Combining Frequency-difference and Ultrasound-modulated EIT

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(joint work with Eunjung Lee and Marcel Ullrich)

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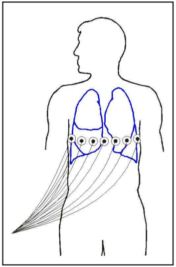
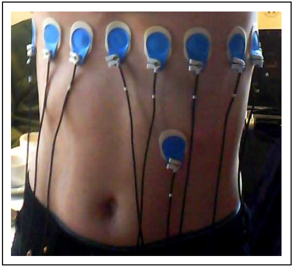
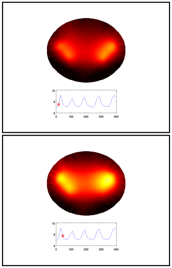
Electrical impedance tomography (EIT)







- Apply electric currents on subject's boundary
- Measure necessary voltages

$\leadsto$ Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)

Mathematical Model

Complex electrical potential $u(x)$ solves

$$\nabla \cdot (\gamma_\omega(x) \nabla u(x)) = 0 \quad x \in \Omega$$

$\Omega \subset \mathbb{R}^n$: imaged body, $n \geq 2$
$\gamma_\omega(x)$: complex conductivity at frequency $\omega \geq 0$
$u(x)$: complex electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_\nu u(x)|_{\partial\Omega}$: applied electric current
$u(x)|_{\partial\Omega}$: measured boundary voltage (potential)

Inverse problem

How can we recover $\gamma_\omega \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\gamma_\omega \nabla u) = 0, \quad x \in \Omega \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves } (1)\}\,?$$

Equivalent: Recover γ_ω from Neumann-to-Dirichlet-Operator

$$\Lambda(\gamma_\omega) : \; L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Inclusion detection

- Consider conductivity anomaly in homogenous medium

$$\gamma_\omega(x) = \begin{cases} \gamma_\omega^{(\Omega)} = \sigma_\Omega + \mathrm{i}\omega\epsilon_\Omega & \text{for } x \in \Omega \\ \gamma_\omega^{(D)} = \sigma_D + \mathrm{i}\omega\epsilon_D & \text{for } x \in D \end{cases}$$

  with constant $\sigma_\Omega, \sigma_D, \epsilon_\Omega, \epsilon_D > 0$ and $\Omega \setminus \overline{D}$ connected.
- Anomaly-free case: $\hat{\gamma}_\omega := \gamma_\omega^{(\Omega)} = \mathrm{const.}$.

Goal: Detect anomaly $D := \mathrm{supp}_{\partial\Omega}(\gamma_\omega - \hat{\gamma}_\omega)$ from NtD $\Lambda(\gamma_\omega)$.

Linearization

Goal: Detect anomaly $D := \mathrm{supp}_{\partial\Omega}(\gamma_\omega - \hat{\gamma}_\omega)$ from NtD $\Lambda(\gamma_\omega)$.

- One-step linearization methods

  e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

  Solve $\Lambda'(\hat{\gamma}_\omega)\kappa \approx \Lambda(\gamma_\omega) - \Lambda(\hat{\gamma}_\omega)$ to obtain $\kappa \approx \gamma_\omega - \hat{\gamma}_\omega$.

- Exact shape reconstruction by one-step linearization

  (**H.**/*Seo 2010, for* $\omega = 0$ *and piecw. anal. conductivities)*

  $$\Lambda'(\hat{\gamma}_0)\kappa = \Lambda(\gamma_0) - \Lambda(\hat{\gamma}_0) \quad \Longrightarrow \quad \mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\gamma_0 - \hat{\gamma}_0)$$

Modelling errors

Major challenge: Modelling errors affect numerical calculations

(boundary shape, electrode positions, ...)

Solve $\Lambda'(\hat{\gamma}_\omega)\kappa \approx \Lambda(\gamma_\omega) - \Lambda(\hat{\gamma}_\omega)$ to obtain $\kappa \approx \gamma_\omega - \hat{\gamma}_\omega$.

Absolute data EIT:

$\Lambda(\gamma_\omega)$:	measured
$\Lambda(\hat{\gamma}_\omega)$, $\Lambda'(\hat{\gamma}_\omega)$:	obtained from numerical forward solver

- Extremely sensitive to modeling errors
- Practical feasibility is a highly discussed topic

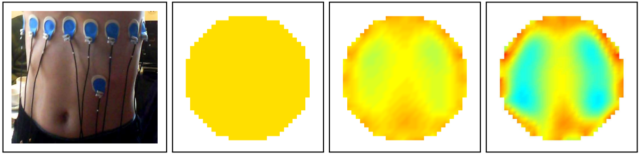
Time-difference EIT

Solve $\Lambda'(\hat{\gamma}_0)\kappa \approx \Lambda(\gamma_0) - \Lambda(\hat{\gamma}_0)$ to obtain $\kappa \approx \gamma_0 - \hat{\gamma}_0$.

Time-difference EIT:

$\Lambda(\gamma_0)$, $\Lambda(\hat{\gamma}_0)$: measured
$\Lambda'(\hat{\gamma}_0)$: obtained from numerical forward solver

- Requires anomaly-free measurement
- Less sensitive to modeling errors



Weighted frequency-difference EIT

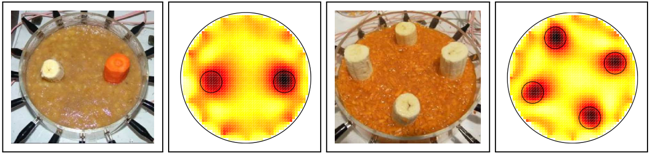
Solve $\Lambda'(\gamma_0)\kappa \approx \alpha\Lambda(\gamma_\omega) - \Lambda(\gamma_0)$ to obtain $\kappa \approx \alpha\gamma_\omega - \gamma_0$.

($\omega > 0$, $\alpha := \gamma_\omega^{(\Omega)}/\gamma_0^{(\Omega)}$ ratio of background conductivities.)

Weighted frequency-difference EIT:

$\Lambda(\gamma_\omega)$, $\Lambda(\gamma_0)$, α: measured
$\Lambda'(\gamma_0)$: obtained from numerical forward solver

- No anomaly-free measurement required
- Less sensitive to modeling errors
- Requires contrast condition: *($\epsilon_D\sigma_\Omega - \epsilon_\Omega\sigma_D \neq 0 \rightsquigarrow \alpha\gamma_\omega - \gamma_0 \neq 0$)*



*(**H.**/Seo/Woo, IEEE Trans. Med. Imaging, 2010)*

US-modulated EIT

Ultrasound-modulated EIT:

- Change conductivity in small area: $\gamma_0 \rightsquigarrow \gamma_0(1+\beta\chi_B)$
- Measure $\Lambda(\gamma_0(1+\beta\chi_B))$

Can we locate anomaly D just from measurements

(e.g., from $\Lambda(\gamma_\omega)$, $\Lambda(\gamma_0)$, $\Lambda(\gamma_0(1+\beta\chi_B))$, *…)*

without any numerical forward solver,

without knowing electrode position, boundary shape, …

US-modulated fdEIT: continuous data

Theorem

Let $\epsilon_D \sigma_\Omega - \epsilon_\Omega \sigma_D > 0$ (contrast condition of weighted fdEIT).
For each suff. small $\beta > 0$ and every open set $B \subseteq \Omega$

$$B \subseteq D \quad \Longleftrightarrow \quad \Re\left(\alpha \Lambda(\gamma_\omega)\right) \leq \Lambda((1 + \beta \chi_B)\gamma_0).$$

(For $\epsilon_D \sigma_\Omega - \epsilon_\Omega \sigma_D < 0$: $\quad B \subseteq D \quad \Longleftrightarrow \Re\left(\alpha \Lambda(\gamma_\omega)\right) \geq \Lambda((1 - \beta \chi_B)\gamma_0)$.)

$\Re\left(\alpha \Lambda(\gamma_\omega)\right) - \Lambda(\gamma_0)$: (weighted) change of frequency
$\Lambda((1 + \beta \chi_B)\gamma_0) - \Lambda(\gamma_0)$: US-modulation focussed to set B

Comparing the effect of a change of frequency to that of a focussed US-modulation shows whether the US-focus is inside an anomaly.

US-modulated fdEIT: continuous data

Theorem

Let $\epsilon_D\sigma_\Omega - \epsilon_\Omega\sigma_D > 0$ (contrast condition of weighted fdEIT).
For each suff. small $\beta > 0$ and every open set $B \subseteq \Omega$

$$B \subseteq D \quad \Longleftrightarrow \quad \Re(\alpha\Lambda(\gamma_\omega)) \leq \Lambda((1+\beta\chi_B)\gamma_0).$$

(For $\epsilon_D\sigma_\Omega - \epsilon_\Omega\sigma_D < 0$: $\quad B \subseteq D \Longleftrightarrow \Re(\alpha\Lambda(\gamma_\omega)) \geq \Lambda((1-\beta\chi_B)\gamma_0)$.)

- $\Lambda(\gamma_\omega)$ and $\Lambda((1+\beta\chi_B)\gamma_0)$ can be measured.
- α can be estimated as in fdEIT *(by minimizing* $\Re(\alpha\Lambda(\gamma_\omega)) - \Lambda(\gamma_0)$*)*.

$\rightsquigarrow$ No forward calculations, knowing Ω is not required

Proof. Monotony & localized potentials

US-modulated fdEIT: electrode measurements

$R(\gamma_\omega)$: Matrix of electrode measurements (shunt model)

Theorem

Let $\epsilon_D\sigma_\Omega - \epsilon_\Omega\sigma_D > 0$ (contrast condition of weighted fdEIT).
For each suff. small $\beta > 0$ and every open set $B \subseteq \Omega$

$$B \subseteq D \quad \Longrightarrow \quad \Re\left(\alpha R(\gamma_\omega)\right) \leq R((1+\beta\chi_B)\gamma_0).$$

(For $\epsilon_D\sigma_\Omega - \epsilon_\Omega\sigma_D < 0$: $\quad B \subseteq D \iff \Re\left(\alpha R(\gamma_\omega)\right) \geq R((1-\beta\chi_B)\gamma_0)$.)

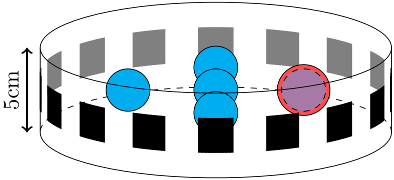
- Roughly speaking, "$\Longleftarrow$" holds if enough electrodes are used.
- Measure $R(\gamma_\omega)$, $R(\gamma_0)$, $R((1+\beta\chi_B)\gamma_0)$. Estimate α as before.

$\leadsto$ No forward calculations.

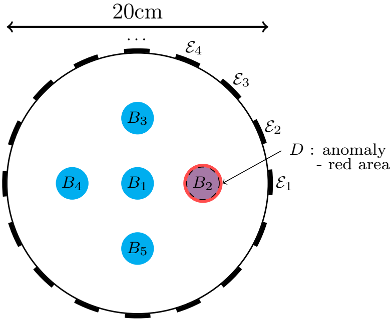
Anomaly can be located without knowing the electrode position.

Numerical results

Example: $\gamma_0 := 1 + \chi_D$, $\gamma_\omega := 1 + \chi_D + \mathrm{i}\omega$.



side view



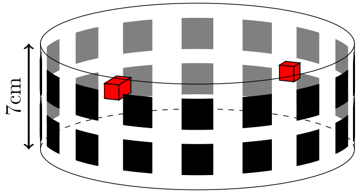
top view

$$\Re\left(\alpha R(\gamma_\omega)\right) \geq R((1-\beta\chi_{B_j})\gamma_0) \qquad \text{for } j = 2$$

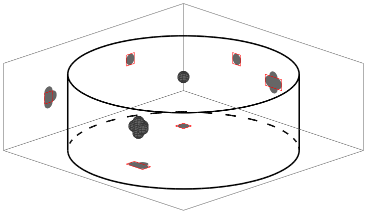
$$\Re\left(\alpha R(\gamma_\omega)\right) \not\geq R((1-\beta\chi_{B_j})\gamma_0) \qquad \text{for } j \in \{1,3,4,5\}$$

Numerical results

Example: $\gamma_0 := 1$, $\gamma_\omega := 1 + (2 - \chi_D)\mathrm{i}\omega$.



side view



red: true inclusion

gray: all balls with $\Re\left(\alpha R(\gamma_\omega)\right) \geq R((1 - \beta\chi_{B_j})\gamma_0)$

Conclusions

In electrical impedance tomography,

- modeling errors present a major challenge,
- time difference data reduces the sensitivity to modelling errors,
- weighted-fdEIT data extends this robustness to static settings.

New idea: Combining w-fdEIT with US-modulated EIT potentially

- eliminates all model dependance
- allows to detect anomaly directly from measurements
  - without any forward calculations
  - without knowing domain shape
  - without knowing electrode position