# The Vanishing Conductivity Limit in Eddy Current Imaging

Bastian von Harrach

harrach@math.uni-stuttgart.de

(joint work with Lilian Arnold)

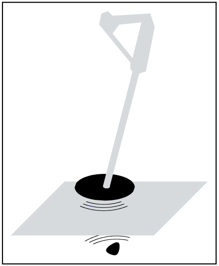
Chair of Optimization and Inverse Problems, University of Stuttgart, Germany

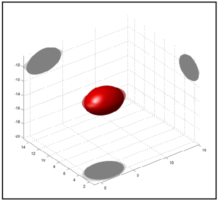
SIAM Conference on Imaging Science
Hong Kong Baptist University,
Hong Kong, May 12–14, 2014.

# Inverse Electromagnetics

## Inverse Electromagnetics:

- Generate EM field
  *(drive excitation current through coil)*
- Measure EM field
  *(induced voltages in meas. coil)*
- Gain information from measurements





## Applications:

- Metal detection *(buried conductor)*
- Non-destructive testing
  *(crack in metal, metal in concrete)*
- . . .

# Maxwell's equations

Classical Electromagnetics: Maxwell's equations

$$\operatorname{curl} H = \epsilon \partial_t E + \sigma E + J \qquad \text{in } \mathbb{R}^3 \times ]0, T[$$
$$\operatorname{curl} E = -\mu \partial_t H \qquad \text{in } \mathbb{R}^3 \times ]0, T[$$

| | | | |
|---|---|---|---|
| $E(x,t)$: | Electric field | $\epsilon(x)$: | Permittivity |
| $H(x,t)$: | Magnetic field | $\mu(x)$: | Permeability |
| $J(x,t)$: | Excitation current | $\sigma(x)$: | Conductivity |

*Knowing $J, \sigma, \mu, \epsilon$ + init. cond. determines $E$ and $H$.*

# Eddy currents

Maxwell's equations

$$\operatorname{curl} H = \epsilon\partial_t E + \sigma E + J \qquad \text{in } \mathbb{R}^3 \times ]0, T[$$

$$\operatorname{curl} E = -\mu\partial_t H \qquad \text{in } \mathbb{R}^3 \times ]0, T[$$

Eddy current approximation: Neglect displacement currents $\epsilon\partial_t E$

- Justified for low-frequency excitations
  *(Alonso 1999, Ammari/Buffa/Nédélec 2000)*

$$\rightsquigarrow \qquad \partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times ]0, T[$$

# Where's Eddy?

- $\sigma = 0$: (Quasi-)Magnetostatics

$$\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -\partial_t J$$

Excitation $\partial_t J$ instantly generates magn. field $\frac{1}{\mu}\operatorname{curl} E = -\partial_t H$.

- $\sigma \neq 0$: Eddy currents

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -\partial_t J$$

$\partial_t J$ generates changing magn. field + currents inside conductor

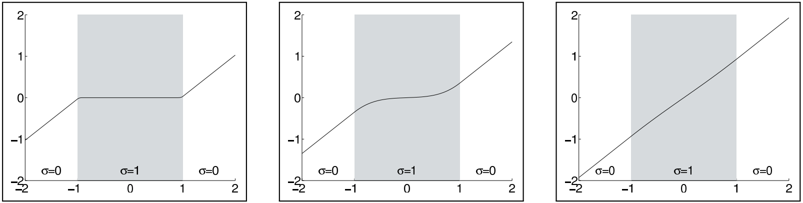
*Induced currents oppose what created them (Lenz law)*

# Parabolic-elliptic equations

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -\partial_t J \quad \text{in } \mathbb{R}^3\times]0,T[$$

- parabolic inside conductor $\Omega = \operatorname{supp}(\sigma)$
- elliptic outside conductor

Scalar example: $(\sigma u)_t = u_{xx},\ u(\cdot,0)=0,\ u_x(-2,\cdot)=u_x(2,\cdot)=1.$



# Vanishing conductivity limit

Parabolic-elliptic eddy current equation

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -\partial_t J \quad \text{in } \mathbb{R}^3\times]0,T[$$

Vanishing conductivity limit required for

- Numerical implementation by parabolic regularization

$$\text{Replace } \sigma(x) \text{ by } \sigma_\epsilon(x) := \min\{\sigma(x),\epsilon\}, \quad \epsilon > 0.$$

- Inversion by linearization: Find $\sigma$ from measurements of $E$ by

$$\text{linearizing } E \text{ w.r.t. } \sigma \text{ around } \sigma = 0.$$

---

In this talk: *How does the solution change*

- *if a parabolic equation becomes elliptic?*
- *if an elliptic equation becomes a little bit parabolic?*

---

# Standard approach

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times ]0, T[$$

Standard approach: *Decouple elliptic and parabolic part*
*(e.g. Bossavit 1999, Acevedo/Meddahi/Rodriguez 2009)*

---

Find $(E_{\mathbb{R}^3\setminus\Omega}, E_\Omega) \in H_{\mathbb{R}^3\setminus\Omega} \times H_\Omega$ s.t.

- $E_\Omega$ solves parabolic equation + init. cond.
- $E_{\mathbb{R}^3\setminus\Omega}$ solves elliptic equation
- interface conditions are satisfied

---

Problem: Theory depends on $\Omega = \operatorname{supp}\sigma$ and $\inf \sigma|_\Omega$.

*Vanishing conductivity limits require unified approach.*

# Rigorous formulation

Rigorous formulation: Let $\mu \in L^\infty_+$, $\sigma \in L^\infty$, $\sigma \geq 0$,

$$J_t \in L^2(0, T, W(\mathrm{curl})') \qquad \text{with } \operatorname{div} J_t = 0$$
$$E_0 \in L^2(\mathbb{R}^3)^3 \qquad \text{with } \operatorname{div}(\sigma E_0) = 0.$$

For $E \in L^2(0, T, W(\mathrm{curl}))$ the eddy current equations

$$\partial_t(\sigma E) + \mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\, E\right) = -J_t \qquad \text{in } \mathbb{R}^3\times]0, T[$$
$$\sqrt{\sigma}E(x,0) = \sqrt{\sigma(x)}E_0(x) \qquad \text{in } \mathbb{R}^3$$

are well-defined and (if solvable) uniquely determine $\mathrm{curl}\, E$, $\sqrt{\sigma}E$.

# Natural variational formulation

Natural unified variational formulation ($E_0 = 0$ for simplicity):

Find $E \in L^2(0, T, W(\mathrm{curl}))$ that solves

$$\int_0^T \int_{\mathbb{R}^3} \left( \sigma E \cdot \partial_t \Phi - \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.$$

for all smooth $\Phi$ with $\Phi(\cdot, T) = 0$.

- equivalent to eddy current equation
- not coercive, does not yield existence results

# Gauged formulation

Gauged unified variational formulation ($E_0 = 0$ for simplicity)

---

Find divergence-free $E \in L^2(0, T, W^1(\mathbb{R}^3))$ that solves

$$\int_0^T \int_{\mathbb{R}^3} \left( \sigma E \cdot \partial_t \Phi - \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.$$

for all smooth divergence-free $\Phi$ with $\Phi(\cdot, T) = 0$.

---

- coercive, yields existence and continuity results
- not equivalent to eddy current equation
  ($\sigma \neq$ const. $\rightsquigarrow \operatorname{div} \sigma E \neq \sigma \operatorname{div} E$)
- does not determine true solution up to gauge (curl-free) field

# Coercive unified formulation

How to obtain coercive + equivalent unified formulation?

- Ansatz $E = A + \nabla\varphi$ with divergence-free $A$.
  *(almost the standard $(A, \varphi)$-formulation with Coulomb gauge)*
- Consider $\nabla\varphi = \nabla\varphi_A$ as function of $A$ by solving
  $$\operatorname{div} \sigma\nabla\varphi_A = -\operatorname{div} \sigma A.$$
  ($\rightsquigarrow \operatorname{div} \sigma E = 0$).
- Obtain coercive formulation for $A$
  (Lions-Lax-Milgram Theorem $\rightsquigarrow$ Solvability and continuity results)
- $A$ determines $E$
  (more precisely: $\operatorname{curl} E$ and $\sqrt{\sigma}E$)

# Unified variational formulation

Unified variational formulation *(Arnold/**H.**, SIAP, 2012)*

---

Find divergence-free $A \in L^2(0, T, W^1(\mathbb{R}^3))$ that solves

$$\int_0^T \int_{\mathbb{R}^3} \left( \sigma(A + \nabla \varphi_A) \cdot \partial_t \Phi - \frac{1}{\mu} \operatorname{curl} A \cdot \operatorname{curl} \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.$$

for all smooth divergence-free $\Phi$ with $\Phi(\cdot, T) = 0$.

---

- coercive, uniquely solvable
- $E := A + \nabla \varphi_A$ is one solution of the eddy current equation

$\rightsquigarrow$ $\operatorname{curl} E$, $\sqrt{\sigma} E$ depend continuously on $J_t$ (uniformly w.r.t. $\sigma$) (for all solutions of the eddy current equation)

# Asymptotic results

## Unified variational formulation

- allows to rigorously linearize $E$ w.r.t. $\sigma$ around $\sigma_0 = 0$
  (elliptic equation becoming a little bit parabolic in some region...)
- easily extends from $\mathbb{R}^3$ to bounded domain $O$
  ($O$ simply conn. with Lipschitz-boundary, $\nu \wedge E|_{\partial O} = 0$)
- justifies parabolic regularization: If $E_\epsilon$ solves

$$\partial_t(\sigma_\epsilon E_\epsilon) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E_\epsilon\right) = -\partial_t J \quad \text{in } O\times ]0,T[,$$

  with $\sigma_\epsilon(x) = \max\{\sigma(x), \epsilon\}$ then

$$\sigma_\epsilon E_\epsilon \to \sigma E, \quad \operatorname{curl} E_\epsilon \to \operatorname{curl} E$$

  *(Arnold/**H.**, Proceedings of IPDO 2013)*
- yields the factorization method for inverse eddy current probl.
  *(Arnold/**H.**, Inverse Problems 2013)*

# Open problem

Unified variation theory

- requires some regularity of $\Omega = \operatorname{supp} \sigma$

  *(finite union of bounded Lipschitz domains with connected complement).*
- requires conductivity jump between conducting and insulating regions, i.e. $\sigma \in L^\infty_+(\Omega)$ in order to determine $\varphi$ from $A$.
- does not cover continuous transitions between conducting and non-conducting parts.

Missing step: Solution theory for

$$\operatorname{div} \sigma \nabla \varphi = -\operatorname{div} \sigma A$$

for general $\sigma \in L^\infty$, $\sigma \geq 0$?