Lecture 3: The Factorization Method for inclusion detection in EIT

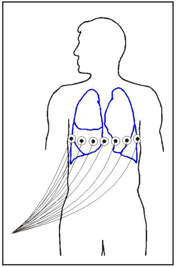
Bastian von Harrach

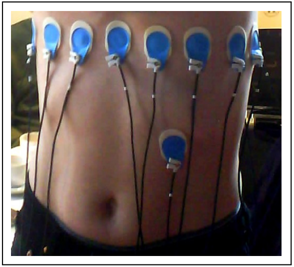
harrach@math.uni-stuttgart.de

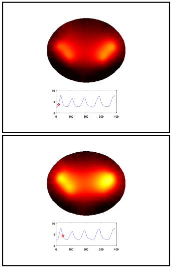
Chair of Optimization and Inverse Problems, University of Stuttgart, Germany

Advanced Instructional School on
Theoretical and Numerical Aspects of Inverse Problems
TIFR Centre For Applicable Mathematics
Bangalore, India, June 16–28, 2014.

Electrical impedance tomography (EIT)







- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)

Mathematical Model

Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega$$

$\Omega \subset \mathbb{R}^n$: imaged body, $n \geq 2$
$\sigma(x)$: conductivity
$u(x)$: electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_\nu u(x)|_{\partial\Omega}$: applied electric current
$u(x)|_{\partial\Omega}$: measured boundary voltage (potential)

PDE theory

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. We define

- $L^\infty_+(\Omega) := \{\sigma \in L^\infty(\Omega) : \ \operatorname{ess\,inf} \sigma(x) > 0\}$
- $H^1_\diamond(\Omega) := \{u \in H^1_\diamond(\Omega) : \ \int_{\partial\Omega} g \, \mathrm{d}s = 0\}$
- $L^2_\diamond(\partial\Omega) := \{g \in L^2_\diamond(\partial\Omega) : \ \int_{\partial\Omega} g \, \mathrm{d}s = 0\}$

Elliptic PDE theory (Lax-Milgram):

For each $g \in L^2_\diamond(\partial\Omega)$ there exists a unique solution $u \in H^1_\diamond(\Omega)$ of

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega \quad \text{and} \quad \sigma \partial_\nu u|_{\partial\Omega} = g.$$

The solution is uniquely detemined by the variational formulation

$$\int_\Omega \sigma \nabla u \cdot \nabla v \, \mathrm{d}x = \int_{\partial\Omega} g v|_{\partial\Omega} \, \mathrm{d}s \quad \forall v \in H^1_\diamond(\Omega).$$

PDE theory

Elliptic PDE theory (Lax-Milgram):

For each $g \in L^2_\diamond(\partial\Omega)$ there exists a unique solution $u \in H^1_\diamond(\Omega)$ of

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega \quad \text{and} \quad \sigma \partial_\nu u|_{\partial\Omega} = g.$$

The solution is uniquely determined by the variational formulation

$$\int_\Omega \sigma \nabla u \cdot \nabla v \, \mathrm{d}x = \int_{\partial\Omega} g v|_{\partial\Omega} \, \mathrm{d}s \quad \forall v \in H^1_\diamond(\Omega). \tag{1}$$

Neumann-to-Dirichlet operator (NtD):

- Define $\Lambda(\sigma) : \; g \mapsto u|_{\partial\Omega}$, where u solves (1).
- $\Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\partial\Omega))$ compact and selfadjoint.

Forward and inverse problem

- (Non-linear) forward operator of EIT:

$$\begin{array}{rcl} \Lambda: \ \sigma \in L^\infty_+(\Omega) & \mapsto & \Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\partial\Omega)) \\ \text{conductivity} & \mapsto & \text{measurements} \end{array}$$

- Inverse problem of EIT:

$$\begin{array}{rcl} \Lambda^{-1}: \ \Lambda(\sigma) & \mapsto & \sigma \\ \text{measurements} & \mapsto & \text{conductivity (image)} \end{array}$$

Problems

- Uniqueness ("Calderón problem"): Is Λ injective?
- Convergent numerical methods to reconstruct σ?

Reconstruction

Convergent numerical methods to reconstruct σ?

- Convergence of generic methods unclear for EIT
  Dobson (1992): (Local) convergence for regularized EIT equation.
  Lechleiter/Rieder(2008): (Local) convergence for discretized setting.
- D-bar method: convergent 2D-implementation for $\sigma \in C^2$
  Knudsen, Lassas, Mueller, Siltanen (2008)

In practice:

- large jumps in conductivity
- large interest in detecting shapes / inclusions / anomalies

---

Inclusion/shape detection problem:

$$\Lambda(\sigma) \mapsto \operatorname{supp}(\sigma - \sigma_0)?, \qquad \sigma_0\text{: reference conductivity.}$$

Factorization method

In this lecture: Factorization method

- *Developed by Kirsch (1998) for inverse scattering problems*
- *FM for EIT (1999–): Brühl, Hakula, Hanke, H., Hyvönen, Kirsch, Lechleiter, Nachman, Päivärinta, Pursiainen, Schappel, Schmitt, Seo, Teirilä*
- *This lecture follows H.: Recent progress on the factorization method for EIT (Comput. Math. Methods Med., vol. 2013, Article ID 425184, 8 pages, 2013)*

Goal: Explicit characterization of $\mathrm{supp}(\sigma - \sigma_0)$:

$$z \in \mathrm{supp}(\sigma - \sigma_0) \quad \text{iff } \Phi_z|_{\partial\Omega} \in \mathcal{R}\left(|\Lambda(\sigma) - \Lambda(\sigma_0)|^{1/2}\right)$$

where $\Phi_z|_{\partial\Omega}$ is a special function with singularity in $z \in \Omega$.

Factorization Method

$$z \in \operatorname{supp}(\sigma - \sigma_0) \quad \text{iff } \Phi_z|_{\partial\Omega} \in \mathcal{R}\left(|\Lambda(\sigma) - \Lambda(\sigma_0)|^{1/2} \right)$$

Outline of the proof:

- Introduce Φ_z
- Introduce virtual measurement operator L
- Show that $\Phi_z|_{\partial\Omega} \in \mathcal{R}(L)$ iff $z \in \operatorname{supp}(\sigma - \sigma_0)$
- Show that $\mathcal{R}(L) = \mathcal{R}\left(|\Lambda(\sigma) - \Lambda(\sigma_0)|^{1/2} \right)$

For simplicity, we will assume that

- $\sigma_0 = 1, \quad \sigma := 1 + \chi_D$, with D open, $\overline{D} \subset \Omega$, $\Omega \setminus \overline{D}$ connected.

The dipole functions Φ_z

Definition 3.1. Let $d \in \mathbb{R}^n$, $|d| = 1$ be an arbitrary direction. Let Φ_z solve

$$\Delta \Phi_z = d \cdot \nabla \delta_z \text{ in } \Omega, \quad \partial_\nu \Phi_z|_{\partial\Omega} = 0$$

and $\int_{\partial\Omega} \Phi_z \, \mathrm{d}s = 0$. ($\Phi_z$ is called dipole function).

Example. For $\Omega = B_1(0) \in \mathbb{R}^2$,

$$\Phi_z(x) = \frac{1}{\pi} \frac{(z - x) \cdot d}{|z - x|^2}$$

Virtual measurements

Definition 3.2. We define the virtual measurements

$$L_D: \; L^2_\diamond(D)^n \to L^2_\diamond(\partial\Omega), \quad F \mapsto v|_{\partial\Omega},$$

where $v \in H^1_\diamond(\Omega)$ solves

$$\int_\Omega \nabla v \cdot \nabla w \, \mathrm{d}x = \int_D F \cdot \nabla w \, \mathrm{d}x \quad \forall w \in H^1_\diamond(\Omega).$$

(Note that Lax-Milgram yields that $L_D \in \mathcal{L}(L^2_\diamond(D)^n, L^2_\diamond(\partial\Omega))$ is well-defined.)

L_D determines D

Theorem 3.3. For all unit vectors $d \in \mathbb{R}^n$, $\|d\| = 1$, and every point $z \in \Omega \setminus \partial D$,

$$z \in D \quad \text{if and only if} \quad \Phi_z|_{\partial\Omega} \in \mathcal{R}(L_D).$$

The adjoint of L_D

Theorem 3.4. The adjoint operator of L_D is given by

$$L_D^* : \; L^2_\diamond(\partial\Omega) \to L^2(D)^n, \qquad g \mapsto \nabla u_0|_D,$$

where $u_0 \in H^1_\diamond(\Omega)$ solves

$$\Delta u_0 = 0 \text{ in } \Omega, \quad \text{and} \qquad \partial_\nu u_0|_{\partial\Omega} = g.$$

A monotonicity result

Theorem 3.5. Let $\sigma_1, \sigma_0 \in L^\infty_+(\Omega)$. Then, for all $g \in L^2_\diamond(\partial\Omega)$,

$$\begin{aligned}\int_\Omega (\sigma_0 - \sigma_1)|\nabla u_0|^2 \,\mathrm{d}x &\leq \int_{\partial\Omega} g\,(\Lambda(\sigma_1) - \Lambda(\sigma_0))\,g \,\mathrm{d}s \\ &\leq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_0 - \sigma_1)|\nabla u_0|^2 \,\mathrm{d}x,\end{aligned}$$

where $u_0 \in H^1_\diamond(\Omega)$ solves $\nabla \cdot (\sigma_0 \nabla u_0) = 0$ in Ω, and $\sigma_0 \partial_\nu u_0|_{\partial\Omega} = g$.

Corollary. For $\sigma_0 = 1$, $\sigma_1 = 1 + \chi_D$

$$\|L_D^* g\|^2_{L^2(D)^n} \geq \underbrace{\int_{\partial\Omega} g\,(\Lambda_0 - \Lambda_1)\,g \,\mathrm{d}s}_{=\left\||\Lambda_1 - \Lambda_0|^{1/2} g\right\|_{\mathcal{L}(L^2_\diamond(\partial\Omega))}} \geq \frac{1}{2}\|L_D^* g\|^2_{L^2(D)^n}$$

A tool from functional analysis

Theorem 3.6. Let X_1, X_2, and Y be three real Hilbert spaces, $A_i \in \mathcal{L}(X_i, Y)$, $i = 1, 2$. If there exists $C > 0$ with

$$\|A_1^* y\|_{X_1} \leq C \|A_2^* y\|_{X_2} \qquad \text{for all } y \in Y$$

then $\mathcal{R}(A_1) \subseteq \mathcal{R}(A_2)$.

Corollary. For all unit vectors $d \in \mathbb{R}^n$, $\|d\| = 1$, and every point $z \in \Omega \setminus \partial D$, it holds that

$$z \in D \quad \text{if and only if} \quad \Phi_z|_{\partial\Omega} \in \mathcal{R}(L_D) = \mathcal{R}(|\Lambda_1 - \Lambda_0|^{1/2}).$$

Conclusions

$$z \in D = \operatorname{supp}(\sigma - \sigma_0) \quad \text{iff } \Phi_z|_{\partial\Omega} \in \mathcal{R}\left(|\Lambda(\sigma) - \Lambda(\sigma_0)|^{1/2}\right)$$

- Original proofs of Kirsch, Hanke and Brühl used a factorization of the operator $\Lambda(\sigma) - \Lambda(\sigma_0)$ (Factorization Method).
- FM shows that conductivity inclusions are uniquely determined from measuring the NtD.
- FM extends to more general (e.g., piecew. anal.) conductivities and inclusions and to partial boundary measurements.
- FM can be implemented numerically, but convergence for noisy data is still an unsolved issue.

  (A stable alternative will be presented in the next lecture...)