Direct and Inverse Eddy Current Problems

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Motivation

Inverse Electromagnetics & Eddy currents
Inverse Electromagnetics

Inverse Electromagnetics:

- Generate EM field
  *(drive excitation current through coil)*
- Measure EM field
  *(induced voltages in meas. coil)*
- Gain information from measurements

Applications:

- Metal detection *(buried conductor)*
- Non-destructive testing
  *(crack in metal, metal in concrete)*
- ...
Maxwell’s equations

Classical Electromagnetics: Maxwell’s equations

curl $H = \epsilon \partial_t E + \sigma E + J$ in $\mathbb{R}^3 \times ]0, T[$

curl $E = -\mu \partial_t H$ in $\mathbb{R}^3 \times ]0, T[$

$E(x, t)$: Electric field $\quad \epsilon(x)$: Permittivity
$H(x, t)$: Magnetic field $\quad \mu(x)$: Permeability
$J(x, t)$: Excitation current $\quad \sigma(x)$: Conductivity

Knowing $J, \sigma, \mu, \epsilon \ + \ init. \ cond. \ determines \ E \ and \ H.$
Eddy currents

Maxwell's equations

\[
\begin{align*}
curl H &= \epsilon \partial_t E + \sigma E + J \\
curl E &= -\mu \partial_t H
\end{align*}
\]

in \( \mathbb{R}^3 \times ]0, T[ \)

Eddy current approximation: Neglect displacement currents \( \epsilon \partial_t E \)

\( \rightarrow \)

Justified for low-frequency excitations


\[
\begin{align*}
\partial_t (\sigma E) + \text{curl} \left( \frac{1}{\mu} \text{curl} E \right) &= -\partial_t J \\
&\text{in } \mathbb{R}^3 \times ]0, T[ \)
\]
Where’s Eddy?

- $\sigma = 0$: (Quasi-)Magnetostatics

$$\text{curl} \left( \frac{1}{\mu} \text{curl } E \right) = -\partial_t J$$

Excitation $\partial_t J$ instantly generates magn. field $\frac{1}{\mu} \text{curl } E = -\partial_t H$.

- $\sigma \neq 0$: Eddy currents

$$\partial_t (\sigma E) + \text{curl} \left( \frac{1}{\mu} \text{curl } E \right) = -\partial_t J$$

$\partial_t J$ generates changing magn. field + currents inside conductor

*Induced currents oppose what created them (Lenz law)*
Parabolic-elliptic equations

\[
\partial_t (\sigma E) + \text{curl} \left( \frac{1}{\mu} \text{curl} E \right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times J_0, T[
\]

- **parabolic** inside conductor \( \Omega = \text{supp} (\sigma) \)
- **elliptic** outside conductor

Scalar example: \((\sigma u)_t = u_{xx}, u(\cdot, 0) = 0, u_x(-2, \cdot) = u_x(2, \cdot) = 1.\)
The direct problem

Unified variational formulation for the parabolic-elliptic eddy current problem
Standard approach

\[ \partial_t (\sigma E) + \text{curl} \left( \frac{1}{\mu} \text{curl} E \right) = -\partial_t J \quad \text{in} \quad \mathbb{R}^3 \times ]0, T[ \]

**Standard approach:** *Decouple elliptic and parabolic part* (e.g. Bossavit 1999, Acevedo/Meddahi/Rodriguez 2009)

Find \((E_{\mathbb{R}^3 \setminus \Omega}, E_\Omega) \in H_{\mathbb{R}^3 \setminus \Omega} \times H_\Omega\) s.t.

- \(E_\Omega\) solves parabolic equation + init. cond.
- \(E_{\mathbb{R}^3 \setminus \Omega}\) solves elliptic equation
- interface conditions are satisfied

**Problem:** Theory (solution spaces, coercivity constants, etc.) depends on \(\Omega = \text{supp} \sigma\) and on lower bounds of \(\sigma|_\Omega\).

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Unified approach?

Parabolic-elliptic eddy current equation

\[
\partial_t (\sigma E) + \text{curl} \left( \frac{1}{\mu} \text{curl} E \right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times ]0, T[
\]

**Inverse problem**: Find \( \sigma \) (or \( \Omega = \text{supp } \sigma \)) from measurements of \( E \)

- requires unified solution theory

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Test for unified theory: Can we linearize \( E \) w.r.t. \( \sigma \)?

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*How does the solution of an elliptic equation change if the equation becomes a little bit parabolic?*

(For scalar analogue: Frühauf/H./Scherzer 2007, H. 2007)
Rigorous formulation: Let $\mu \in L^\infty_+$, $\sigma \in L^\infty$, $\sigma \geq 0$,

\[ J_t \in L^2(0, T, W(\text{curl}')) \quad \text{with} \quad \text{div } J_t = 0 \]
\[ E_0 \in L^2(\mathbb{R}^3)^3 \quad \text{with} \quad \text{div}(\sigma E_0) = 0. \]

For $E \in L^2(0, T, W(\text{curl}))$ the eddy current equations

\[
\partial_t (\sigma E) + \text{curl} \left( \frac{1}{\mu} \text{curl } E \right) = -J_t \quad \text{in } \mathbb{R}^3 \times ]0, T[ \\
\sqrt{\sigma} E(x, 0) = \sqrt{\sigma(x)} E_0(x) \quad \text{in } \mathbb{R}^3
\]

are well-defined and (if solvable) uniquely determine $\text{curl } E$, $\sqrt{\sigma E}$.  

Natural variational formulation

Natural unified variational formulation ($E_0 = 0$ for simplicity):

Find $E \in L^2(0, T, W(\text{curl}))$ that solves

$$\int_0^T \int_{\mathbb{R}^3} \left( \sigma E \cdot \partial_t \Phi - \frac{1}{\mu} \text{curl} E \cdot \text{curl} \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.$$ 

for all smooth $\Phi$ with $\Phi(\cdot, T) = 0$.

- equivalent to eddy current equation
- not coercive, does not yield existence results
Gauged formulation

Gauged unified variational formulation ($E_0 = 0$ for simplicity)

Find divergence-free $E \in L^2(0, T, W(\text{curl}))$ that solves

$$
\int_0^T \int_{\mathbb{R}^3} \left( \sigma E \cdot \partial_t \Phi - \frac{1}{\mu} \text{curl } E \cdot \text{curl } \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.
$$

for all smooth divergence-free $\Phi$ with $\Phi(\cdot, T) = 0$.

- coercive, yields existence and continuity results
- not equivalent to eddy current equation
  ($\sigma \neq \text{const.} \implies \text{div } \sigma E \neq \sigma \text{ div } E$)
- does not determine true solution up to gauge (curl-free) field
Coercive unified formulation

How to obtain coercive + equivalent unified formulation?

- Ansatz $E = A + \nabla \varphi$ with divergence-free $A$.
  *(almost the standard $(A, \varphi)$-formulation with Coulomb gauge)*

- Consider $\nabla \varphi = \nabla \varphi_A$ as function of $A$ by solving
  $\text{div } \sigma \nabla \varphi_A = - \text{div } \sigma A$.
  *(\Rightarrow \text{div } \sigma E = 0)*.

- Obtain coercive formulation for $A$
  *(Lions-Lax-Milgram Theorem \Rightarrow Solvability and continuity results)*

- $A$ determines $E$
  *(more precisely: curl $E$ and $\sqrt{\sigma}E$)*
Unified variational formulation

Unified variational formulation (Arnold/H., SIAP, 2012)

Find divergence-free \( A \in L^2(0, T, W(\text{curl})) \) that solves

\[
\int_0^T \int_{\mathbb{R}^3} \left( \sigma (A + \nabla \varphi_A) \cdot \partial_t \Phi - \frac{1}{\mu} \text{curl} A \cdot \text{curl} \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.
\]

for all smooth divergence-free \( \Phi \) with \( \Phi(\cdot, T) = 0 \).

- coercive, uniquely solvable
- \( E := A + \nabla \varphi_A \) is one solution of the eddy current equation
- \( \text{curl} E, \sqrt{\sigma}E \) depend continuously on \( J_t \) (uniformly w.r.t. \( \sigma \))
  (for all solutions of the eddy current equation)
Solved and open problems

Unified variational formulation

- allows to study inverse problems w.r.t. $\sigma$
- allows to rigorously linearize $E$ w.r.t. $\sigma$ around $\sigma_0 = 0$
  (elliptic equation becoming a little bit parabolic in some region...)

Open problem:

- Theory requires some regularity of $\Omega = \text{supp} \sigma$ and $\sigma \in L^\infty_+ (\Omega)$ in order to determine $\varphi$ from $A$.
- Solution theory for $\text{div} \, \sigma \nabla \varphi = - \text{div} \, \sigma A$
  for general $\sigma \in L^\infty$, $\sigma \geq 0$?

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The inverse problem
Setup

Detecting conductors:

- Apply surface currents \( J \) on \( S \) (divergence-free, no electrostatic effects)
- Measure electric field \( E \) on \( S \) (tangential component, up to grad. fields)
- Measurement operator

\[
\Lambda_\sigma : J_t \mapsto \gamma_\tau E := (\nu \wedge E|_S) \wedge \nu
\]

Locate \( \Omega = \text{supp } \sigma \) in

\[
\partial_t (\sigma E) + \text{curl} \left( \frac{1}{\mu} \text{curl } E \right) = -J_t \quad \text{in } \mathbb{R}^3 \times [0, T[ \\
(+ \text{ zero IC}) \text{ from all possible surface currents and measured values.}
\]
Measurement operator

\[ S \subset \mathbb{R}^3 \]

\[ TL^2 := \{ u \in L^2(S)^3 \mid u \cdot \nu = 0 \} \]

\[ TL^2_\ominus := \{ u \in TL^2 \mid \int_S u \cdot \nabla \psi = 0 \quad \forall \text{smooth} \ \psi \} \]

Measurement operator

\[ \Lambda_\sigma : L^2(0, T, TL^2_\ominus) \to L^2(0, T, TL^2'_\ominus), \quad J_t \mapsto \gamma_T E, \]

where \( E \) solves eddy current eq. with \( [\nu \times \text{curl} \ E]_S = J_t \) on \( S \).

Remark

\[ TL^2'_\ominus \approx TL^2 / TL^2_{\ominus \perp} \leadsto E \text{ not unique, but } \Lambda_\sigma \text{ well-defined}. \]
Sampling methods

Non-iterative shape detection methods:

- **Linear Sampling Method (Colton/Kirsch 1996)**
  - characterizes subset of scatterer by range test
  - allows fast numerical implementation

- **Factorization Method (Kirsch 1998)**
  - characterizes scatterer by range test
  - yields uniqueness under definiteness assumptions
  - allows fast numerical implementation

- Beyond LSM/FM?
Sampling ingredients

Ingredients for LSM and FM:

- **Reference measurements:** $\Lambda := \Lambda_\sigma - \Lambda_0$, 
  $\Lambda_0 : J_t \mapsto \gamma_t F$, 
  $F$ solves $\text{curl} \ \text{curl} \ F = -J_t$ in $\mathbb{R}^3 \times ]0, T[$.

- **Time-integration:** Consider $I\Lambda$, 
  with $I : E(\cdot, \cdot) \mapsto \int_0^T E(\cdot, t) \, dt$

- **Singular test functions** 
  $$G_{z,d}(x) := \text{curl} \frac{d}{4\pi|x - z|}, \quad x \in \mathbb{R}^3 \setminus \{z\}$$
LSM and FM

*Arnold/H. (submitted)*:
For every $z$ below $S$, $z \not\in \Omega$ and direction $d \in \mathbb{R}^3$.

**Theorem (LSM)**

\[
\gamma_T G_{z,d} \in \mathcal{R}(I\Lambda) \quad \Rightarrow \quad z \in \Omega
\]

**Theorem (FM)**

If, additionally, $\sup \mu|_{\Omega} < 1$ (diamagnetic scatterer)

\[
\gamma_T G_{z,d} \in \mathcal{R}(I(\Lambda + \Lambda')^{1/2}) \quad \Leftrightarrow \quad z \in \Omega
\]
Beyond LSM/FM?

- Beyond LSM/FM?: Monotony methods
- For EIT: $\Lambda_{\sigma}$ NtD-operator for conductivity $\sigma = 1 + \chi_D$
  \[
  D = \text{Union of all balls } B \text{ where } \Lambda_{1+\chi_B} \leq \Lambda_{\sigma} \text{ (H./Ullrich)}
  \]
  (under the assumptions of the FM)
- stable test criterion (no infinity tests)
- allows fast numerical implementation
- allows extensions to indefinite cases
Conclusions

Inverse transient eddy current problems
▷ require unified parabolic-elliptic theory
▷ can be approached by sampling methods (LSM/FM)

Open problems
▷ Solution theory for
\[ \text{div } \sigma \nabla \varphi = - \text{div } \sigma A \]
for general \( \sigma \in L^\infty, \sigma \geq 0 \)?
▷ Monotony based methods beyond EIT? Parabolic-elliptic problems? Inverse Scattering?