Combining Frequency-difference and Ultrasound-modulated EIT

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AIP - Applied Inverse Problems 2015
Electrical impedance tomography (EIT)

- Apply electric currents on subject’s boundary
- Measure necessary voltages
- Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)

B. Harrach: Combining Frequency-difference and Ultrasound-modulated EIT
Complex electrical potential $u(x)$ solves

$$\nabla \cdot (\gamma(x) \nabla u(x)) = 0 \quad x \in \Omega$$

$\Omega \subset \mathbb{R}^n$: imaged body, $n \geq 2$

$\gamma(x)$: complex conductivity at frequency $\omega \geq 0$

$u(x)$: complex electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_{\nu} u(x)|_{\partial \Omega}$: applied electric current

$u(x)|_{\partial \Omega}$: measured boundary voltage (potential)
Inverse problem

How can we recover $\gamma_\omega \in L^\infty_+ (\Omega)$ in

$$\nabla \cdot (\gamma_\omega \nabla u) = 0, \quad x \in \Omega \quad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial \Omega}, \sigma \partial_\nu u|_{\partial \Omega}) \, : \, u \text{ solves (1)}\}?$$

Equivalent: Recover $\gamma_\omega$ from Neumann-to-Dirichlet-Operator

$$\Lambda(\gamma_\omega) : L^2_\diamond (\partial \Omega) \to L^2_\diamond (\partial \Omega), \quad g \mapsto u|_{\partial \Omega},$$

where $u$ solves (1) with $\sigma \partial_\nu u|_{\partial \Omega} = g$. 
Inclusion detection

- Consider conductivity anomaly in homogenous medium

\[
\gamma_\omega(x) = \begin{cases} 
\gamma_\omega^{(\Omega)} = \sigma_\Omega + i\omega\epsilon_\Omega & \text{for } x \in \Omega \\
\gamma_\omega^{(D)} = \sigma_D + i\omega\epsilon_D & \text{for } x \in D
\end{cases}
\]

with constant $\sigma_\Omega, \sigma_D, \epsilon_\Omega, \epsilon_D > 0$ and $\Omega \setminus \overline{D}$ connected.

- Anomaly-free case: $\hat{\gamma}_\omega := \gamma_\omega^{(\Omega)} = \text{const.}$

**Goal:** Detect anomaly $D := \text{supp}_\partial \Omega (\gamma_\omega - \hat{\gamma}_\omega)$ from $\text{NtD} \ \Lambda(\gamma_\omega)$. 
Linearization

Goal: Detect anomaly $D := \text{supp}_{\partial \Omega} (\gamma_\omega - \hat{\gamma}_\omega)$ from $\text{NtD } \Lambda(\gamma_\omega)$.

- **One-step linearization methods**
  
e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

  Solve $\Lambda'(\hat{\gamma}_\omega) \kappa \approx \Lambda(\gamma_\omega) - \Lambda(\hat{\gamma}_\omega)$ to obtain $\kappa \approx \gamma_\omega - \hat{\gamma}_\omega$.

- **Exact shape reconstruction by one-step linearization**
  
  (H. Seo 2010, for $\omega = 0$ and piecw. anal. conductivities)

  $\Lambda'(\hat{\gamma}_0) \kappa = \Lambda(\gamma_0) - \Lambda(\hat{\gamma}_0) \quad \implies \quad \text{supp}_{\partial \Omega} \kappa = \text{supp}_{\partial \Omega} (\gamma_0 - \hat{\gamma}_0)$
Modelling errors

**Major challenge:** Modelling errors affect numerical calculations 
*boundary shape, electrode positions, ...*

\[
\Lambda'(\hat{\gamma}_\omega) \kappa \approx \Lambda(\gamma_\omega) - \Lambda(\hat{\gamma}_\omega) \quad \text{to obtain} \quad \kappa \approx \gamma_\omega - \hat{\gamma}_\omega.
\]

**Absolute data EIT:**

\[
\begin{align*}
\Lambda(\gamma_\omega) & : \text{measured} \\
\Lambda(\hat{\gamma}_\omega), \Lambda'(\hat{\gamma}_\omega) & : \text{obtained from numerical forward solver}
\end{align*}
\]

- Extremely sensitive to modeling errors
- Practical feasibility is a highly discussed topic
Time-difference EIT

Solve $\Lambda'(\hat{\gamma}_0) \kappa \approx \Lambda(\gamma_0) - \Lambda(\hat{\gamma}_0)$ to obtain $\kappa \approx \gamma_0 - \hat{\gamma}_0$.

Time-difference EIT:

\begin{align*}
\Lambda(\gamma_0), ~ \Lambda(\hat{\gamma}_0): & \quad \text{measured} \\
\Lambda'(\hat{\gamma}_0): & \quad \text{obtained from numerical forward solver}
\end{align*}

- Requires anomaly-free measurement
- Less sensitive to modeling errors
Weighted frequency-difference EIT

Solve $\Lambda'(\gamma_0) \kappa \approx \alpha \Lambda(\gamma_\omega) - \Lambda(\gamma_0)$ to obtain $\kappa \approx \alpha \gamma_\omega - \gamma_0$.

($\omega > 0$, $\alpha := \gamma_\omega^{(\Omega)} / \gamma_0^{(\Omega)}$ ratio of background conductivities.)

Weighted frequency-difference EIT:

- $\Lambda(\gamma_\omega)$, $\Lambda(\gamma_0)$, $\alpha$: measured
- $\Lambda'(\gamma_0)$: obtained from numerical forward solver

- No anomaly-free measurement required
- Less sensitive to modeling errors
- Requires contrast condition: $(\epsilon_D \sigma_\Omega - \epsilon_\Omega \sigma_D \neq 0 \sim \alpha \gamma_\omega - \gamma_0 \neq 0)$


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US-modulated EIT

Ultrasound-modulated EIT:

- Change conductivity in small area: \( \gamma_0 \sim \gamma_0(1 + \beta \chi_B) \)
- Measure \( \Lambda(\gamma_0(1 + \beta \chi_B)) \)

Can we locate anomaly \( D \) just from measurements (e.g., from \( \Lambda(\gamma_\omega), \Lambda(\gamma_0), \Lambda(\gamma_0(1 + \beta \chi_B)), \ldots \)) without any numerical forward solver, without knowing electrode position, boundary shape, \ldots
US-modulated fdEIT: continuous data

Theorem
Let \( \epsilon_D \sigma_\Omega - \epsilon_\Omega \sigma_D > 0 \) (contrast condition of weighted fdEIT).
For each suff. small \( \beta > 0 \) and every open set \( B \subseteq \Omega \)

\[
B \subseteq D \iff \Re(\alpha \Lambda(\gamma_\omega)) \leq \Lambda((1 + \beta \chi_B)\gamma_0).
\]

(For \( \epsilon_D \sigma_\Omega - \epsilon_\Omega \sigma_D < 0 \): \( B \subseteq D \iff \Re(\alpha \Lambda(\gamma_\omega)) \geq \Lambda((1 - \beta \chi_B)\gamma_0) \).

\[
\Re(\alpha \Lambda(\gamma_\omega)) - \Lambda(\gamma_0): \text{(weighted) change of frequency}
\]

\[
\Lambda((1 + \beta \chi_B)\gamma_0) - \Lambda(\gamma_0): \text{US-modulation focussed to set } B
\]

Comparing the effect of a change of frequency to that of a focussed US-modulation shows whether the US-focus is inside an anomaly.

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US-modulated fdEIT: continuous data

Theorem
Let $\epsilon_D \sigma_\Omega - \epsilon_\Omega \sigma_D > 0$ (contrast condition of weighted fdEIT).
For each suff. small $\beta > 0$ and every open set $B \subseteq \Omega$

$$B \subseteq D \iff \Re (\alpha \Lambda(\gamma_\omega)) \leq \Lambda((1 + \beta \chi_B)\gamma_0).$$

(For $\epsilon_D \sigma_\Omega - \epsilon_\Omega \sigma_D < 0$: $B \subseteq D \iff \Re (\alpha \Lambda(\gamma_\omega)) \geq \Lambda((1 - \beta \chi_B)\gamma_0).$)

- $\Lambda(\gamma_\omega)$ and $\Lambda((1 + \beta \chi_B)\gamma_0)$ can be measured.
- $\alpha$ can be estimated as in fdEIT (by minimizing $\Re (\alpha \Lambda(\gamma_\omega)) - \Lambda(\gamma_0)$).
- No forward calculations, knowing $\Omega$ is not required

Proof. Monotony & localized potentials
US-modulated fdEIT: electrode measurements

\( R(\gamma_\omega) \): Matrix of electrode measurements (shunt model)

**Theorem**

Let \( \epsilon_D \sigma_\Omega - \epsilon_\Omega \sigma_D > 0 \) (contrast condition of weighted fdEIT).

For each suff. small \( \beta > 0 \) and every open set \( B \subseteq \Omega \)

\[
B \subseteq D \quad \iff \quad R(\alpha R(\gamma_\omega)) \leq R((1 + \beta \chi_B)\gamma_0).
\]

(For \( \epsilon_D \sigma_\Omega - \epsilon_\Omega \sigma_D < 0 \): \( B \subseteq D \quad \iff \quad R(\alpha R(\gamma_\omega)) \geq R((1 - \beta \chi_B)\gamma_0) \).

- Roughly speaking, ”\( \iff \)” holds if enough electrodes are used.
- Measure \( R(\gamma_\omega), R(\gamma_0), R((1 + \beta \chi_B)\gamma_0) \). Estimate \( \alpha \) as before.
- No forward calculations.

*Anomaly can be located without knowing the electrode position.*
Numerical results

Example: $\gamma_0 := 1 + \chi_D$, $\gamma_\omega := 1 + \chi_D + i\omega$.

\[
\Re (\alpha R(\gamma_\omega)) \geq R((1 - \beta \chi_{B_j}) \gamma_0) \quad \text{for } j = 2
\]
\[
\Re (\alpha R(\gamma_\omega)) \notin R((1 - \beta \chi_{B_j}) \gamma_0) \quad \text{for } j \in \{1, 3, 4, 5\}
\]

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Numerical results

Example: $\gamma_0 := 1$, $\gamma_\omega := 1 + (2 - \chi_D)i\omega$.

red: true inclusion
gray: all balls with $\Re(\alpha R(\gamma_\omega)) \geq R((1 - \beta \chi_{B_j})\gamma_0)$
Conclusions

In electrical impedance tomography,

- modeling errors present a major challenge,
- time difference data reduces the sensitivity to modelling errors,
- weighted-fdEIT data extends this robustness to static settings.

**New idea:** Combining w-fdEIT with US-modulated EIT potentially

- eliminates all model dependence
- allows to detect anomaly directly from measurements
  - without any forward calculations
  - without knowing domain shape
  - without knowing electrode position