Lecture 4: The Monotonicity Method for inclusion detection in EIT

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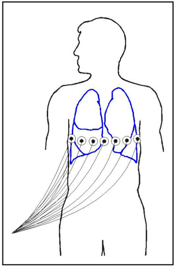
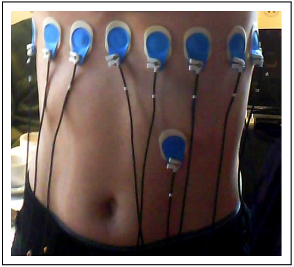
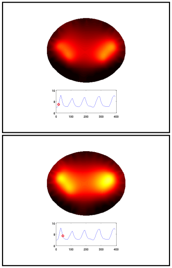
Electrical impedance tomography (EIT)







- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)

Mathematical Model

Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega$$

$\Omega \subset \mathbb{R}^n$:	imaged body, $n \geq 2$
$\sigma(x)$:	conductivity
$u(x)$:	electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_\nu u(x)\vert_{\partial\Omega}$:	applied electric current
$u(x)\vert_{\partial\Omega}$:	measured boundary voltage (potential)

PDE theory

Elliptic PDE theory (Lax-Milgram):

For each $g \in L^2_\diamond(\partial\Omega)$ there exists a unique solution $u \in H^1_\diamond(\Omega)$ of

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega \quad \text{and} \quad \sigma \partial_\nu u|_{\partial\Omega} = g.$$

The solution is uniquely determined by the variational formulation

$$\int_\Omega \sigma \nabla u \cdot \nabla v \, \mathrm{d}x = \int_{\partial\Omega} g v|_{\partial\Omega} \, \mathrm{d}s \quad \forall v \in H^1_\diamond(\Omega). \tag{1}$$

Neumann-to-Dirichlet operator (NtD):

- Define $\Lambda(\sigma): \; g \mapsto u|_{\partial\Omega}$, where u solves (1).
- $\Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\partial\Omega))$ compact and selfadjoint.

Forward and inverse problem

- (Non-linear) forward operator of EIT:

$$\begin{array}{rcl} \Lambda: \ \sigma \in L^\infty_+(\Omega) & \mapsto & \Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\partial\Omega)) \\ \text{conductivity} & \mapsto & \text{measurements} \end{array}$$

- Inverse problem of EIT:

$$\begin{array}{rcl} \Lambda^{-1}: \ \Lambda(\sigma) & \mapsto & \sigma \\ \text{measurements} & \mapsto & \text{conductivity (image)} \end{array}$$

Inclusion/shape detection problem:

$$\Lambda(\sigma) \mapsto \mathrm{supp}(\sigma - \sigma_0)?, \qquad \sigma_0\text{: reference conductivity.}$$

Monotonicity (from Lecture 3)

Theorem 3.3. Let $\sigma_1, \sigma_0 \in L^\infty_+(\Omega)$. Then, for all $g \in L^2_\diamond(\partial\Omega)$,

$$\int_\Omega (\sigma_0 - \sigma_1)|\nabla u_0|^2 \, \mathrm{d}x \quad \leq \quad \int_{\partial\Omega} g \left(\Lambda(\sigma_1) - \Lambda(\sigma_0)\right) g \, \mathrm{d}s$$

where $u_0 \in H^1_\diamond(\Omega)$ solves $\nabla \cdot (\sigma_0 \nabla u_0) = 0$ in Ω, and $\sigma_0 \partial_\nu u_0|_{\partial\Omega} = g$.

Corollary.

$$\sigma_0 \geq \sigma_1 \quad \Longrightarrow \quad \Lambda(\sigma_1) \leq \Lambda(\sigma_0)$$

Monotonicity-based inclusion detection

$$\sigma_0 \geq \sigma_1 \quad \Longrightarrow \quad \Lambda(\sigma_1) \leq \Lambda(\sigma_0)$$

For simplicity, assume for the true conductivity

- $\sigma(x) = 1 + \chi_D(x)$, with D open, $\overline{D} \subset \Omega$, $\Omega \setminus \overline{D}$ connected.

Introduce test conductivity

- $\tau(x) = 1 + \chi_B(x)$ with a small ball $B \subset \Omega$.

By monotonicity,

$$B \subseteq D \quad \Longrightarrow \quad \tau \leq \sigma \quad \Longrightarrow \quad \Lambda(\tau) \geq \Lambda(\sigma)$$

Monotonicity-based inclusion detection

Simple monotonicity-based inclusion detection *(formulated for $\sigma = 1 + \chi_D$)*

For each ball $B \subseteq \Omega$

- Calculate Test-NtD $\Lambda(\tau)$ for $\tau := 1 + \chi_B$
- Mark ball if $\Lambda(\tau) \geq \Lambda(\sigma)$

Result: Each ball $B \subseteq D$ will be marked.

Problems:

- Does this algorithm mark balls $B \not\subseteq D$?
- Calculating $\Lambda(\tau)$ is computationally expensive.

Monotonicity-based inclusion detection

- Does this algorithm mark balls $B \not\subseteq D$?
  - Show that $B \subseteq D$ if and only if $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$.
  - ⇝ Monotonicity Algorithm precisely marks D.
- Calculating $\Lambda(1 + \chi_B)$ is computationally expensive.
  - Show that

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma)$$

where

$$\int_{\partial\Omega} g \left(\Lambda'(1)\chi_B\right) h \, \mathrm{d}s = -\int_B \nabla u_0^g \cdot \nabla u_0^h \, \mathrm{d}x$$

⇝ Algorithm only requires homogeneous solutions

$$\Delta u_0^g = 0, \quad \partial_\nu u_0^g|_{\partial\Omega} = g.$$

Localized potentials

Theorem 4.1. Let

- D be open, $\overline{D} \subset \Omega$, and $\Omega \setminus \overline{D}$ connected,
- B be open, $\overline{B} \subset \Omega$, and $B \not\subseteq D$.

Then there exists $(g_m)_{m\in\mathbb{N}} \subset L^2_\diamond(\partial\Omega)$ s.t. the solutions $(u_m)_{m\in\mathbb{N}}$ of

$$\Delta u_m = 0 \quad \text{in } \Omega, \qquad \partial_\nu u_m|_{\partial\Omega} = g_m,$$

fulfill

$$\lim_{m\to\infty} \int_B |\nabla u_m|^2 \,\mathrm{d}x = \infty \quad \text{and} \quad \lim_{m\to\infty} \int_D |\nabla u_m|^2 \,\mathrm{d}x = 0.$$

Monotonicity-based shape reconstruction

Theorem 4.2. Let

- $\sigma(x) = 1 + \chi_D(x)$, with
- D open, $\overline{D} \subset \Omega$, and $\Omega \setminus \overline{D}$ connected.

Then for each open ball $B \subseteq \Omega$,

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma)$$

Corollary.

D is the union of all balls $B \subseteq \Omega$ with $\Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma)$.

Stability / Regularization / Convergence

- Let $\Lambda^\delta \in \mathcal{L}(L^2_\diamond(\partial\Omega))$ (compact & self-adjoint) with
$$\left\|\Lambda^\delta - \Lambda(\sigma)\right\|_{\mathcal{L}(L^2_\diamond(\partial\Omega))} \leq \delta.$$
- Regularized definiteness test: For $\alpha > 0$, and a ball $B \subseteq \Omega$ define
$$R_\alpha(\Lambda^\delta, B) := \begin{cases} 1 & \text{if } \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B - \Lambda(\sigma) \geq -\alpha I, \\ 0 & \text{else.} \end{cases}$$
- Then,
$$R_\delta(\Lambda^\delta, B) := \begin{cases} 1 & \text{if } B \subseteq D, \\ 0 & \text{if } B \not\subseteq D \text{ and } \delta \text{ is suff. small.} \end{cases}$$

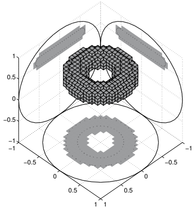
Ball B is correctly marked, if noise is below some (B-depend.) level.

Conclusions

$$D \text{ is the union of all balls } B \subseteq \Omega \text{ with } \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

The montonicity method

- shows that conductivity inclusions are uniquely determined from measuring the NtD.
- can be extended to more general cases, even further than FM.
- allows convergent implementation for noisy data.



Literature: **H.**/*Ullrich: Monotonicity-based shape reconstruction in EIT, SIMA 2013.*