Detecting stochastic inclusions in electrical impedance tomography

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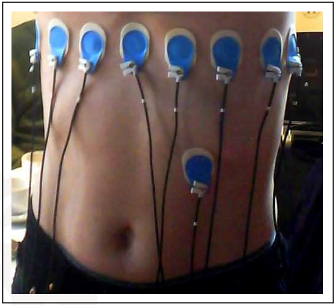
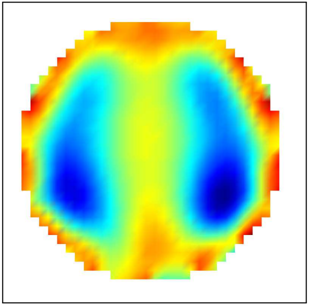
Electrical impedance tomography (EIT)





- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject.

Mathematical Model (deterministic)

Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in D$$

$D \subset \mathbb{R}^n$:	imaged body, $n \geq 2$
$\sigma(x)$:	conductivity
$u(x)$:	electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_\nu u(x)\|_{\partial D}$:	applied electric current
$u(x)\|_{\partial D}$:	measured boundary voltage (potential)

Calderón problem (deterministic)

Can we recover $\sigma \in L^\infty_+(D)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in D \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial D}, \sigma \partial_\nu u|_{\partial D}) \quad : \quad u \text{ solves (1)}\}\,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma):\ L^2_\diamond(\partial D) \to L^2_\diamond(\partial D), \quad g \mapsto u|_{\partial D},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial D} = g$.

Inclusion detection in EIT

σ: Actual (unknown) conductivity
σ_0: Initial guess or reference state (e.g. exhaled state)

- $\operatorname{supp}(\sigma - \sigma_0)$ often relevant in practice

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Inclusion detection problem (aka shape reconstruction or anomaly detection)

Can we recover $\operatorname{supp}(\sigma - \sigma_0)$ *from* $\Lambda(\sigma)$, $\Lambda(\sigma_0)$*?*

- Generic approach: parametrize $\operatorname{supp}(\sigma - \sigma_0)$ (e.g., Level-Set-Methods)
- Problems:
  - PDE solutions required in each iteration
  - convergence unclear

Linearization and inclusion detection

Theorem (**H.**/Seo, SIAM J. Math. Anal. 2010)
Let κ, σ, σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Longrightarrow \quad \mathrm{supp}_{\partial D}\kappa = \mathrm{supp}_{\partial D}(\sigma - \sigma_0)$$

$\mathrm{supp}_{\partial D}$: outer support ($=\mathrm{supp} +$ parts unreachable from ∂D)

- ⇝ Inclusion detection is essentially a linear problem.
- ⇝ Fast, rigorous and globally convergent inclusion detection methods are possible.
  - Next slides: Monotonicity method.

Monotonicity

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For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\sigma_0 \leq \sigma_1 \quad \Longrightarrow \quad \Lambda(\sigma_0) \geq \Lambda(\sigma_1)$$

This follows from

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \geq \int_{\partial\Omega} g\,(\Lambda(\sigma_0) - \Lambda(\sigma_1))\,g \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2$$

for all solutions u_0 of

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_\nu u_0|_{\partial\Omega} = g.$$

(e.g., Kang/Seo/Sheen 1997, Ikehata 1998)

Monotonicity based imaging

- Monotonicity:

$$\tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

- Idea: Simulate $\Lambda(\tau)$ for test cond. τ and compare with $\Lambda(\sigma)$.

  (Tamburrino/Rubinacci 02, Lionheart, Soleimani, Ventre, ...)

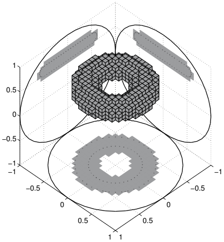
- Inclusion detection: For $\sigma = 1 + \chi_A$ with unknown anomaly A, use $\tau = 1 + \chi_B$, with small ball B.

$$B \subseteq A \quad \Longrightarrow \quad \tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

- Algorithm: Mark all balls B with $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$
- Result: upper bound of anomaly A.

Only an upper bound? Converse monotonicity relation?

Monotonicity method



Theorem *(**H.**/Ullrich, SIAM J. Math. Anal. 2013)*

$D \setminus \overline{A}$ connected. $\sigma = 1 + \chi_A$.

$$B \subseteq A \quad \Longleftrightarrow \quad \Lambda(1 + \chi_B) \geq \Lambda(\sigma).$$

For faster implementation:

$$B \subseteq A \quad \Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

Inclusion can be reconstructed by linearized monotonicity tests.

⇝ Fast, rigorous, allows globally convergent implementation

- Ideas of proof evolved from the similar Factorization Method

  *(For EIT: Arridge, Betcke, Brühl, Chaulet, Choi, Hakula, Hanke, **H.**, Holder, Hyvönen, Kirsch, Lechleiter, Nachman, Päivärinta, Pursiainen, Schappel, Schmitt, Seo, Teirilä, ...)*

Calderón problem

Deterministic Calderón Problem: Can we recover σ from NtD

$$\Lambda(\sigma): L^2_\diamond(\partial D) \to L^2_\diamond(\partial D), \quad g \mapsto u|_{\partial D},$$

where u solves $\nabla \cdot (\sigma \nabla u) = 0$ with $\sigma \partial_\nu u|_{\partial D} = g$?

- Stochastic Calderón problem:

  Can we recover $\mathbb{E}(\sigma)$ from $\mathbb{E}(\Lambda(\sigma))$?

- Stochastic inclusion detection in hom. background ($\sigma_0 = 1$):

  Can we recover $\mathrm{supp}(\mathbb{E}(\sigma) - 1)$ from $\mathbb{E}(\Lambda(\sigma))$?

- (Possible) Application: Biomedical anomaly detection from temporally averaged measurements.

NtD-operator is of finite expectation

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Theorem *(Barth/**H.**/Hyvönen/Mustonen, submitted)*

If $\sigma, \sigma^{-1} \in L^1(\Omega, L^\infty_+(D))$ then

- $\Lambda(\sigma) \in L^1(\Omega, L^\infty_+(D))$,
- $\mathbb{E}(\Lambda(\sigma))$ is well-defined,
- $\mathbb{E}(\Lambda(\sigma)) : L^2_\diamond(\partial D) \to L^2_\diamond(\partial D)$ is compact and self-adjoint.

Proof.

- $\Lambda(\sigma) : \Omega \to \mathcal{L}(L^2_\diamond(\partial D))$ is concatenation of strongly meas. function and continuous function and thus strongly measurable.
- Integrability bound on $\Lambda(\sigma)$ follows from monotonicity inequality.

Detecting stochastic inclusions

Theorem *(Barth/**H.**/Hyvönen/Mustonen, submitted)*

Consider a domain with with a stochastic inclusion A,

- $\sigma = \begin{cases} 1 & \text{in } D \setminus A, \\ \sigma_A(x,\omega) & \text{in } A, \end{cases}$
- $\sigma_A : \Omega \to L^\infty_+(A)$, Ω probability space,
- $\sigma_A, \sigma_A^{-1} \in L^1(\Omega, L^\infty_+(A))$

If there exists $\alpha > 0$ with

$$\mathbb{E}(\sigma_A) > 1 + \alpha \quad \text{and} \quad \mathbb{E}(\sigma_A^{-1})^{-1} > 1 + \alpha,$$

then $\mathbb{E}(\Lambda(\sigma))$ uniquely determines A.

Applying FM or MM to $\mathbb{E}(\Lambda(\sigma))$ recovers the true inclusion A.

Monotonicity for stochastic inclusions

Main idea of the proof. Monotonicity for stochastic inclusions:

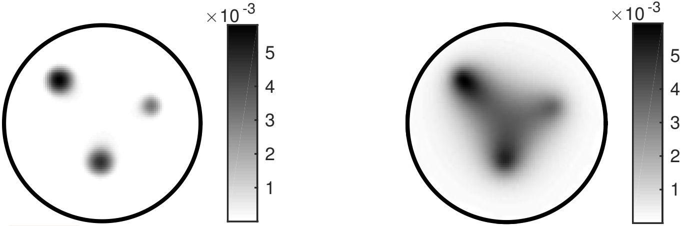
For deterministic σ_0 and stochastic σ:

$$\int_D (\mathbb{E}(\sigma) - \sigma_0)) |\nabla u_0|^2 \,\mathrm{d}x \geq \int_{\partial D} g\big(\Lambda(\sigma_0) - \mathbb{E}(\Lambda(\sigma))\big) g \,\mathrm{d}s$$
$$\geq \int_D \sigma_0^2 \left(\sigma_0^{-1} - \mathbb{E}(\sigma^{-1})\right) |\nabla u_0|^2 \,\mathrm{d}x.$$

In particular,

$$\sigma_0 \leq \mathbb{E}(\sigma) \text{ and } \sigma_0 \leq \mathbb{E}(\sigma^{-1})^{-1} \quad \Longrightarrow \quad \Lambda(\sigma_0) \geq \mathbb{E}(\Lambda(\sigma))$$

Example



- Background conductivity $\sigma_0 = 1$
- Inclusions conductivity uniformly distributed in $[0.5, 3.5]$

$$\mathbb{E}(\sigma_A) \geq \mathbb{E}(\sigma_A^{-1})^{-1} \approx 1.54 > 1 = \sigma_0$$

- Images show result of Factorization Method applied to $\mathbb{E}(\sigma)$ (Left Image: no noise, Right Image: 0.1% noise)

Open problems / Outlook

Stochastic background?

- Roughly speaking (for monotonicity-based algorithms):

  stoch. $\sigma(\omega) \Longleftrightarrow$ determ. uncertainty in $[\mathbb{E}(\sigma^{-1})^{-1}, \mathbb{E}(\sigma)]$.
- Stochastic anomaly in stochastic background can be detected if deterministic anomaly in deterministic (unknown!) background can be detected.
- Problem may be treatable with worst-case tests

  (Resolution guarantees for deterministic case: ***H.**, Ullrich, IEEE TMI 2015*)

Open problems / Outlook

Stochastic anomaly shape?

- Problem formulation requires $\sigma \in L^1(\Omega, L^\infty_+(D))$.

$\rightsquigarrow$ $\sigma : \Omega \to L^\infty_+(D)$ must be essentially separably valued.

(Banach-space valued integration, Lebesgue-Bochner spaces)

- Conductivity $\sigma(\omega) = 1 + \chi_{B_{r(\omega)}}$
  where anomaly $B_{r(\omega)}$ is ball of random radius $r(\omega)$
  (e.g. uniformly distibuted in $[r_{\text{min}}, r_{\text{max}}]$)

$$\|\sigma(\omega_1) - \sigma(\omega_2)\|_{L^\infty} = 1 \quad \text{for all} \quad \omega_1 \neq \omega_2.$$

$\rightsquigarrow$ $\sigma : \Omega \to L^\infty_+(D)$ is not essentially separably valued.

- Different functional analytic setting?

Conclusions

In EIT, stochastic inclusions in a deterministic background

- can be detected by deterministic Factorization or Monotonicity Method applied to the measurement's expectation value,
- if, both, $\mathbb{E}(\sigma_A)$ and $\mathbb{E}(\sigma_A^{-1})^{-1}$ are larger than bg conductivity
  (or both are smaller than background conductivity)

Roughly speaking,

- stochastic conductivity uncertainty in σ is analogous to deterministic uncertainty in $[\mathbb{E}(\sigma^{-1})^{-1}, \mathbb{E}(\sigma)]$

Open Problems / Outlook:

- Stochastic inclusions in stochastic backgrounds may be treatable by resolution guarantees.
- Unclear how to treat stochastic inclusion shapes in this functional analytic setting.