Monotonicity-based methods for elliptic inverse coefficient problems

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Electrical impedance tomography (EIT)

- Apply electric currents on subject’s boundary
- Measure necessary voltages
- Reconstruct conductivity inside subject.

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Mathematical Model

Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega$$

$\Omega \subset \mathbb{R}^n$: imaged body, $n \geq 2$

$\sigma(x)$: conductivity

$u(x)$: electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_\nu u(x)|_{\partial \Omega}$: applied electric current

$u(x)|_{\partial \Omega}$: measured boundary voltage (potential)
Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \quad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial \Omega}, \sigma \partial_\nu u|_{\partial \Omega}) : u \text{ solves (1)}\}?$$

Equivalent: Recover $\sigma$ from Neumann-to-Dirichlet-Operator

$$\Lambda(\sigma) : L^2_\phi(\partial \Omega) \rightarrow L^2_\phi(\partial \Omega), \quad g \mapsto u|_{\partial \Omega},$$

where $u$ solves (1) with $\sigma \partial_\nu u|_{\partial \Omega} = g$. 

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Inversion of $\sigma \mapsto \Lambda(\sigma) = \Lambda_{\text{meas}}$?

Generic solvers for non-linear inverse problems:

- **Linearize and regularize:**
  \[
  \Lambda_{\text{meas}} = \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).
  \]
  $\sigma_0$: Initial guess or reference state (e.g. exhaled state)

- **Linear inverse problem for $\sigma$**
  (Solve using linear regularization method, repeat for Newton-type algorithm.)

- **Regularize and linearize:**
  E.g., minimize non-linear Tikhonov functional
  \[
  \|\Lambda_{\text{meas}} - \Lambda(\sigma)\|^2 + \alpha \|\sigma - \sigma_0\|^2 \rightarrow \text{min}!
  \]

Advantages of generic solvers:

- Very flexible, additional data/unknowns easily incorporated
- Problem-specific regularization can be applied
  (e.g., total variation penalization, stochastic priors, etc.)
Inversion of $\sigma \mapsto \Lambda(\sigma) = \Lambda_{\text{meas}}$?

Problems with generic solvers

- High computational cost
  (Evaluations of $\Lambda(\cdot)$ and $\Lambda'(\cdot)$ require PDE solutions)

- Convergence unclear
  (Validity of TCC/Scherzer-condition is a long-standing open problem for EIT.)
  - Convergence against true solution for exact meas. $\Lambda_{\text{meas}}$?
    (in the limit of infinite computation time)
  - Convergence against true solution for noisy meas. $\Lambda_{\delta_{\text{meas}}}$?
    (in the limit of $\delta \to 0$ and infinite computation time)
  - Global convergence? Resolution estimates for realistic noise?

D-bar method

- Convergent 2D-implementation for $\sigma \in C^2$ and full bndry data
  (Knudsen, Lassas, Mueller, Siltanen 2008)
Linearization and shape reconstruction

Let $\kappa$, $\sigma$, $\sigma_0$ p.cw. analytic.

$$\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \implies \text{supp}_{\partial\Omega}\kappa = \text{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

$\text{supp}_{\partial\Omega}$: outer support ( = supp + parts unreachable from $\partial\Omega$)

$\sim$ Linearized EIT equation contains correct shape information

**Next slides**: Idea of proof using monotonicity & localized potentials.

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Monotonicity

For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\sigma_0 \leq \sigma_1 \implies \Lambda(\sigma_0) \geq \Lambda(\sigma_1)$$

This follows from

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \geq \int_{\partial \Omega} g \left( \Lambda(\sigma_0) - \Lambda(\sigma_1) \right) g \geq \int_\Omega \frac{\sigma_0}{\sigma_1} (\sigma_1 - \sigma_0)|\nabla u_0|^2$$

for all solutions $u_0$ of

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_\nu u_0|_{\partial \Omega} = g.$$

(e.g., Kang/Seo/Sheen 1997, Ikehata 1998)

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Localized potentials

**Theorem (H., 2008)**

Let $\sigma_0$ fulfill unique continuation principle (UCP),

\[
\overline{D_1} \cap \overline{D_2} = \emptyset, \quad \text{and} \quad \Omega \setminus (\overline{D_1} \cup \overline{D_2}) \text{ be connected with } \Sigma.
\]

Then there exist solutions $u_0^{(k)}$, $k \in \mathbb{N}$ with

\[
\int_{D_1} |\nabla u_0^{(k)}|^2 \, dx \to \infty \quad \text{and} \quad \int_{D_2} |\nabla u_0^{(k)}|^2 \, dx \to 0.
\]
Proof of shape invariance under linearization

- **Linearization:** \( \Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \)
- **Monotonicity:** For all "reference solutions" \( u_0 \):

\[
\int_\Omega (\sigma - \sigma_0)|\nabla u_0|^2 \, dx \\
\geq \int_{\partial \Omega} g (\Lambda(\sigma_0) - \Lambda(\sigma)) \, g \\
= -\int_{\partial \Omega} g (\Lambda'(\sigma_0)\kappa) \, g = \int_\Omega \frac{\sigma_0}{\sigma} (\sigma - \sigma_0)|\nabla u_0|^2 \, dx.
\]

- Use localized potentials to control \( |\nabla u_0|^2 \)

\( \supp_{\partial \Omega} \kappa = \supp_{\partial \Omega} (\sigma - \sigma_0) \)

\[ \square \]
Linearization and shape reconstruction

Let \( \kappa, \sigma, \sigma_0 \) pcw. analytic.

\[
\Lambda'(\sigma_0) \kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \quad \implies \quad \text{supp}_{\partial \Omega} \kappa = \text{supp}_{\partial \Omega}(\sigma - \sigma_0)
\]

\( \text{supp}_{\partial \Omega} \): outer support (\( = \text{supp} + \text{parts unreachable from } \partial \Omega \) )

\( \Rightarrow \) Linearized EIT equation contains correct shape information

*Can we recover conductivity changes (anomalies, inclusions, . . . ) in a fast, rigorous and globally convergent way?*

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Monotonicity based imaging

- Monotonicity:

\[ \tau \leq \sigma \implies \Lambda(\tau) \geq \Lambda(\sigma) \]

- Idea: Simulate \( \Lambda(\tau) \) for test cond. \( \tau \) and compare with \( \Lambda(\sigma) \).

\( (\text{Tamburrino/Rubinacci 02, Lionheart, Soleimani, Ventre, ...}) \)

- Inclusion detection: For \( \sigma = 1 + \chi_D \) with unknown \( D \), use \( \tau = 1 + \chi_B \), with small ball \( B \).

\[ B \subseteq D \implies \tau \leq \sigma \implies \Lambda(\tau) \geq \Lambda(\sigma) \]

- Algorithm: Mark all balls \( B \) with \( \Lambda(1 + \chi_B) \geq \Lambda(\sigma) \)

- Result: upper bound of \( D \).

Only an upper bound? Converse monotonicity relation?

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Monotonicity method (for simple test example)

Theorem (H./Ullrich, 2013)
\[ \Omega \setminus \overline{D} \text{ connected. } \sigma = 1 + \chi_D. \]
\[ B \subseteq D \iff \Lambda(1 + \chi_B) \geq \Lambda(\sigma). \]

For faster implementation:
\[ B \subseteq D \iff \Lambda(1) + \frac{1}{2} \Lambda'(1) \chi_B \geq \Lambda(\sigma). \]

Proof: Monotonicity & localized potentials

Shape can be reconstructed by linearized monotonicity tests.

\[ \sim \text{ fast, rigorous, allows globally convergent implementation} \]

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Improving residuum-based methods

**Theorem** ([H./Minh, preprint])
Let $\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

- Pixel partition $\Omega = \bigcup_{k=1}^{m} P_k$
- Monotonicity tests

$$\beta_k \in [0, \infty] \text{ max. values s.t. } \beta_k \Lambda'(1) \chi_{P_k} \geq \Lambda(\sigma) - \Lambda(1)$$

- $R(\kappa) \in \mathbb{R}^{s \times s}$: Discretization of lin. residual $\Lambda(\sigma) - \Lambda(1) - \Lambda'(1)\kappa$
  (e.g. Galerkin proj. to fin.-dim. space)

Then, the monotonicity-constrained residuum minimization problem

$$\|R(\kappa)\|_F \rightarrow \min! \quad \text{s.t. } \kappa|_{P_k} = \text{const.}, \ 0 \leq \kappa|_{P_k} \leq \min\left\{\frac{1}{2}, \beta_k\right\}$$

possesses a unique solution $\kappa$, and $P_k \subseteq \text{supp } \kappa$ iff $P_k \subseteq \text{supp } (\sigma - 1)$.
Phantom experiment

Enhancing standard methods by monotonicity-based constraints

(Zhou/H./Seo, submitted)

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Realistic data & Uncertainties

- Finite number of electrodes, CEM, noisy data $\Lambda^\delta(\sigma)$
- Unknown background, e.g., $1 - \epsilon \leq \sigma_0(x) \leq 1 + \epsilon$
- Anomaly with some minimal contrast to background, e.g.,
  \[ \sigma(x) = \sigma_0(x) + \kappa(x)\chi_D, \quad \kappa(x) \geq 1 \]
- Can we rigorously guarantee to find inclusion $D$?

**H./Ullrich (IEEE TMI 2015):**

Rigorous Resolution Guarantee

- If $D = \emptyset$, methods return $\emptyset$.
- If $D \supset \omega_i$ then it is detected.

(Here: 32 electrodes, $\epsilon = 1\%$, $\delta = 1.4\%$)
Conclusions

EIT is a highly ill-posed, non-linear inverse problem.

- Convergence of generic solvers unclear.
- But: Shape reconstruction in EIT is essentially a linear problem.

Monotonicity-based methods for EIT shape reconstruction

- allow fast, rigorous, globally convergent implementations.
- work in any dimensions $n \geq 2$, full or partial boundary data.
- can enhance standard residual-based methods.
- yield rigorous resolution guarantees for realistic settings.

Open problems / challenges:

- Method requires voltages on current-driven electrodes
  \( (H., \textit{submitted}: \text{Missing electrode data may be replaced by interpolation.}) \)
- Method applicable without definiteness, but more complicated.