Lecture 1: Introduction to Inverse Problems

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Motivation and examples

Laplace's demon

Laplace's demon: *(Pierre Simon Laplace 1814)*

"An intellect which (. . .) would know all forces (. . .) and all positions of all items (. . .), if this intellect were also vast enough to submit these data to analysis, (. . .); for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes."

Computational Science

Computational Science / Simulation Technology:

If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by solving mathematical formulas).

Goals:

- Prediction
- Optimization
- Inversion/Identification

Computational Science

Generic simulation problem:

Given input x calculate outcome $y = F(x)$.

$x \in X$: parameters / input
$y \in Y$: outcome / measurements
$F : X \to Y$: functional relation / model

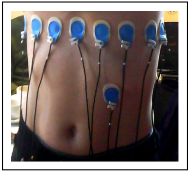
Goals:

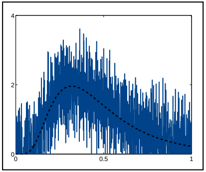
- Prediction: Given x, calculate $y = F(x)$.
- Optimization: Find x, such that $F(x)$ is optimal.
- Inversion/Identification: Given $F(x)$, calculate x.

Examples

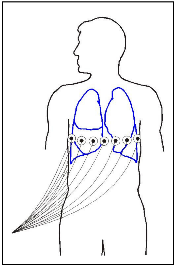
Examples of inverse problems:

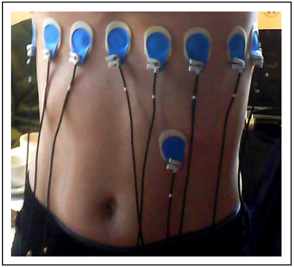
- Electrical impedance tomography
- Computerized tomography
- Image Deblurring
- Numerical Differentiation

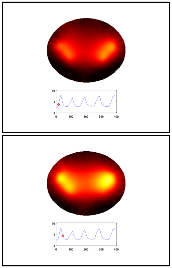


Electrical impedance tomography (EIT)







- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject.

Electrical impedance tomography (EIT)

x Image	$\overset{F}{\longmapsto}$	$y = F(x)$ Measurements

x: Interior conductivity distribution (image)
y: Voltage and current measurements

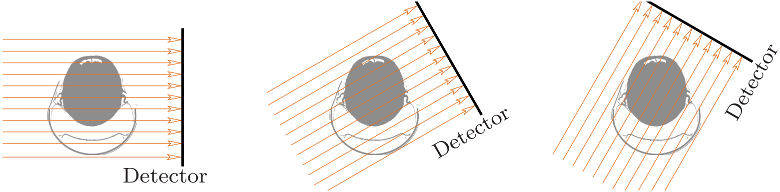
Direct problem: Simulate/predict the measurements
(from knowledge of the interior conductivity distribution)
Given x calculate $F(x) = y$!

Inverse problem: Reconstruct/image the interior distribution
(from taking voltage/current measurements)
Given y solve $F(x) = y$!

X-ray computed tomography

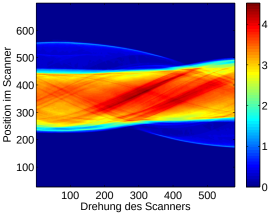
Nobel Prize in Physiology or Medicine 1979:
Allan M. Cormack and Godfrey N. Hounsfield

Idea: Take x-ray images from several directions



Computed tomography (CT)

$\overset{F}{\longmapsto}$



Image

Measurements

Direct problem: Simulate/predict the measurements
(from knowledge of the interior density distribution)
Given x *calculate* $F(x) = y$*!*

Inverse problem: Reconstruct/image the interior distribution
(from taking x-ray measurements)
Given y *solve* $F(x) = y$*!*

Image deblurring

$$\xrightarrow{F}$$

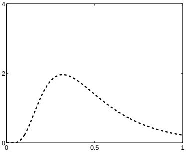
x
True image

$y = F(x)$
Blurred image

Direct problem: Simulate/predict the blurred image
(from knowledge of the true image)
Given x calculate $F(x) = y$!

Inverse problem: Reconstruct/image the true image
(from the blurred image)
Given y solve $F(x) = y$!

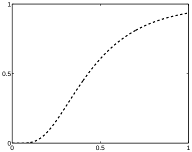
Numerical differentiation



x

Function

$\overset{F}{\longmapsto}$



$y = F(x)$

Primitive Function

Direct problem: Calculate the primitive
Given x calculate $F(x) = y$!

Inverse problem: Calculate the derivative
Given y solve $F(x) = y$!

Ill-posedness

Well-posedness

Hadamard (1865–1963): A problem is called well-posed, if

- a solution exists,
- the solution is unique,
- the solution depends continuously on the given data.

Inverse Problem: *Given y solve $F(x) = y$!*

- F surjective?
- F injective?
- F^{-1} continuous?

Ill-posed problems

Ill-posedness: $F^{-1}: \; Y \to X$ not continuous.

$$
\begin{aligned}
\hat{x} \in X: &\quad \text{true solution} \\
\hat{y} = F(\hat{x}) \in Y: &\quad \text{exact measurement} \\
y^\delta \in Y: &\quad \text{real measurement containing noise } \delta > 0, \\
&\quad \text{e.g. } \|y^\delta - \hat{y}\|_Y \leq \delta
\end{aligned}
$$

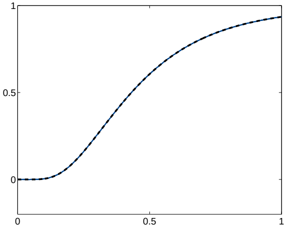
For $\delta \to 0$

$y^\delta \to \hat{y}$, but (generally) $F^{-1}(y^\delta) \not\to F^{-1}(\hat{y}) = \hat{x}$

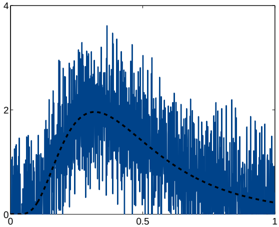
Even the smallest amount of noise will corrupt the reconstructions.

Numerical differentiation

Numerical differentiation example ($h = 10^{-3}$)



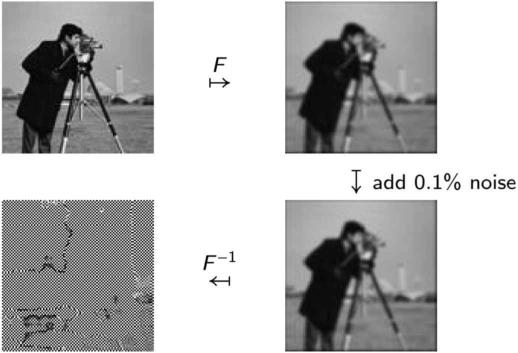
$y(t)$ and $y^\delta(t)$



$\frac{y(t+h)-y(t)}{h}$ and $\frac{y^\delta(t+h)-y^\delta(t)}{h}$

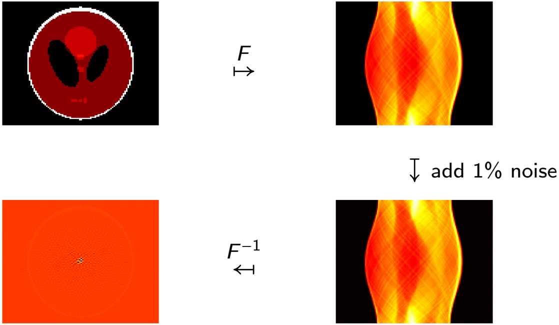
Differentiation seems to be an ill-posed (inverse) problem.

Image deblurring



Deblurring seems to be an ill-posed (inverse) problem.

Image deblurring



CT seems to be an ill-posed (inverse) problem.

Compactness and ill-posedness

Compactness

Consider the general problem

$$F: X \to Y, \quad F(x) = y$$

with X, Y real Hilbert spaces.
Assume that F is linear, bounded and injective with left inverse

$$F^{-1}: F(X) \subseteq Y \to X.$$

Definition 1.1. $F \in \mathcal{L}(X, Y)$ is called compact, if

$$\overline{F(U)} \text{ is compact for alle bounded } U \subseteq X,$$

i.e. if $(x_n)_{n\in\mathbb{N}} \subset X$ is a bounded sequence then $(F(x_n))_{n\in\mathbb{N}} \subset Y$ contains a bounded subsequence.

Compactness

Theorem 1.2. Let

- $F \in \mathcal{L}(X, Y)$ be compact and injective, and
- $\dim X = \infty$,

then the left inverse F^{-1} is not continuous, i.e. the inverse problem

$$Fx = y$$

is ill-posed.

Compactness

Theorem 1.3. Every limit[1] of compact operators is compact.

Theorem 1.4. If $\dim \mathcal{R}(F) < \infty$ then F is compact.

Corollary. Every operator that can be approximated[1] by finite dimensional operators is compact.

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[1] in the uniform operator topology

Compactness

Theorem 1.5. Let $F \in \mathcal{L}(X, Y)$ possess an unbounded left inverse F^{-1}, and let $R_n \in \mathcal{L}(Y, X)$ be a sequence with

$$R_n y \to F^{-1} y \quad \text{for all } y \in \mathcal{R}(F).$$

Then $\|R_n\| \to \infty$.

Corollary. If we discretize an ill-posed problem, the better we discretize, the more unbounded our discretizations become.

Compactness and ill-posedness

Discretization: Approximation by finite-dimensional operators.

Consequences for discretizing infinite-dimensional problems:

If an infinite-dimensional direct problem can be discretized[1], then

- the direct operator is compact.
- the inverse problem is ill-posed, i.e. the smallest amount of measurement noise may completely corrupt the outcome of the (exact, infinite-dimensional) inversion.

If we discretize the inverse problem, then

- the better we discretize, the larger the noise amplification is.

[1] in the uniform operator topology

Examples

- The operator

$$F: \quad \text{function} \; \mapsto \; \text{primitive function}$$

is a linear, compact operator.

$\leadsto$ The inverse problem of differentiation is ill-posed.

- The operator

$$F: \quad \text{exact image} \; \mapsto \; \text{blurred image}$$

is a linear, compact operator.

$\leadsto$ The inverse problem of image deblurring is ill-posed.

Examples

- In computerized tomography, the operator

$$F: \quad \text{image} \ \mapsto \ \text{measurements}$$

  is a linear, compact operator.

  $\rightsquigarrow$ The inverse problem of CT is ill-posed.

- In EIT, the operator

$$F: \quad \text{image} \ \mapsto \ \text{measurements}$$

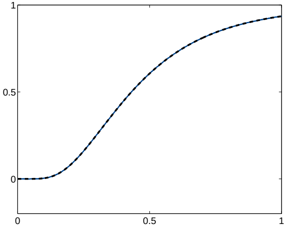
  is a non-linear operator. Its Fréchet derivative is a compact linear operator.

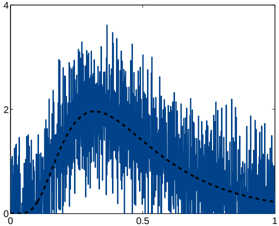
  $\rightsquigarrow$ The (linearized) inverse problem of EIT is ill-posed.

Regularization

Numerical differentiation

Numerical differentiation example





$y(t)$ and $y^\delta(t)$ $\qquad$ $\frac{y(t+h)-y(t)}{h}$ and $\frac{y^\delta(t+h)-y^\delta(t)}{h}$

Differentiation is an ill-posed (inverse) problem

Regularization

Numerical differentiation:

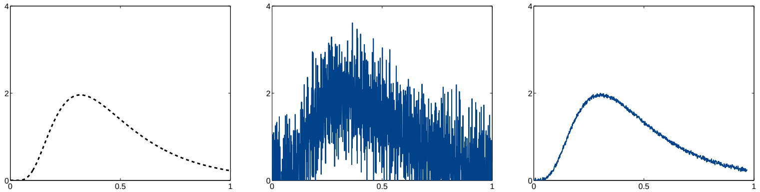
- $y \in C^2$, $C := 2\sup_\tau |g''(\tau)| < \infty$, $|y^\delta(t) - y(t)| \leq \delta \ \forall t$

$$
\begin{aligned}
&\left| y'(t) - \frac{y^\delta(t+h) - y^\delta(t)}{h} \right| \\
&\leq \left| y'(x) - \frac{y(t+h) - y(t)}{h} \right| \\
&\qquad + \left| \frac{y(t+h) - y(t)}{h} - \frac{y^\delta(t+h) - y^\delta(t)}{h} \right| \\
&\leq Ch + \frac{2\delta}{h} \to 0.
\end{aligned}
$$

for $\delta \to 0$ and adequately chosen $h = h(\delta)$, e.g., $h := \sqrt{\delta}$.

Numerical differentiation

Numerical differentiation example



$y'(t)$ | $\frac{y^\delta(t+h)-y^\delta(t)}{h}$ with h very small | $\frac{y^\delta(t+h)-y^\delta(t)}{h}$ with $h \approx \sqrt{\delta}$

Idea of regularization: Balance noise amplification and approximation

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Regularization

Regularization of inverse problems:

- F^{-1} not continuous, so that generally $F^{-1}(y^\delta) \not\to F^{-1}(y) = x$ for $\delta \to 0$
- R_h continuous approximations of F^{-1},
  $R_h \to F^{-1}$ (pointwise) for $h \to 0$

$$R_{h(\delta)} y^\delta \to F^{-1} y = x \quad \text{for } \delta \to 0$$

if the parameter $h = h(\delta)$ is correctly chosen.

Inexact but continuous reconstruction (regularization)
\+ Information on measurement noise (parameter choice rule)
= Convergence

Conclusions

Ill-posed inverse problems

- Inverse problems are of great importance in comput. science
  (parameter identification, medical tomography, etc.)
- Infinite-dimensionality often leads to ill-posed inverse problems (infinite noise amplification)
- The better we discretize an ill-posed inverse problems, the larger the noise amplification gets.

Regularization

- Balancing noise-amplification and approximation may still yield convergence for noisy data.
  (More on this in the second lecture. . .)