Monotonicity-based regularization of inverse coefficient problems

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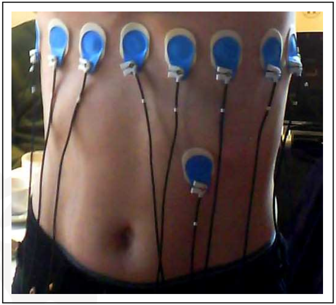
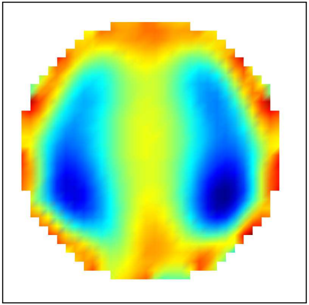
Electrical impedance tomography (EIT)





- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject.

Mathematical Model

Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega$$

$\Omega \subset \mathbb{R}^n$:	imaged body, $n \geq 2$
$\sigma(x)$:	conductivity
$u(x)$:	electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_\nu u(x)\|_{\partial\Omega}$:	applied electric current
$u(x)\|_{\partial\Omega}$:	measured boundary voltage (potential)

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves } (1)\}\,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma) : \; L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Uniqueness results

- Measurements on complete boundary $\partial\Omega$ (full data):

  Calderón (1980), Druskin (1982+85), Kohn/Vogelius (1984+85), Sylvester/Uhlmann (1987), Nachman (1996), Astala/Päivärinta (2006)

- Measurements on part of the boundary (local data):

  *Bukhgeim/Uhlmann (2002), Knudsen (2006), Isakov (2007), Kenig/Sjöstrand/Uhlmann (2007), **H**. (2008), Imanuvilov/Uhlmann/Yamamoto (2009+10), Kenig/Salo (2012+13)*

Rough summary of known results:

- L^∞ coefficients are uniquely determined from full data in 2D.
- In all cases, piecew.-anal. coefficients are uniquely determined.
- Sophisticated research on uniqueness for $\approx C^2$-coefficients

  (based on CGO-solutions for Schrödinger eq. $-\Delta u + qu = 0$, $q = \frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}}$).

Inversion of $\sigma \mapsto \Lambda(\sigma)$?

Generic iterative solvers for non-linear inverse problems:

- Linearize and regularize:

$$\Lambda_{\text{meas}} \approx \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).$$

  σ_0: Initial guess or reference state (e.g. exhaled state)

  $\rightsquigarrow$ Linear inverse problem for σ

  (Solve using linear regularization method, repeat for Newton-type algorithm.)

- Regularize and linearize:

  E.g., minimize non-linear Tikhonov functional

$$\|\Lambda_{\text{meas}} - \Lambda(\sigma)\|^2 + \alpha \|\sigma - \sigma_0\|^2 \to \text{min!}$$

Advantages of generic solvers:

- Very flexible, additional data/unknowns easily incorporated
- Problem-specific regularization can be applied

  (e.g., total variation penalization, stochastic priors, etc.)

Inversion of $\sigma \mapsto \Lambda(\sigma)$?

Problems with generic iterative solvers

- High computational cost
  - Evaluations of $\Lambda(\cdot)$ and $\Lambda'(\cdot)$ require PDE solutions.
  - PDE solutions too expensive for real-time imaging

- Convergence unclear

  (Validity of TCC/Scherzer-condition is a long-standing open problem for EIT.)
  - Convergence against true solution for exact meas. Λ_{meas}?

    (in the limit of infinite computation time)
  - Convergence against true solution for noisy meas. $\Lambda_{\text{meas}}^{\delta}$?

    (in the limit of $\delta \to 0$ and infinite computation time)
  - Global convergence? Resolution estimates for realistic noise?

- Influence of modelling errors
  - Evaluations of $\Lambda(\cdot)$ affected by large modelling errors

    (boundary geometry, electrode position, etc.)

Linearized methods

Popular approach in practice:

- Measure difference data $\Lambda_{\text{meas}} \approx \Lambda(\sigma) - \Lambda(\sigma_0)$.

  (e.g. $\Lambda(\sigma_0)$ measurement at exhaled state)
- Calculate $\sigma - \sigma_0$ from Λ_{meas} by single linearization step.

Standard linearized method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

$$\text{Solve } \Lambda'(\sigma_0)\kappa = \Lambda_{\text{meas}}, \text{ then } \kappa \approx \sigma - \sigma_0.$$

After discretization and regularization:

$$\|\mathbf{S}\kappa - \mathbf{V}\|^2 + \alpha \|\kappa\|^2 \to \text{min!}$$

$\mathbf{S}$: sensitivity matrix, $\mathbf{V}$: vector of EIT measurements.

Linearization and shape reconstruction

Theorem (**H.**/Seo, SIMA 2010)
Let κ, σ, σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Longrightarrow \quad \mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

$\mathrm{supp}_{\partial\Omega}$: outer support ($=$ supp $+$ parts unreachable from $\partial\Omega$)

$\rightsquigarrow$ Linearized EIT equation contains correct shape information

Next slides: Idea of proof using monotonicity & localized potentials.

Monotonicity

For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\sigma_0 \leq \sigma_1 \quad \Longrightarrow \quad \Lambda(\sigma_0) \geq \Lambda(\sigma_1)$$

This follows from

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \geq \int_{\partial\Omega} g\,(\Lambda(\sigma_0) - \Lambda(\sigma_1))\,g \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2$$

for all solutions u_0 of

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_\nu u_0|_{\partial\Omega} = g.$$

(e.g., Kang/Seo/Sheen 1997, Ikehata 1998)

Theorem *(**H.**, IPI 2008)*

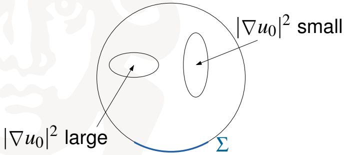
Let σ_0 fulfill unique continuation principle (UCP),

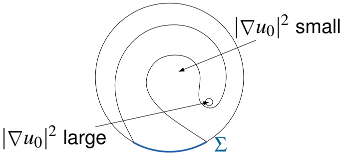
$$\overline{D_1} \cap \overline{D_2} = \emptyset, \quad \text{and} \quad \Omega \setminus (\overline{D_1} \cup \overline{D_2}) \text{ be connected with } \Sigma.$$

(Σ: open part of $\partial\Omega$)

Then there exist solutions $u_0^{(k)}$, $k \in \mathbb{N}$ with

$$\int_{D_1} \left|\nabla u_0^{(k)}\right|^2 \mathrm{d}x \to \infty \quad \text{and} \quad \int_{D_2} \left|\nabla u_0^{(k)}\right|^2 \mathrm{d}x \to 0.$$





Proof of shape invariance under linearization

- Linearization: $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$
- Monotonicity: For all "reference solutions" u_0:

$$\int_\Omega (\sigma - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x$$

$$\geq \underbrace{\int_{\partial\Omega} g\left(\Lambda(\sigma_0) - \Lambda(\sigma)\right) g}_{= -\int_{\partial\Omega} g\left(\Lambda'(\sigma_0)\kappa\right) g = \int_\Omega \kappa |\nabla u_0|^2 \,\mathrm{d}x} \geq \int_\Omega \frac{\sigma_0}{\sigma}(\sigma - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x.$$

- Use localized potentials to control $|\nabla u_0|^2$

$\leadsto$ $\mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$ $\square$

Linearization and shape reconstruction

Theorem (**H.**/Seo, SIMA 2010)
Let κ, σ, σ_0 pcw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Longrightarrow \quad \operatorname{supp}_{\partial\Omega}\kappa = \operatorname{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

$\operatorname{supp}_{\partial\Omega}$: outer support ($=$ supp $+$ parts unreachable from $\partial\Omega$)

- Linearized EIT equation contains correct shape information.
  (in the continuous version for noise-free measurements on infinitely many electrodes)
- Practitioners use heuristic regularization of linearized EIT equ.
  (in the discretized version for noisy measurements on finitely many electrodes)

Can we find a regularization that rigorously guarantees convergence of reconstructed shapes?

Monotonicity based imaging

- Monotonicity:

$$\tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

- Idea: Simulate $\Lambda(\tau)$ for test cond. τ and compare with $\Lambda(\sigma)$.
  (Tamburrino/Rubinacci 02, Lionheart, Soleimani, Ventre, ...)
- Inclusion detection: For $\sigma = 1 + \chi_D$ with unknown D, use $\tau = 1 + \chi_B$, with small ball B.

$$B \subseteq D \quad \Longrightarrow \quad \tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

- Algorithm: Mark all balls B with $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$
- Result: upper bound of D.

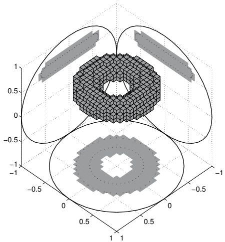
Only an upper bound? Converse monotonicity relation?

Monotonicity method (for simple test example)

Theorem *(**H.**/Ullrich, SIMA 2013)*

$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1 + \chi_B) \geq \Lambda(\sigma).$$



For faster implementation:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

Proof: Monotonicity & localized potentials

Shape can be reconstructed by linearized monotonicity tests.

Idea: Use monotonicity tests for regularizing linearized EIT equation.

Monotonicity method

Quantitative, pixel-based variant of monotonicity method:

(for $\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$)

- Pixel partition $\Omega = \bigcup_{k=1}^{m} P_k$
- Monotonicity tests

$$\beta_k \in [0, \infty] \text{ max. values s.t. } \beta_k \Lambda'(1)\chi_{P_k} \geq \Lambda(\sigma) - \Lambda(1)$$

- By theory of monotonicity method:

$$\beta_k \text{ fulfills } \begin{cases} \beta_k = 0 & \text{if } P_k \nsubseteq D \\ \beta_k \geq \frac{1}{2} & \text{if } P_k \subseteq D \end{cases}$$

Raise conductivity in each pixel until monotonicity test fails.

- Plot of β_k shows inclusions for perfect data but is very noise-sensitive since it ignores residuum information.

Monotonicity-based regularization

- Standard linearized methods for EIT: Minimize

$$\|\Lambda'(1)\kappa - (\Lambda(\sigma) - \Lambda(1))\|^2 + \alpha \|\kappa\|^2 \to \text{min!}$$

Choice of norms heuristic. No convergence theory!

- Monotonicity-based regularization: Minimize

$$\|\Lambda'(1)\kappa - (\Lambda(\sigma) - \Lambda(1))\|_{\mathrm{F}} \to \text{min!}$$

under the constraint $\kappa|_{P_k} = \text{const.}$, $0 \leq \kappa|_{P_k} \leq \min\{\frac{1}{2}, \beta_k\}$.

($\|\cdot\|_F$: Frobenius norm of Galerkin projektion to finite-dimensional space)

Theorem (**H.**/Mach, submitted)

- There exists unique minimizer $\hat{\kappa}$ and

$$P_k \subseteq \operatorname{supp} \hat{\kappa} \quad \Longleftrightarrow \quad P_k \subseteq \operatorname{supp}(\sigma - 1).$$

- Minimizer fulfills $\hat{\kappa} = \sum_{k=1}^{m} \min\{1/2, \beta_k\} \chi_{P_k}$

Monotonicity-based regularization

For noisy measurements $\Lambda^\delta_{\text{meas}} \approx \Lambda(\sigma) - \Lambda(1)$:

- Use regularized monotonicity tests

$$\beta_k^\delta \in [0,\infty] \text{ max. values s.t. } \beta_k^\delta \Lambda'(1)\chi_{P_k} \geq \Lambda^\delta_{\text{meas}} - \delta I$$

($\delta > 0$: noise level in $\mathcal{L}(L^2_\diamond(\partial\Omega))$-norm)

- Minimize

$$\|\Lambda'(1)\kappa^\delta - \Lambda^\delta_{\text{meas}}\|_{\text{F}} \to \text{min!}$$

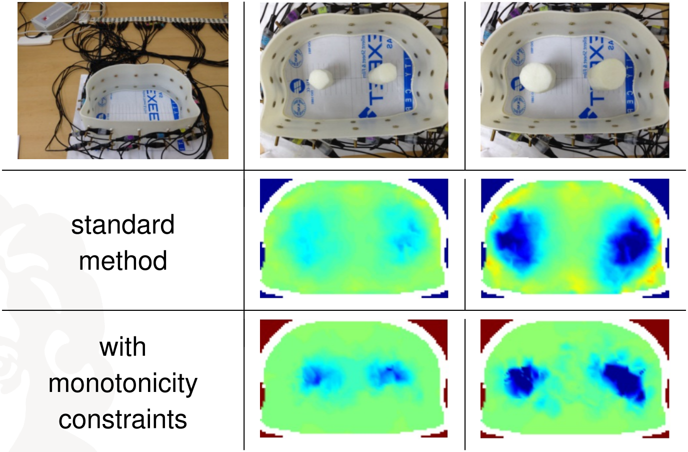
under the constraint $\kappa^\delta|_{P_k} = \text{const.}$, $0 \leq \kappa^\delta|_{P_k} \leq \min\{\frac{1}{2}, \beta_k^\delta\}$.

Theorem (**H.**/Mach, submitted)

- There exist minimizers κ^δ and $\kappa^\delta \to \hat{\kappa}$ for $\delta \to 0$.

Monotonicity-regularized solutions converge against correct shape.

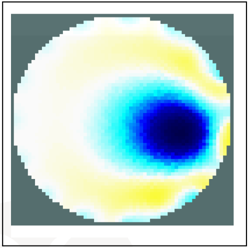
Phantom experiment



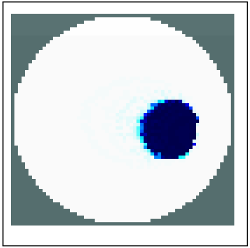
Enhancing standard methods by monotonicity-based constraints

(Zhou/**H.**/Seo, 2016)

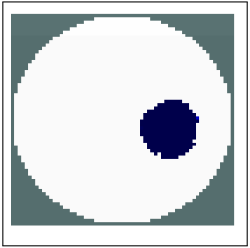
Benchmark example



standard



monoton.-regularized
(Matlab quadprog)



monoton.-regularized
(cvx package)

Monotonicity-regularization vs. community standard

(**H.**/Mach)

- EIT community standard: GREIT in EIDORS
- EIDORS: `http://eidors3d.sourceforge.net` (Adler/Lionheart)
- GREIT: Graz consensus Reconstruction algorithm for EIT (Adler et al.)
- Dataset: `iirc_data_2006` (Woo et al.): 2cm insulated inclusion in 20cm tank
  - using interpolated data on active electrodes (**H.**, Inverse Problems 2015)

Conclusions

EIT is a highly ill-posed, non-linear inverse problem.

- Convergence of generic solvers unclear.
- Practitioners use single linearization step with heuristic regularization and no theoretical justification.

Monotonicity-based regularization of linearized EIT equation

- uses that shape reconstr. in EIT is (essentially) a linear problem,
- yields solutions that rigorously converge against correct shape,
- combines rigorous theory of monotonicity method with practical robustness of residuum-based methods.

Approach (monotonicity + localized potentials) can be extended

- to partial boundary data, independently of dimension $n \geq 2$
- to other linear elliptic problems *(diffuse optical tomography, magnetostatics)*
- at least partially to closely related problems

  (eddy-current equations, p-Laplacian)