Inverse parameter identification problems

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Contents

- Parameter identification problems in diffuse optical tomography (DOT)
- Linearized reconstruction algorithms in electrical impedance tomography (EIT)
Parameter identification problems

in diffuse optical tomography (DOT)
Diffuse optical tomography

Diffuse optical tomography (DOT):

- Transilluminate biological tissue with visible/near-infrared light
- **Goal:** Reconstruct spatial image of interior physical properties.

  Relevant quantities (in diffusive regime):
  - Scattering
  - Absorption

- **Applications:**
  - Breast cancer detection
  - Bedside-imaging of neonatal brain function
Diffuse optical tomography

Figure 1 from Gibson, Hebden and Arridge (Phys. Med. Biol. 50, R1–R43, 2005) Imaging Diagnostic System Inc.’s computed tomography laser breast imaging system. See www.imds.com for more details.

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Mathematical Model

- **General Forward Model:**
  Photon transport models (Boltzmann transport equation)
  cf., e.g., Bal, Inverse Problems 25, 053001 (48pp), 2009.

- **For highly scattering media:**
  - DC diffusion approximation for photon density $u$:
    $$ -\nabla \cdot (a \nabla u) + cu = 0 \quad \text{in } B \subset \mathbb{R}^n, $$
    $u : B \to \mathbb{R}$: photon density
    $a : B \to \mathbb{R}$: diffusion/scattering coefficient
    $c : B \to \mathbb{R}$: absorption coefficient

- **Boundary measurements (idealized):**
  Neumann and Dirichlet data $u|_S$, $a\partial_\nu u|_S$ on $S \subseteq \partial B$.
  Remaining boundary assumed to be insulated, $a\partial_\nu u|_{\partial B \setminus \bar{S}} = 0$. 

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Forward & inverse problem

DC diffuse optical tomography:

$$-\nabla \cdot (a \nabla u) + cu = 0 \quad \text{in } B \subset \mathbb{R}^n, \ n \geq 2,$$

$B$ bounded with smooth bndry, $S \subseteq \partial B$ open part, $a, c \in L^\infty(B)$.

$\forall g \in L^2(S) \ \exists! \ \text{sol. } u \in H^1(B) : \ a \partial_\nu u|_{\partial B} = \begin{cases} 
g & \text{on } S, \\
0 & \text{on } B \setminus \overline{S}. \end{cases}$

$\triangleright$ (Local) Neumann-to-Dirichlet map

$$\Lambda_{a,c} : g \mapsto u|_S, \quad L^2(S) \rightarrow L^2(S)$$

is linear, compact and self-adjoint.

Inverse Problem: Can we reconstruct $a$ and $c$ from $\Lambda_{a,c}$?
Non-uniqueness

DC diffuse optical tomography:

\[-\nabla \cdot (a \nabla u) + cu = 0\]


- \( v := \sqrt{a} u \) solves
  \[-\Delta v + \eta v = 0, \quad \text{with} \quad \eta = \frac{\Delta \sqrt{a}}{\sqrt{a}} + \frac{c}{a}.\]

- \( a = 1 \) around \( S \) \( \leadsto (u|_S, a \partial_\nu u|_S) = (v|_S, \partial_\nu v|_S).\)

\( \leadsto \Lambda_{a,c} \) only depends on effective absorption \( \eta = \eta(a, c). \)

Absorption and scattering effects cannot be distinguished.

(Note: Argument requires smooth scattering coefficient \( a \)).

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Experimental results

Theory: Absorption and scattering effects cannot be distinguished.

Practice:
Successful separate reconstructions of absorption and scattering (from phantom experiment using dc diffusion model!)
Pei et al. (2001), Jiang et al. (2002), Schmitz et al. (2002), Xu et al. (2002)

⇝ Practice contradicts theory!

Pei et al. (2001):
"As a matter of established methodological principle (...) empirical facts have the right-of-way; if a theoretical derivation yields a conclusion that is at odds with experimental results, the reconciliatory burden falls on the theorist, not on the experimentalist."

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New uniqueness result

Theorem (H., Inverse Problems 2009)

- \( a_1, a_2 \in L_+^\infty(B) \) piecewise constant
- \( c_1, c_2 \in L_+^\infty(B) \) piecewise analytic

If \( \Lambda_{a_1,c_1} = \Lambda_{a_2,c_2} \) then \( a_1 = a_2 \) and \( c_1 = c_2 \).

- Piecewise constantness seems fulfilled for phantom experiments.
- Result reconciles theory with practice.
- Measurements contain more than just the effective absorption!

Next slides: Idea of the proof using monotony and loc. potentials.

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Monotony result

Lemma
Let $a_1, a_2, c_1, c_2 \in L^\infty_+(B)$. Then for all $g \in L^2(S)$,

$$
\int_B \left( (a_2 - a_1)|\nabla u_1|^2 + (c_2 - c_1)|u_1|^2 \right) \, dx \\
\geq \langle (\Lambda_{a_1, c_1} - \Lambda_{a_2, c_2})g, g \rangle \\
\geq \int_B \left( (a_2 - a_1)|\nabla u_2|^2 + (c_2 - c_1)|u_2|^2 \right) \, dx,
$$

$u_1, u_2 \in H^1(B)$: solutions for $(a_1, c_1)$, resp., $(a_2, c_2)$.

Can we control $|u_j|^2$ and $|\nabla u_j|^2$?

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Localized potentials

Lemma There exist solutions $u$ with

- $\|u\|_{H^1(B \setminus \overline{O})}$ small
- $\|u\|_{H^1(O)}$ large
- $\|u\|_{L^2(O)}$ small

- $\|u\|_{H^1(B \setminus \overline{O})}$ small
- $\|u\|_{L^2(O')} = \|u\|_{L^2(O')} = \text{large}$
Localized potentials

Lemma There exist solutions $u$ with

- $\|u\|_{H^1(B \setminus \overline{O \cup \Omega})}$ small
- $\|u\|_{H^1(\Omega)}$ large
- $\|u\|_{L^2(\Omega)}$ small
- $\|u\|_{L^2(\Omega')}$ large

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Proof of uniqueness

Monotony

\[ \int_B ((a_2 - a_1)|\nabla u_1|^2 + (c_2 - c_1)|u_1|^2) \, dx \geq \langle (\Lambda a_1, c_1 - \Lambda a_2, c_2)g, g \rangle \]

\[ \geq \int_B ((a_2 - a_1)|\nabla u_2|^2 + (c_2 - c_1)|u_2|^2) \, dx, \]

Proof of the uniqueness result \(\text{(very sketchy \ldots)}\)

Start with region next to \(S\)

- Use loc. pot. with \(|\nabla u|^2 \to \infty\) in that region \(\rightsquigarrow a_1 = a_2\)
- Then use loc. pot. with \(|u|^2 \to \infty\) in that region \(\rightsquigarrow c_1 = c_2\)
- Repeat over all regions.
Uniqueness or not?

- Arridge/Lionheart (1998): Non-uniqueness for general smooth \((a, c)\).
- H. (2009): Uniqueness for piecew. constant \(a\), piecew. analytic \(c\).

What information about \((a, c)\) does \(\Lambda_{a,c}\) really contain?

Formally(!), \(\Lambda_{a,c}\) can only determine \(\eta = \frac{\Delta \sqrt{a}}{\sqrt{a}} + \frac{c}{a}\).

Jumps in \(a\) or \(\nabla a\) \(\leadsto\) distributional singularities in \(\Delta \sqrt{a}\).

Bold guess: Maybe \(\Lambda_{a,c}\) determines
- \(\eta\) where \(a\) and \(c\) are smooth,
- jumps in \(a\) and \(\nabla a\).

(However, note that \(\Delta \sqrt{a}/\sqrt{a}\) is not well-defined for non-smooth \(a\) . . . )

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Exact characterization

Theorem (H., Inverse Probl. Imaging 2012)

Let \( a_1, a_2, c_1, c_2 \in L_+^\infty(B) \) piecewise analytic on joint partition
\[
B = O_1 \cup \ldots \cup O_J \cup \Gamma, \quad \partial O_1 \cup \ldots \cup \partial O_J = \partial B \cup \Gamma.
\]

Then, \( \Lambda_{a_1,c_1} = \Lambda_{a_2,c_2} \) if and only if
\[
\begin{align*}
\text{(a)} & \quad a_1|_S = a_2|_S, \quad \text{and} \quad \partial_\nu a_1|_S = \partial_\nu a_2|_S \quad \text{on } S, \\
\text{(b)} & \quad \frac{\partial_\nu a_1}{a_1}\bigg|_{\partial B \setminus \overline{S}} = \frac{\partial_\nu a_2}{a_2}\bigg|_{\partial B \setminus \overline{S}} \quad \text{on } \partial B \setminus \overline{S}, \\
\text{(c)} & \quad \eta_1 := \frac{\Delta \sqrt{a_1}}{\sqrt{a_1}} + \frac{c_1}{a_1} = \frac{\Delta \sqrt{a_2}}{\sqrt{a_2}} + \frac{c_2}{a_2} =: \eta_2 \quad \text{on } B \setminus \Gamma, \\
\text{(d)} & \quad \frac{a_1^+|_\Gamma}{a_1^-|_\Gamma} = \frac{a_2^+|_\Gamma}{a_2^-|_\Gamma}, \quad \text{and} \quad \frac{[\partial_\nu a_2]|_\Gamma}{a_2^-|_\Gamma} = \frac{[\partial_\nu a_1]|_\Gamma}{a_1^-|_\Gamma} \quad \text{on } \Gamma.
\end{align*}
\]
Linearized reconstruction algorithms

in electrical impedance tomography (EIT)
Electrical impedance tomography (EIT):

- Apply currents $\sigma \partial \nu u |_{\partial B}$ (Neumann boundary data)
  \implies Electric potential $u$ solves
  \[ \nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } B \]

- Measure voltages $u |_{\partial B}$ (Dirichlet boundary data)

**Current-Voltage-Measurements** $\implies$ Neumann-to-Dirichlet map $\Lambda(\sigma)$

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Inverse problem

Non-linear forward operator of EIT

\[ \Lambda: \sigma \mapsto \Lambda(\sigma), \quad L^\infty_+ (B) \to \mathcal{L}(L^2_\diamond(\partial B)) \]

Inverse problem of EIT: \( \Lambda(\sigma) \mapsto \sigma? \)

Localized potentials \( \leadsto \) Uniqueness for piecew. analytic conductivities
already known: Druskin (1982+85), Kohn/Vogelius (1984+85)
Linearization

Generic approach: Linearization

\[ \Lambda(\sigma) - \Lambda(\sigma_0) \approx \Lambda'(\sigma_0)(\sigma - \sigma_0) \]

\(\sigma_0\): known reference conductivity / initial guess / . . .

\(\Lambda'(\sigma_0)\): Fréchet-Derivative / sensitivity matrix.

\[ \Lambda'(\sigma_0) : L^\infty_+(B) \to \mathcal{L}(L^2_\diamond(\partial B)) \]

\(\supp(\sigma - \sigma_0) \subset \subset B\) compact. ("shape" / "inclusion")

\(\rightsquigarrow\) Solve linearized equation for difference \(\sigma - \sigma_0\).

Often: \(\supp(\sigma - \sigma_0) \subset \subset B\) compact. ("shape" / "inclusion")
Linearization

Linear reconstruction method
e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)
Solve $\Lambda'(\sigma_0) \kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- Multiple possibilities to measure residual norm and to regularize.
- No rigorous theory for single linearization step.
- Almost no theory for Newton iteration:
  - Dobson (1992): (Local) convergence for regularized EIT equation.
  - No (local) convergence theory for non-discretized case!
Linearization

Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- Seemingly, no rigorous results possible for single lineariz. step.
- Seemingly, only justifiable for small $\sigma - \sigma_0$ (local results).

Here: Rigorous and global(!) result about the linearization error.
Exact Linearization


Let $\kappa, \sigma, \sigma_0$ piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

$$\text{supp}_{\partial B}\kappa = \text{supp}_{\partial B}(\sigma - \sigma_0)$$

$\text{supp}_{\partial B}$: outer support ($=$ support, if support is compact and has conn. complement)

- Exact solution of lin. equation yields correct (outer) shape.
- No assumptions on $\sigma - \sigma_0$!
- Linearization error does not lead to shape errors.

**Proof**: Combination of monotony and localized potentials.

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Proof

- Exact linearization: \( \Lambda'(\sigma_0) \kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \)

- Monotony: For all "reference solutions" \( u_0 \):

\[
\int_B (\sigma - \sigma_0) |\nabla u_0|^2 \, dx \\
\geq \langle g, (\Lambda(\sigma) - \Lambda(\sigma_0)) g \rangle \geq \int_B \frac{\sigma_0}{\sigma} (\sigma - \sigma_0) |\nabla u_0|^2 \, dx.
\]

\[
= \int_B \kappa |\nabla u_0|^2 \, dx
\]

- Use localized potentials to control \( |\nabla u_0|^2 \)

\sim \text{supp}_{\partial\Omega} \kappa = \text{supp}_{\partial\Omega} (\sigma - \sigma_0)
Interpretation

$\sigma_0$: reference conductivity
$\sigma$: true conductivity

- Current paths depend on unknown conductivity $\sigma$
  (Non-linearity of EIT)
- Linearizing EIT around ref. conductivity $\sigma_0$ corresponds to assuming that currents take the same paths as in a ref. body.
- Paths may differ considerably when $\sigma - \sigma_0$ is large.
- However, taking the (wrong) reference paths for reconstruction still yields the correct shape information!

Practical issues: Theorem requires exact sol. of linearized equation

- Does an approximate solution of the linearized equation show approximately correct shapes?
Experimental result (frequency difference data)

*Heuristic* combination of linearization and localized potentials for fdEIT.

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Planar EIT

Ts/Lee/Seo/H./Kim (submitted): Partial inversion plus linearization

Sensing Electrodes

\[ \Gamma \]

\[ E_1 \]

\[ E_2 \]

Distance from sensing surface = \( d \) cm

Sensing electrodes are placed on \( z = 8 \) surface

Graph of \( \Gamma_1 \)

\( d = 1 : 'E' \) is located at \( z = 6 \)

\( d = 2 : 'E' \) is located at \( z = 5 \)

\( d = 3 : 'E' \) is located at \( z = 4 \)

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Lung monitoring - real human data

*BMBF-project on regularizing EIT in the medical and geophysical sciences.*
Conclusion

- Novel tomography techniques lead to mathematical parameter identification problems.
- Uniqueness questions may have non-trivial answers.
  In diffuse optical tomography:
  - Effective absorption plus jumps in diffusion coefficient (and its derivative) can be reconstructed simultaneously.
    ⇝ Uniqueness for piecewise constant coefficients, but not for general smooth coefficients
- Non-linearity is main challenge for stable/convergent algos.
  - Shape information is not affected by linearization errors.
  - Stable shape reconstruction is possible.
- Close interplay between uniqueness arguments and convergent reconstruction algorithms.

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