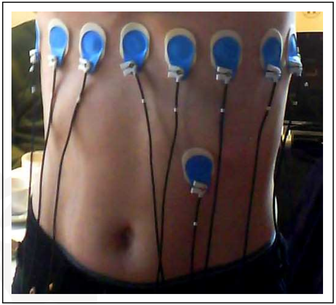
Regularizing the optimization-based solution of inverse coefficient problems with monotonicity constraints

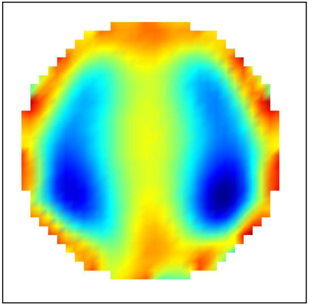
Bastian von Harrach
(joint work with Mach Nguyet Minh)
http://numerical.solutions

Institute of Mathematics, Goethe University Frankfurt, Germany

The 10th International Conference on
Optimization: Techniques and Applications (ICOTA 10)
Ulaanbaatar, Mongolia, July 23–26, 2016.

Electrical impedance tomography (EIT)





- Apply electric currents on subject's boundary
- Measure necessary voltages

$\rightsquigarrow$ Reconstruct conductivity inside subject.

Mathematical Model

Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega$$

$\Omega \subset \mathbb{R}^n$:	imaged body, $n \geq 2$
$\sigma(x)$:	conductivity
$u(x)$:	electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_\nu u(x)\|_{\partial\Omega}$:	applied electric current
$u(x)\|_{\partial\Omega}$:	measured boundary voltage (potential)

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves (1)}\}\,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma) : L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Optimization-based Inversion

Popular approach in practice:

- Measure difference data $\Lambda_{\text{meas}} \approx \Lambda(\sigma) - \Lambda(\sigma_0)$.

  (e.g., $\Lambda(\sigma_0)$: measurement at exhaled state)

- Minimize (regularized and linearized) data fit functional

Standard linearized method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Approximate $\kappa \approx \sigma - \sigma_0$ by minimizing

$$\|\Lambda'(\sigma_0)\kappa - \Lambda_{\text{meas}}\|^2 + \alpha\|\kappa\|^2 \to \min!$$

Problem: Choice of norms heuristic. No convergence theory!

Linearization and shape reconstruction

Theorem (**H.**/Seo, SIMA 2010)

Let κ, σ, σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda_{\text{meas}} = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Longrightarrow \quad \text{supp}_{\partial\Omega}\kappa = \text{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

$\text{supp}_{\partial\Omega}$: outer support ($=$ supp $+$ parts unreachable from $\partial\Omega$)

- Linearized EIT equation contains correct shape information.

  (in the continuous version for noise-free measurements on infinitely many electrodes)
- Practitioners use heuristic regularization of linearized EIT equ.

  (in the discretized version for noisy measurements on finitely many electrodes)

Can we find a regularization that rigorously guarantees convergence of reconstructed shapes?

Monotonicity based imaging

- Monotonicity:

$$\tau \le \sigma \quad \Longrightarrow \quad \Lambda(\tau) \ge \Lambda(\sigma)$$

- Pixel-wise inclusion detection:

  For $\sigma = \sigma_0 + \chi_D$ with unknown D, use $\tau = \sigma_0 + \chi_P$ on pixel P

$$P \subseteq D \quad \Longrightarrow \quad \tau \le \sigma \quad \Longrightarrow \quad \Lambda(\tau) \ge \Lambda(\sigma)$$

  Quantitative and linearized version: For k-th pixel P_k, maximize

$$\beta_k \to \max! \quad \text{s.t.} \quad \beta_k \Lambda'(\sigma_0)\chi_{P_k} \ge \Lambda(\sigma) - \Lambda(\sigma_0).$$

- Theorem. *(**H.**/Ullrich, SIMA 2013)*

$$\exists a > 0 : \quad \begin{cases} \beta_k = 0 & \text{if } P_k \not\subseteq D \\ \beta_k \ge a & \text{if } P_k \subseteq D \end{cases}$$

Monotonicity-based regularization

- Monotonicity-based regularization: Minimize

$$\|\Lambda'(\sigma_0)\kappa - (\Lambda(\sigma) - \Lambda(\sigma_0))\|_{\mathrm{F}} \to \text{min!}$$

under the constraint $\kappa|_{P_k} = \text{const.},\ 0 \leq \kappa|_{P_k} \leq \min\{a, \beta_k\}$.

($\|\cdot\|_F$: Frobenius norm of Galerkin projektion to finite-dimensional space)

Theorem (**H.**/Mach, submitted)

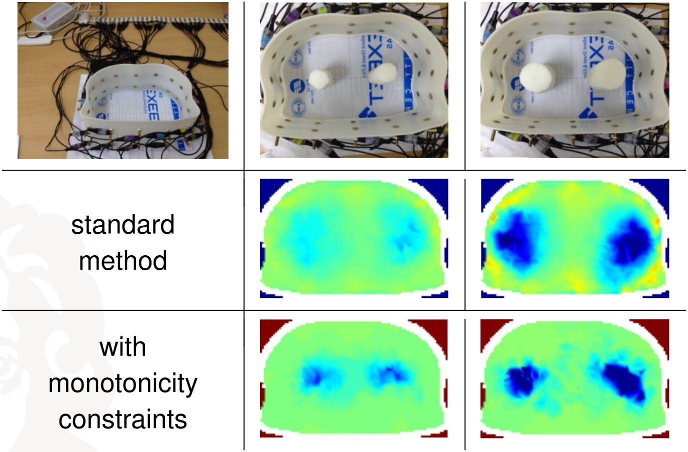
- There exists unique minimizer $\hat{\kappa}$ and

$$P_k \subseteq \operatorname{supp} \hat{\kappa} \quad \Longleftrightarrow \quad P_k \subseteq \operatorname{supp}(\sigma - \sigma_0).$$

- Convergent regularization for noisy data, $\kappa^\delta \to \kappa$ pointwise.

Monotonicity-regularized solutions converge against correct shape.

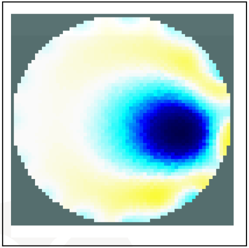
Phantom experiment



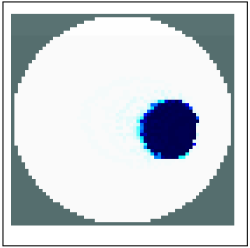
Enhancing standard methods by monotonicity-based constraints

(Zhou/**H.**/Seo, 2016)

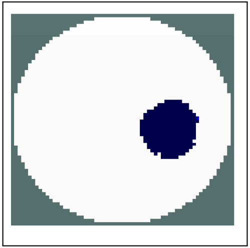
Benchmark example



standard



monoton.-regularized
(Matlab quadprog)



monoton.-regularized
(cvx package)

Monotonicity-regularization vs. community standard

(**H.**/Mach)

- EIT community standard: GREIT in EIDORS
- EIDORS: `http://eidors3d.sourceforge.net` (Adler/Lionheart)
- GREIT: Graz consensus Reconstruction algorithm for EIT (Adler et al.)
- Dataset: `iirc_data_2006` (Woo et al.): 2cm insulated inclusion in 20cm tank
  - using interpolated data on active electrodes (**H.**, Inverse Problems 2015)

Conclusions

EIT is a highly ill-posed, non-linear inverse problem.

- Convergence of generic solvers unclear.
- Practitioners minimize linearized data-fit functional with heuristic regularization and no theoretical justification.

Monotonicity-based regularization of linearized EIT equation

- uses that shape reconstr. in EIT is (essentially) a linear problem,
- yields solutions that rigorously converge against correct shape,
- combines rigorous theory of monotonicity method with practical robustness of residuum-based methods.