Detecting stochastic inclusions in electrical impedance tomography

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Electrical impedance tomography (EIT)

- Apply electric currents on subject’s boundary
- Measure necessary voltages
- Reconstruct conductivity inside subject.

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Mathematical Model (deterministic)

Electrical potential $u(x)$ solves
\[ \nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in D \]

$D \subset \mathbb{R}^n$: imaged body, $n \geq 2$
$\sigma(x)$: conductivity
$u(x)$: electrical potential

Idealistic model for boundary measurements (continuum model):
\[ \sigma \partial_{\nu} u(x) \big|_{\partial D}: \text{applied electric current} \]
\[ u(x) \big|_{\partial D}: \text{measured boundary voltage (potential)} \]
Calderón problem (deterministic)

Can we recover $\sigma \in L_+^\infty(D)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in D \quad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial D}, \sigma \partial_{\nu} u|_{\partial D}) : u \text{ solves } (1)\}?$$

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Equivalent: Recover $\sigma$ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma) : L^2_\diamond(\partial D) \to L^2_\diamond(\partial D), \quad g \mapsto u|_{\partial D},$$

where $u$ solves (1) with $\sigma \partial_{\nu} u|_{\partial D} = g$. 

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Inclusion detection in EIT

\[ \sigma: \text{Actual (unknown) conductivity} \]
\[ \sigma_0: \text{Initial guess or reference state (e.g. exhaled state)} \]

- \( \text{supp}(\sigma - \sigma_0) \) often relevant in practice

**Inclusion detection problem** (aka shape reconstruction or anomaly detection)

*Can we recover \( \text{supp}(\sigma - \sigma_0) \) from \( \Lambda(\sigma), \Lambda(\sigma_0) \)?*

- Generic approach: parametrize \( \text{supp}(\sigma - \sigma_0) \), Level-Set-Methods
- Problems:
  - PDE solutions required in each iteration
  - convergence unclear
Linearization and inclusion detection

**Theorem** ([H. Seo, SIAM J. Math. Anal. 2010])

Let $\kappa$, $\sigma$, $\sigma_0$ piecewise analytic.

$$\Lambda'(\sigma_0) \kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \Rightarrow \quad \text{supp}_{\partial D} \kappa = \text{supp}_{\partial D}(\sigma - \sigma_0)$$

$\text{supp}_{\partial D}$: outer support ($= \text{supp} + \text{parts unreachable from } \partial D$)

$\sim$ Inclusion detection is essentially a linear problem.

$\sim$ Fast, rigorous and globally convergent inclusion detection methods are possible.

- Next slides: Monotonicity method.
Monotonicity

For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\sigma_0 \leq \sigma_1 \implies \Lambda(\sigma_0) \geq \Lambda(\sigma_1)$$

This follows from

$$\int_{\Omega} (\sigma_1 - \sigma_0)|\nabla u_0|^2 \geq \int_{\partial \Omega} g (\Lambda(\sigma_0) - \Lambda(\sigma_1)) g \geq \int_{\Omega} \frac{\sigma_0}{\sigma_1} (\sigma_1 - \sigma_0)|\nabla u_0|^2$$

for all solutions $u_0$ of

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_{\nu} u_0 |_{\partial \Omega} = g.$$ 

(e.g., Kang/Seo/Sheen 1997, Ikehata 1998)
Monotonicity based imaging

- Monotonicity:
  \[ \tau \leq \sigma \implies \Lambda(\tau) \geq \Lambda(\sigma) \]

- Idea: Simulate \( \Lambda(\tau) \) for test cond. \( \tau \) and compare with \( \Lambda(\sigma) \).
  (Tamburrino/Rubinacci 02, Lionheart, Soleimani, Ventre, . . .)

- Inclusion detection: For \( \sigma = 1 + \chi_A \) with unknown anomaly \( A \),
  use \( \tau = 1 + \chi_B \), with small ball \( B \).
  \[ B \subseteq A \implies \tau \leq \sigma \implies \Lambda(\tau) \geq \Lambda(\sigma) \]

- Algorithm: Mark all balls \( B \) with \( \Lambda(1 + \chi_B) \geq \Lambda(\sigma) \)
- Result: upper bound of anomaly \( A \).

*Only an upper bound? Converse monotonicity relation?*
Monotonicity method

Theorem (H./Ullrich, 2013)

\[ D \setminus \overline{A} \text{ connected. } \sigma = 1 + \chi_A. \]

\[ B \subseteq A \iff \Lambda(1 + \chi_B) \geq \Lambda(\sigma). \]

For faster implementation:

\[ B \subseteq A \iff \Lambda(1) + \frac{1}{2} \Lambda'(1) \chi_B \geq \Lambda(\sigma). \]

Inclusion can be reconstructed by linearized monotonicity tests.

\[ \sim \text{ Fast, rigorous, allows globally convergent implementation} \]

- Ideas of proof evolved from the similar Factorization Method

(For EIT: Arridge, Betcke, Brühl, Chaulet, Choi, Hakula, Hanke, H., Holder, Hyvönen, Kirsch, Lechleiter, Nachman, Päivärinta, Pursiainen, Schappel, Schmitt, Seo, Teirilä, . . . )

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Calderón problem

Deterministic Calderón Problem: Can we recover $\sigma$ from $\text{NtD} \Lambda(\sigma) : L^2_\partial(\partial D) \to L^2_\partial(\partial D)$, $g \mapsto u|_{\partial D}$, where $u$ solves $\nabla \cdot (\sigma \nabla u) = 0$ with $\sigma \partial_\nu u|_{\partial D} = g$?

- Stochastic Calderón problem:
  
  *Can we recover $E(\sigma)$ from $E(\Lambda(\sigma))$?*

- Stochastic inclusion detection in hom. background ($\sigma_0 = 1$):
  
  *Can we recover $\text{supp}(E(\sigma) - 1)$ from $E(\Lambda(\sigma))$?*

- (Possible) Application: Biomedical anomaly detection from temporally averaged measurements.

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Detecting stochastic inclusions

Theorem (Barth/H./Hyvönen/Mustonen, submitted)
Consider a domain with a stochastic inclusion \( A \),

\[
\sigma = \begin{cases} 
1 & \text{in } D \setminus A, \\
\sigma_A(x, \omega) & \text{in } A,
\end{cases}
\]

- \( \sigma_A : \Omega \rightarrow L^\infty_+ (A) \), \( \Omega \) probability space,
- \( \sigma_A, \sigma_A^{-1} \in L^1(\Omega, L^\infty_+(A)) \)

If

\[
\mathbb{E}(\sigma_A) > 1 \quad \text{and} \quad \mathbb{E}(\sigma_A^{-1})^{-1} > 1,
\]

then, both, the Factorization Method and the Monotonicity Method applied to \( \mathbb{E}(\sigma) \) recover the inclusion \( A \).
Monotonicity for stochastic inclusions

Main idea of the proof: Monotonicity for stochastic inclusions:

For deterministic \( \sigma_0 \) and stochastic \( \sigma \):

\[
\int_D (\mathbb{E}(\sigma) - \sigma_0) |\nabla u_0|^2 \, dx \geq \int_{\partial D} g(\Lambda(\sigma_0) - \mathbb{E}(\Lambda(\sigma))) g \, ds \\
\geq \int_D \sigma_0^2 (\sigma_0^{-1} - \mathbb{E}(\sigma^{-1})) |\nabla u_0|^2 \, dx.
\]

In particular,

\[
\sigma_0 \leq \mathbb{E}(\sigma) \text{ and } \sigma_0 \leq \mathbb{E}(\sigma^{-1})^{-1} \implies \Lambda(\sigma_0) \geq \mathbb{E}(\Lambda(\sigma))
\]
Example

- Background conductivity \( \sigma_0 = 1 \)
- Inclusions conductivity uniformly distributed in \([0.5, 3.5]\)

\[
\mathbb{E}(\sigma_A) \geq \mathbb{E}(\sigma_A^{-1})^{-1} \approx 1.54 > 1 = \sigma_0
\]

- Images show result of Factorization Method applied to \(\mathbb{E}(\sigma)\)
  (Left Image: no noise, Right Image: 0.1% noise)

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Conclusions

In EIT, stochastic inclusions in a deterministic background

- can be detected by deterministic Factorization or Monotonicity Method applied to the measurement’s expectation value,

- if, both, $\mathbb{E}(\sigma_A)$ and $\mathbb{E}(\sigma_A^{-1})^{-1}$ are larger than bg conductivity (or both are smaller than background conductivity)

Roughly speaking,

- stochastic conductivity uncertainty in $\sigma$ is analogous to deterministic uncertainty in $[\mathbb{E}(\sigma^{-1})^{-1}, \mathbb{E}(\sigma)]$

Outlook:

- Stochastic inclusions in stochastic backgrounds may be treatable by resolution guarantees 
  (Deterministic case: H., Ullrich, IEEE TMI 2015)