_{D_2}) = 0.$$

Sources on different domains yield different virtual measurements.

Dual operator:

$$L_D' : L^2_\diamond(\Sigma) \to L^2(D)^n, \quad g \mapsto \nabla u_0|_D, \text{ with}$$

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_\nu u_0|_\Sigma = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$



Evaluating solutions on D is dual operation to virtual measurements.

Proof 3/3

Functional analysis:

X, Y_1, Y_2 reflexive Banach spaces, $L_1 \in \mathcal{L}(Y_1, X)$, $L_2 \in \mathcal{L}(Y_1, X)$.

$$\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2) \quad \Longleftrightarrow \quad \|L_1' x\| \lesssim \|L_2' x\| \ \ \forall x \in X'.$$

Here: $\mathcal{R}(L_{D_1}) \nsubseteq \mathcal{R}(L_{D_2}) \quad \Longrightarrow \quad \|\nabla u_0|_{D_1}\|_{L^2} \not\lesssim \|\nabla u_0|_{D_2}\|_{L^2}.$

If sources on different subdomains do not generate the same data, then the respective evaluations are not bounded by each other.

Proof of shape invariance under linearization

- Linearization: $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$
- Monotonicity: For all "reference solutions" u_0:

$$\int_\Omega (\sigma - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x$$

$$\geq \underbrace{\int_\Sigma g\left(\Lambda(\sigma_0) - \Lambda(\sigma)\right) g}_{= -\int_\Sigma g\left(\Lambda'(\sigma_0)\kappa\right) g = \int_\Omega \kappa |\nabla u_0|^2 \, \mathrm{d}x} \geq \int_\Omega \frac{\sigma_0}{\sigma}(\sigma - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x.$$

- Use localized potentials to control $|\nabla u_0|^2$

$\rightsquigarrow$ $\mathrm{supp}_\Sigma \kappa = \mathrm{supp}_\Sigma(\sigma - \sigma_0)$ □

Linearization and shape reconstruction

Theorem (**H.**/Seo, SIMA 2010)
Let κ, σ, σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Longrightarrow \quad \mathrm{supp}_\Sigma \kappa = \mathrm{supp}_\Sigma(\sigma - \sigma_0)$$

supp_Σ: outer support ($= \mathrm{supp} +$ parts unreachable from Σ)

$\leadsto$ Linearized EIT equation contains correct shape information

Can we recover conductivity changes (anomalies, inclusions, . . .) in a fast, rigorous and globally convergent way?

Monotonicity based imaging

- Monotonicity:

$$\tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

- Idea: Simulate $\Lambda(\tau)$ for test cond. τ and compare with $\Lambda(\sigma)$.

  (Tamburrino/Rubinacci 02, Lionheart, Soleimani, Ventre, ...)

- Inclusion detection: For $\sigma = 1 + \chi_D$ with unknown D, use $\tau = 1 + \chi_B$, with small ball B.

$$B \subseteq D \quad \Longrightarrow \quad \tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

- Algorithm: Mark all balls B with $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$
- Result: upper bound of D.

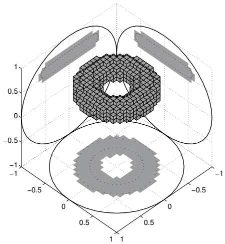
Only an upper bound? Converse monotonicity relation?

Monotonicity method (for simple test example)

Theorem *(**H.**/Ullrich, SIMA 2013)*

$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1 + \chi_B) \geq \Lambda(\sigma).$$



For faster implementation:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

Proof: Monotonicity & localized potentials

Shape can be reconstructed by linearized monotonicity tests.

$\rightsquigarrow$ fast, rigorous, allows globally convergent implementation

Improving residuum-based methods

Let $\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

- Pixel partition $\Omega = \bigcup_{k=1}^m P_k$
- Monotonicity tests

$$\beta_k \in [0,\infty] \text{ max. values s.t. } \beta_k \Lambda'(1)\chi_{P_k} \geq \Lambda(\sigma) - \Lambda(1)$$

- Monotonicity-constrained residuum minimization

$$\|\Lambda'(1)\kappa - \Lambda(\sigma) - \Lambda(1)\|_{\mathrm{F}} \to \text{min!}$$
$$\text{such that} \quad \kappa|_{P_k} = \text{const.},\ 0 \leq \kappa|_{P_k} \leq \min\{\tfrac{1}{2}, \beta_k\}$$

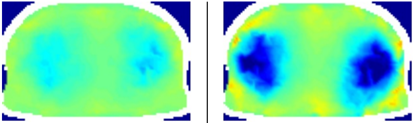
($\|\cdot\|_F$: Frobenius norm of Galerkin projektion to finite-dimensional space)

Theorem (**H.**/Mach, submitted)

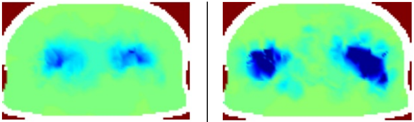
- There exists unique minimizer κ and
$$P_k \subseteq \operatorname{supp}\kappa \iff P_k \subseteq \operatorname{supp}(\sigma - 1).$$
- Convergent regularization for noisy data, $\kappa^\delta \to \kappa$ pointwise.

Phantom experiment

standard method



with monotonicity constraints



Enhancing standard methods by monotonicity-based constraints

(Zhou/**H.**/Seo)

Conclusions

EIT is a highly ill-posed, non-linear inverse problem.

- Convergence of generic solvers unclear.
- But: Shape reconstruction in EIT is essentially a linear problem.

Monotonicity-based methods for EIT shape reconstruction

- allow fast, rigorous, globally convergent implementations.
- work in any dimensions $n \geq 2$, full or partial boundary data.
- can enhance standard residual-based methods.
- yield rigorous resolution guarantees for realistic settings.

Approach (monotonicity $+$ localized potentials) can be extended

- to other linear elliptic problems *(diffuse optical tomography, magnetostatics)*
- at least partially to closely related problems

  (eddy-current equations, p-Laplacian)