Introduction to Inverse Problems

Bastian von Harrach
harrach@math.uni-stuttgart.de

Chair of Optimization and Inverse Problems, University of Stuttgart, Germany

Department of Computational Science & Engineering
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Motivation and examples
Laplace’s demon:  (Pierre Simon Laplace 1814)

"An intellect which (…) would know all forces (…) and all positions of all items (…), if this intellect were also vast enough to submit these data to analysis, (…); for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.”
If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by solving mathematical formulas).

Goals:
- Prediction
- Optimization
- Inversion/Identification
Generic simulation problem:

Given input $x$ calculate outcome $y = F(x)$.

$x \in X$: parameters / input
$y \in Y$: outcome / measurements
$F: X \rightarrow Y$: functional relation / model

Goals:

- **Prediction**: Given $x$, calculate $y = F(x)$.
- **Optimization**: Find $x$, such that $F(x)$ is optimal.
- **Inversion/Identification**: Given $F(x)$, calculate $x$. 

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Examples of inverse problems:

- Electrical impedance tomography
- Computerized tomography
- Image Deblurring
- Numerical Differentiation
Electrical impedance tomography (EIT)

- Apply electric currents on subject’s boundary
- Measure necessary voltages
- Reconstruct conductivity inside subject.
Electrical impedance tomography (EIT)

\[ x \quad \rightarrow \quad F \quad \rightarrow \quad y = F(x) \quad \text{Measurements} \]

- **Image**
- **Measurements**

**x:** Interior conductivity distribution (image)

**y:** Voltage and current measurements

**Direct problem:** Simulate/predict the measurements (from knowledge of the interior conductivity distribution)

*Given* \( x \) calculate \( F(x) = y \)!

**Inverse problem:** Reconstruct/image the interior distribution (from taking voltage/current measurements)

*Given* \( y \) solve \( F(x) = y \)!
X-ray computerized tomography

Nobel Prize in Physiology or Medicine 1979: Allan M. Cormack and Godfrey N. Hounsfield
(Photos: Copyright © The Nobel Foundation)

Idea: Take x-ray images from several directions
Computerized tomography (CT)

Image

Measurements

Direct problem: Simulate/predict the measurements
(from knowledge of the interior density distribution)
Given $x$ calculate $F(x) = y!$

Inverse problem: Reconstruct/image the interior distribution
(from taking x-ray measurements)
Given $y$ solve $F(x) = y!$
Image deblurring

Direct problem: Simulate/predict the blurred image
(from knowledge of the true image)
Given \( x \) calculate \( F(x) = y \)!

Inverse problem: Reconstruct/image the true image
(from the blurred image)
Given \( y \) solve \( F(x) = y \)!

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Numerical differentiation

Direct problem: Calculate the primitive
Given $x$ calculate $F(x) = y!$

Inverse problem: Calculate the derivative
Given $y$ solve $F(x) = y!$
Ill-posedness
Well-posedness

Hadamard (1865–1963): A problem is called well-posed, if

- a solution exists,
- the solution is unique,
- the solution depends continuously on the given data.

Inverse Problem: \textit{Given } y \textit{ solve } F(x) = y!

- $F$ surjective?
- $F$ injective?
- $F^{-1}$ continuous?
Ill-posed problems

Ill-posedness: \( F^{-1} : Y \rightarrow X \) not continuous.

\[
\begin{align*}
\hat{x} \in X : & \quad \text{true solution} \\
\hat{y} = F(\hat{x}) \in Y : & \quad \text{exact measurement} \\
y^\delta \in Y : & \quad \text{real measurement containing noise } \delta > 0, \\
\text{e.g. } & \quad \| y^\delta - \hat{y} \|_Y \leq \delta
\end{align*}
\]

For \( \delta \rightarrow 0 \)

\[
y^\delta \rightarrow \hat{y}, \quad \text{but (generally)} \quad F^{-1}(y^\delta) \nrightarrow F^{-1}(\hat{y}) = \hat{x}
\]

Even the smallest amount of noise will corrupt the reconstructions.
Numerical differentiation

Numerical differentiation example \((h = 10^{-3})\)

\[ y(t) \quad \text{and} \quad y^\delta(t) \]

\[ \frac{y(t+h) - y(t)}{h} \quad \text{and} \quad \frac{y^\delta(t+h) - y^\delta(t)}{h} \]

Differentiation seems to be an ill-posed (inverse) problem.
Image deblurring

Deblurring seems to be an ill-posed (inverse) problem.
CT seems to be an ill-posed (inverse) problem.
Compactness and ill-posedness
Compactness

Consider the general problem

\[ F : X \to Y, \quad F(x) = y \]

with \( X, Y \) real Hilbert spaces. Assume that \( F \) is linear, bounded and injective with left inverse

\[ F^{-1} : F(X) \subseteq Y \to X. \]

Definition 1.1. \( F \in \mathcal{L}(X, Y) \) is called compact, if

\[ \overline{F(U)} \text{ is compact for all bounded } U \subseteq X, \]

i.e. if \( (x_n)_{n \in \mathbb{N}} \subset X \) is a bounded sequence then \( (F(x_n))_{n \in \mathbb{N}} \subset Y \) contains a bounded subsequence.
Compactness

Theorem 1.2. Let

- \( F \in \mathcal{L}(X, Y) \) be compact and injective, and
- \( \dim X = \infty \),

then the left inverse \( F^{-1} \) is not continuous, i.e. the inverse problem

\[
Fx = y
\]

is ill-posed.
Compactness

**Theorem 1.3.** Every limit$^1$ of compact operators is compact.

**Theorem 1.4.** If $\dim \mathcal{R}(F) < \infty$ then $F$ is compact.

**Corollary.** Every operator that can be approximated$^1$ by finite dimensional operators is compact.

$^1$in the uniform operator topology
Compactness

**Theorem 1.5.** Let $F \in \mathcal{L}(X, Y)$ possess an unbounded left inverse $F^{-1}$, and let $R_n \in \mathcal{L}(Y, X)$ be a sequence with

$$R_n y \to F^{-1} y \quad \text{for all } y \in \mathcal{R}(F).$$

Then $\|R_n\| \to \infty$.

**Corollary.** If we discretize an ill-posed problem, the better we discretize, the more unbounded our discretizations become.
Compactness and ill-posedness

**Discretization:** Approximation by finite-dimensional operators.

**Consequences for discretizing infinite-dimensional problems:**
If an infinite-dimensional direct problem can be discretized\(^1\), then
- the direct operator is compact.
- the inverse problem is ill-posed, i.e. the smallest amount of measurement noise may completely corrupt the outcome of the (exact, infinite-dimensional) inversion.

If we discretize the inverse problem, then
- the better we discretize, the larger the noise amplification is.

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\(^1\)in the uniform operator topology

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Examples

- The operator

  \[ F : \text{function} \mapsto \text{primitive function} \]

  is a linear, compact operator.

  \[ \sim \text{The inverse problem of differentiation is ill-posed}. \]

- The operator

  \[ F : \text{exact image} \mapsto \text{blurred image} \]

  is a linear, compact operator.

  \[ \sim \text{The inverse problem of image deblurring is ill-posed}. \]
Examples

- In computerized tomography, the operator

\[ F : \text{image} \mapsto \text{measurements} \]

is a linear, compact operator.

\[ \sim \text{The inverse problem of CT is ill-posed.} \]

- In EIT, the operator

\[ F : \text{image} \mapsto \text{measurements} \]

is a non-linear operator. Its Fréchet derivative is a compact linear operator.

\[ \sim \text{The (linearized) inverse problem of EIT is ill-posed.} \]
Regularization
Numerical differentiation

Numerical differentiation example

\[ y(t) \text{ and } y^\delta(t) \]

\[ \frac{y(t+h) - y(t)}{h} \text{ and } \frac{y^\delta(t+h) - y^\delta(t)}{h} \]

Differentiation is an ill-posed (inverse) problem
Regularization

Numerical differentiation:

- \( y \in C^2, \ C := 2 \sup_{\tau} |g''(\tau)| < \infty, \ |y^\delta(t) - y(t)| \leq \delta \ \forall \ t \)

\[
\left| y'(t) - \frac{y^\delta(t + h) - y^\delta(t)}{h} \right| \\
\leq \left| y'(x) - \frac{y(t + h) - y(t)}{h} \right| \\
+ \left| \frac{y(t + h) - y(t)}{h} - \frac{y^\delta(t + h) - y^\delta(t)}{h} \right| \\
\leq Ch + \frac{2\delta}{h} \to 0.
\]

for \( \delta \to 0 \) and adequately chosen \( h = h(\delta) \), e.g., \( h := \sqrt{\delta} \).
Numerical differentiation

Numerical differentiation example

\[ y'(t) \]

\[ \frac{y(\delta(t+h) - y(\delta(t))}{h} \text{ with } h \text{ very small} \]

\[ \frac{y(\delta(t+h) - y(\delta(t))}{h} \text{ with } h \approx \sqrt{\delta} \]

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Idea of regularization: Balance noise amplification and approximation

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Regularization of inverse problems:

- $F^{-1}$ not continuous, so that generally $F^{-1}(y^\delta) \not= F^{-1}(y) = x$ for $\delta \to 0$
- $R_\alpha$ continuous approximations of $F^{-1}$
  
  $R_\alpha \to F^{-1}$ (pointwise) for $\alpha \to 0$

$$R_\alpha(\delta)y^\delta \to F^{-1}y = x \quad \text{for } \delta \to 0$$

if the parameter $\alpha = \alpha(\delta)$ is correctly chosen.

Inexact but continuous reconstruction (regularization) + Information on measurement noise (parameter choice rule) = Convergence
Prominent regularization methods

- Tikhonov regularization
  
  \[ R_\alpha = (F^* F + \alpha I)^{-1} F^* \]

  - \( R_\alpha \) continuous, \( \| R_\alpha \| \leq \frac{1}{\sqrt{\alpha}} \)
  - \( R_\alpha y \rightarrow F^{-1} y \) for \( \alpha \rightarrow 0, \ y \in \mathcal{R}(F) \)
  - \( R_\alpha y^\delta \) minimizes

  \[ \| Fx - y^\delta \|^2 + \alpha \| x \|^2 \rightarrow \text{min!} \]

- Truncated singular value decomposition (TSVD)
- Landweber method
- ...
Parameter choice rule

Convergence of Tikhonov-regularization

\[ \| R_\alpha y^\delta - F^{-1}y \| \leq \| R_\alpha (y^\delta - y) \| + \| R_\alpha y - F^{-1}y \| \leq \| R_\alpha \| \delta \rightarrow 0 \text{ for } \alpha \rightarrow 0 \]

Choose \( \alpha(\delta) \) such that (for \( \delta \rightarrow 0 \))

\[ \alpha(\delta) \rightarrow 0 \]
\[ \| R_\alpha(\delta) \| \delta = \frac{\delta}{\sqrt{\alpha(\delta)}} \rightarrow 0 \]

then \( R_\alpha(\delta)y^\delta \rightarrow F^{-1}y \). E.g., set \( \alpha(\delta) := \delta \).

Exakt inversion does not converge, \( F^{-1}y^\delta \not\xrightarrow{} F^{-1}y \).
Tikhonov-regularization converges, \( R_\delta y^\delta \rightarrow F^{-1}y \).

Better parameter choice rule:
Choose \( \alpha \) such that \( \| FR_\alpha y^\delta - y^\delta \| \approx \delta \) (discrepancy principle)
Image deblurring

\[ \hat{x}, \quad \hat{y} = F\hat{x}, \quad y^\delta \]

\[ F^{-1}y^\delta \]

\[ (F^* F + \delta I)^{-1}F^*y^\delta \]

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Computerized tomography

\[ \hat{\chi} \quad \hat{y} = F\hat{\chi} \quad y^\delta \]

\[ F^{-1}y^\delta \]

\[ (F^*F + \delta I)^{-1}F^*y^\delta \]
Conclusions and remarks

Conclusions

- Inverse problems are of great importance in comput. science. (parameter identification, medical tomography, etc.)
- For ill-posed inverse problems, the best data-fit solutions generally do not converge against the true solution.
- The regularized solutions do converge against the true solution.

Strategies for non-linear inverse problems $F(x) = y$:

- First linearize, then regularize.
- First regularize, then linearize.

A-priori information

- Regularization can be used to incorporate a-priori knowledge (promote sparsity or sharp edges, include stochastic priors, etc.)