Inverse coefficient problems in elliptic partial differential equations

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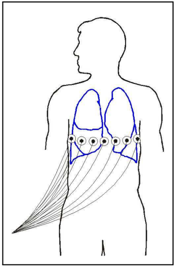
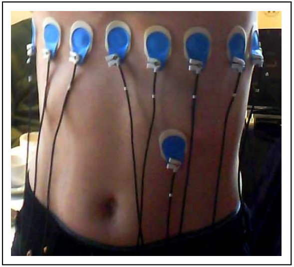
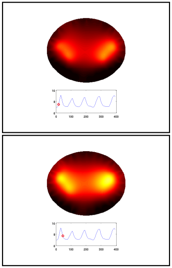
Electrical impedance tomography (EIT)







- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)

Mathematical Model

Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega$$

$\Omega \subset \mathbb{R}^n$: imaged body, $n \geq 2$
$\sigma(x)$: conductivity
$u(x)$: electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_\nu u(x)|_{\partial\Omega}$: applied electric current
$u(x)|_{\partial\Omega}$: measured boundary voltage (potential)

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves } (1)\}\,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma): \; L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Partial/local data

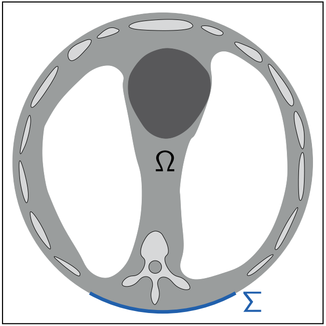
Measurements on open part of boundary $\Sigma \subset \partial\Omega$:
($\partial\Omega \setminus \Sigma$ is kept insulated.)

Recover σ from

$$\Lambda(\sigma): \; L^2_\diamond(\Sigma) \to L^2_\diamond(\Sigma), \quad g \mapsto u|_\Sigma,$$

where u solves $\nabla \cdot (\sigma \nabla u) = 0$ with

$$\sigma \partial_\nu u|_\Sigma = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$



Optimization and Inverse Problems

Given measurements $\Lambda_{\text{measured}}$

- Inverse Problem: Solve $\Lambda(\sigma) = \Lambda_{\text{measured}}$
- Optimization: Minimize $\|\Lambda(\sigma) - \Lambda_{\text{measured}}\|^2$ *(+ regularization)*

Special challenges in inverse problems:

- Uniqueness is crucial.
- Local minima are usually useless.
- Convergence of iterates to true solution is crucial.
- Additional assumptions can often not be justified.

  (sufficient optimality conditions, constraint qualifications, source conditions, . . .)

Inverse coeff. problems pose major challenges even for simple PDEs.

Challenges

Challenges in inverse coefficient problems such as EIT:

- Uniqueness
  - Is σ uniquely determined from the NtD $\Lambda(\sigma)$?
- Non-linearity and ill-posedness
  - Reconstruction algorithms to determine σ from $\Lambda(\sigma)$?
  - Local/global convergence results?
- Realistic data
  - What can we recover from real measurements?
    (finite number of electrodes, realistic electrode models, ...)
  - Measurement and modelling errors? Resolution?

In this talk: A simple strategy (monotonicity + localized potentials) to attack these challenges.

Uniqueness

Uniqueness results

- Measurements on complete boundary (full data):
  Calderón (1980), Druskin (1982+85), Kohn/Vogelius (1984+85), Sylvester/Uhlmann (1987), Nachman (1996), Astala/Päivärinta (2006)
- Measurements on part of the boundary (local data):
  Bukhgeim/Uhlmann (2002), Knudsen (2006), Isakov (2007), Kenig/Sjöstrand/Uhlmann (2007), **H**. *(2008), Imanuvilov/Uhlmann/Yamamoto (2009+10), Kenig/Salo (2012+13)*

- L^∞ coefficients are uniquely determined from full data in 2D.
- In all cases, piecew.-anal. coefficients are uniquely determined.
- Sophisticated research on uniqueness for $\approx C^2$-coefficients
  (based on CGO-solutions for Schrödinger eq. $-\Delta u + qu = 0$, $q = \frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}}$).

Next: Uniqueness proof using monotonicity and loc. potentials.

Monotonicity

For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\sigma_0 \leq \sigma_1 \quad \Longrightarrow \quad \Lambda(\sigma_0) \geq \Lambda(\sigma_1)$$

This follows from

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \geq \int_\Sigma g \left(\Lambda(\sigma_0) - \Lambda(\sigma_1)\right) g \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2$$

for all solutions u_0 of

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_\nu u_0|_\Sigma = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$

(e.g., Kang/Seo/Sheen 1997, Ikehata 1998)

Can we prove uniqueness by controlling $|\nabla u_0|^2$?

Localized potentials

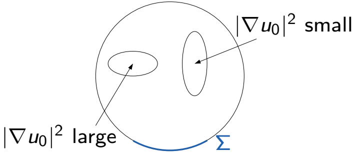
Theorem *(**H.**, 2008)*

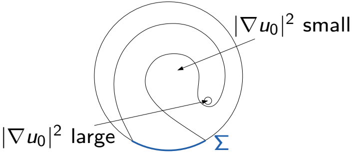
Let σ_0 fulfill unique continuation principle (UCP),

$$\overline{D_1} \cap \overline{D_2} = \emptyset, \quad \text{and} \quad \Omega \setminus (\overline{D}_1 \cup \overline{D}_2) \text{ be connected with } \Sigma.$$

Then there exist solutions $u_0^{(k)}$, $k \in \mathbb{N}$ with

$$\int_{D_1} \left| \nabla u_0^{(k)} \right|^2 \, \mathrm{d}x \to \infty \quad \text{and} \quad \int_{D_2} \left| \nabla u_0^{(k)} \right|^2 \, \mathrm{d}x \to 0.$$



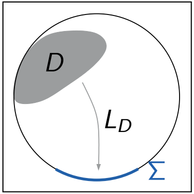


Proof 1/3

Virtual measurements:

$L_D: \ H^1_\diamond(D)' \to L^2_\diamond(\Sigma), \quad f \mapsto u|_\Sigma,$ with

$$\int_\Omega \sigma \nabla u \cdot \nabla v \, \mathrm{d}x = \langle f, v|_D \rangle \quad \forall v \in H^1_\diamond(D).$$



By (UCP): If $\overline{D_1} \cap \overline{D_2} = \emptyset$ and $\Omega \setminus (\overline{D}_1 \cup \overline{D}_2)$ is connected with Σ, then $\mathcal{R}(L_{D_1}) \cap \mathcal{R}(L_{D_2}) = 0$.

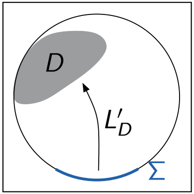
Sources on different domains yield different virtual measurements.

Proof 2/3

Dual operator:

$$L_D' : L^2_\diamond(\Sigma) \to H^1_\diamond(D), \quad g \mapsto u|_D, , \text{ with}$$

$$\nabla \cdot (\sigma \nabla u) = 0, \quad \sigma \partial_\nu u|_\Sigma = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$



Evaluating solutions on D is dual operation to virtual measurements.

Proof 3/3

Functional analysis:

X, Y_1, Y_2 reflexive Banach spaces, $L_1 \in \mathcal{L}(Y_1, X)$, $L_2 \in \mathcal{L}(Y_1, X)$.

$$\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2) \quad \Longleftrightarrow \quad \|L_1' x\| \lesssim \|L_2' x\| \ \forall x \in X'.$$

Here: $\mathcal{R}(L_{D_1}) \not\subseteq \mathcal{R}(L_{D_2}) \quad \Longrightarrow \quad \|u_0|_{D_1}\|_{H^1_\diamond} \not\lesssim \|u_0|_{D_2}\|_{H^1_\diamond}.$

If two sources do not generate the same data, then the respective evaluations are not bounded by each other.

Note: $H^1_\diamond(D)'$-source $\quad \longleftrightarrow \quad$ $H^1_\diamond(D)$-evaluation.

Consequences

- Back to Calderón: Let $\Lambda(\sigma_0) = \Lambda(\sigma_1)$, σ_0 fulfills (UCP).
- By monotonicity,

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x \geq 0 \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x \quad \forall u_0$$

- Assume: $\exists$ neighbourhood U of Σ where $\sigma_1 \geq \sigma_0$ but $\sigma_1 \not\equiv \sigma_0$

$\rightsquigarrow$ Potential with localized energy in U contradicts monotonicity

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Higher conductivity reachable by the bndry cannot be balanced out.

Corollary *(Druskin 1982+85, Kohn/Vogelius, 1984+85)*
Calderón problem is uniquely solvable for piecw.-anal. conductivities.

Diffuse optical tomography

Same strategy shows uniqueness for two coefficients *(**H.** 2009+12)*:

$$-\nabla \cdot (a \nabla u) + cu = 0 \quad \text{in } \Omega.$$

- Let a, c pcw. constant, then

$$\Lambda(a_1, c_1) = \Lambda(a_2, c_2) \quad \Longleftrightarrow \quad a_1 = a_2 \text{ and } c_1 = c_2.$$

- Let a, c pcw. anal., then $\Lambda(a_1, c_1) = \Lambda(a_2, c_2)$

$$\Longleftrightarrow \begin{cases} \frac{\Delta \sqrt{a_1}}{\sqrt{a_1}} + \frac{c_1}{a_1} = \frac{\Delta \sqrt{a_2}}{\sqrt{a_2}} + \frac{c_2}{a_2} \text{ on smooth parts} \\ \frac{a_1^+|_\Gamma}{a_1^-|_\Gamma} = \frac{a_2^+|_\Gamma}{a_2^-|_\Gamma}, \text{ and } \frac{[\partial_\nu a_2]_\Gamma}{a_2^-|_\Gamma} = \frac{[\partial_\nu a_1]_\Gamma}{a_1^-|_\Gamma} \text{ on discontinuity set} \end{cases}$$

Proof: Monotonicity + localized potentials

Non-linearity

Non-linearity

Back to the non-linear forward operator of EIT

$$\Lambda: \ \sigma \mapsto \Lambda(\sigma), \quad L^\infty_+(\Omega) \to \mathcal{L}(L^2_\diamond(\Sigma))$$

Generic approach for inverting Λ: Linearization

$$\Lambda(\sigma) - \Lambda(\sigma_0) \approx \Lambda'(\sigma_0)(\sigma - \sigma_0)$$

σ_0: known reference conductivity / initial guess / ...

$\Lambda'(\sigma_0)$: Fréchet-Derivative / sensitivity matrix.

$$\Lambda'(\sigma_0) : L^\infty_+(\Omega) \to \mathcal{L}(L^2_\diamond(\Sigma)).$$

$\rightsquigarrow$ Solve linearized equation for difference $\sigma - \sigma_0$.

Often: $\operatorname{supp}(\sigma - \sigma_0) \subset \Omega$ (*"shape" / "inclusion"*)

Linearization

Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- Multiple possibilities to measure residual norm and to regularize.
- No rigorous theory for single linearization step.
- Almost no theory for Newton iteration:
    - Dobson (1992): (Local) convergence for regularized EIT equation.
    - Lechleiter/Rieder(2008): (Local) convergence for discretized setting.
    - No (local) convergence theory for non-discretized case!
      Non-linearity condition (Scherzer / tangential cone cond.) still open problem
- D-bar method: convergent 2D-implementation for $\sigma \in C^2$ and full bndry data *(Knudsen, Lassas, Mueller, Siltanen 2008)*

Linearization

Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- Seemingly, no rigorous results possible for single lineariz. step.
- Seemingly, only justifiable for small $\sigma - \sigma_0$ (local results).

Here: Rigorous and global(!) result about the linearization error.

Linearization and shape reconstruction

Theorem *(H./Seo 2010)*

Let κ, σ, σ_0 piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

$$\mathrm{supp}_\Sigma \kappa = \mathrm{supp}_\Sigma(\sigma - \sigma_0)$$

supp_Σ: outer support (= support, if support is compact and has conn. complement)

- Solution of lin. equation yields correct (outer) shape.
- No assumptions on $\sigma - \sigma_0$!

$\rightsquigarrow$ Linearization error does not lead to shape errors.

Taking the (wrong) reference current paths for reconstruction still yields the correct shape information!

Proof

- Linearization: $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$
- Monotonicity: For all "reference solutions" u_0:

$$\int_\Omega (\sigma - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x$$

$$\geq \underbrace{\int_\Sigma g\left(\Lambda(\sigma_0) - \Lambda(\sigma)\right) g}_{} \geq \int_\Omega \frac{\sigma_0}{\sigma}(\sigma - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x.$$

$$= \int_\Sigma g\left(\Lambda'(\sigma_0)\kappa\right) g = \int_\Omega \kappa |\nabla u_0|^2 \, \mathrm{d}x$$

- Use localized potentials to control $|\nabla u_0|^2$

$\leadsto$ $\mathrm{supp}_\Sigma \kappa = \mathrm{supp}_\Sigma(\sigma - \sigma_0)$ □

In shape reconstruction problems we can avoid non-linearity.

Reconstruction from realistic data

Monotonicity based imaging

- Monotonicity:

$$\tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

- Idea: Simulate $\Lambda(\tau)$ for test cond. τ and compare with $\Lambda(\sigma)$.
  (Tamburrino/Rubinacci 02, Lionheart, Soleimani, Ventre, ...)
- Inclusion detection: For $\sigma = 1 + \chi_D$ with unknown D, use $\tau = 1 + \chi_B$, with small ball B.

$$B \subseteq D \quad \Longrightarrow \quad \tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

- Algorithm: Mark all balls B with $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$
- Result: upper bound of D.

Only an upper bound? Converse monotonicity relation?

Converse monotonicity relation

Theorem *(**H.**/Ullrich, 2013)*
$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1 + \chi_B) \geq \Lambda(\sigma).$$

$\rightsquigarrow$ Monotonicity method detects exact shape.

For faster implementation:

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

$\rightsquigarrow$ Linearized monotonicity method detects exact shape.

Proof: Monotonicity + localized potentials

General case

Theorem *(**H.**/Ullrich, 2013)*. Let $\sigma \in L^\infty_+(\Omega)$ be piecewise analytic. The intersection of all *hole-free* $C \subseteq \overline{\Omega}$ with

$$\exists \alpha > 1 : \ \Lambda(1 + \alpha\chi_C) \leq \Lambda(\sigma) \leq \Lambda(1 - \chi_C/\alpha)$$

is identical to the *(outer)* support of $\sigma - 1$.

- Result also holds with linearized condition

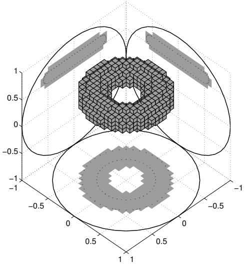
$$\exists \alpha > 1 : \ \Lambda(1) + \alpha\Lambda'(1)\chi_C \leq \Lambda(\sigma) \leq \Lambda(1) - \alpha\Lambda'(1)\chi_C.$$

- Result covers indefinite case,
  e.g., $\sigma = 1 + \chi_{D_1} - \frac{1}{2}\chi_{D_2}$

Monotonicity based shape reconstruction

Monotonicity based reconstruction

- is intuitive, yet rigorous
- is stable (no infinity or range tests)
- works for pcw. anal. conductivities (no definiteness conditions)
- requires only the reference solution



Approach is closely related to (and heavily inspired by)

- Factorization Method of Kirsch and Hanke
  (in EIT: Brühl, Hakula, **H.***, Hyvönen, Lechleiter, Nachman, Päivärinta, Pursiainen, Schappel, Schmitt, Seo, Teirilä, Woo, . . .)*
- Ikehata's Enclosure Method and probing with Sylvester-Uhlmann-CGOs *(Ide, Isozaki, Nakata, Siltanen, Wang, . . .)*
- Classic inclusion detection results *(Friedmann, Isakov, . . .)*

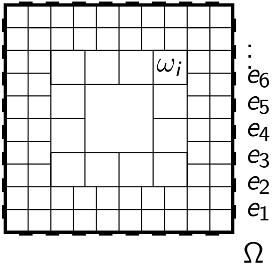
Realistic data & Uncertainties

- Finite number of electrodes, CEM, noisy data $\Lambda^\delta(\sigma)$
- Unknown background, e.g., $1-\epsilon \leq \sigma_0(x) \leq 1+\epsilon$
- Anomaly with some minimal contrast to background, e.g.,
$$\sigma(x) = \sigma_0(x) + \kappa(x)\chi_D, \quad \kappa(x) \geq 1$$
- Can we rigorously guarantee to find inclusion D?

Monotonicity-based
Rigorous Resolution Guarantee
*(**H.**/Ullrich, to appear):*

- If $D = \emptyset$, method returns $\emptyset$.
- If $D \supset \omega_i$ then it is detected.

(Here: 32 electrodes, $\epsilon = 1\%$, $\delta = 1.4\%$)



Conclusions

Using monotonicity and localized potentials we can show

- uniqueness results for piecewiese anal. coefficients.
- invariance of shape information under linearization.
- resolution guarantees for locating anomalies in unknown backgrounds with realistic finite precision data.

Major limitations / open problems for our approach

- Piecewise analyticity required to prevent infinite oscillations.
- Approach requires operator/matrix-structure of measurements.
  ($\rightsquigarrow$ Voltage has to be measured on current-driven electrodes.)