Direct and Inverse Eddy Current Problems

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Motivation

Inverse Electromagnetics & Eddy currents
Inverse Electromagnetics

Inverse Electromagnetics:

- Generate EM field
  \textit{(drive excitation current through coil)}
- Measure EM field
  \textit{(induced voltages in meas. coil)}
- Gain information from measurements

Applications:

- Metal detection \textit{(buried conductor)}
- Non-destructive testing
  \textit{(crack in metal, metal in concrete)}
- ...
Maxwell’s equations

Classical Electromagnetics: Maxwell’s equations

\[ \text{curl } H = \varepsilon \partial_\text{t} E + \sigma E + J \quad \text{in } \mathbb{R}^3 \times ]0, T[ \]
\[ \text{curl } E = -\mu \partial_\text{t} H \quad \text{in } \mathbb{R}^3 \times ]0, T[ \]

\( E(x, t) \): Electric field \quad \varepsilon(x) \): Permittivity
\( H(x, t) \): Magnetic field \quad \mu(x) \): Permeability
\( J(x, t) \): Excitation current \quad \sigma(x) \): Conductivity

Knowing \( J, \sigma, \mu, \varepsilon \) + init. cond. determines \( E \) and \( H \).
Eddy currents

Maxwell’s equations

\[
\text{curl } H = \varepsilon \partial_t E + \sigma E + J \quad \text{in } \mathbb{R}^3 \times ]0, T[ \\
\text{curl } E = -\mu \partial_t H \quad \text{in } \mathbb{R}^3 \times ]0, T[
\]

Eddy current approximation: Neglect displacement currents \( \varepsilon \partial_t E \)

- Justified for low-frequency excitations \\
  \((\text{Alonso 1999, Ammari/Buffa/Nédélec 2000})\)

\[
\partial_t (\sigma E) + \text{curl} \left( \frac{1}{\mu} \text{curl } E \right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times ]0, T[ 
\]

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Where’s Eddy?

- \( \sigma = 0 \): (Quasi-)Magnetostatics

\[
\text{curl} \left( \frac{1}{\mu} \text{curl} \ E \right) = -\partial_t J
\]

Excitation \( \partial_t J \) instantly generates magn. field \( \frac{1}{\mu} \text{curl} \ E = -\partial_t H \).

- \( \sigma \neq 0 \): Eddy currents

\[
\partial_t (\sigma E) + \text{curl} \left( \frac{1}{\mu} \text{curl} \ E \right) = -\partial_t J
\]

\( \partial_t J \) generates changing magn. field + currents inside conductor

*Induced currents oppose what created them (Lenz law)*
Parabolic-elliptic equations

\[ \partial_t (\sigma E) + \text{curl} \left( \frac{1}{\mu} \text{curl} E \right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times [0, T[ \]

- parabolic inside conductor \( \Omega = \text{supp}(\sigma) \)
- elliptic outside conductor

Scalar example: \( (\sigma u)_t = u_{xx}, \ u(\cdot, 0) = 0, \ u_x(-2, \cdot) = u_x(2, \cdot) = 1. \)
The direct problem

Unified variational formulation for the parabolic-elliptic eddy current problem
Standard approach

\[
\partial_t(\sigma E) + \text{curl} \left( \frac{1}{\mu} \text{curl} E \right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times ]0, T[ 
\]

**Standard approach:** *Decouple elliptic and parabolic part* (e.g. Bossavit 1999, Acevedo/Meddahi/Rodriguez 2009)

Find \((E_{\mathbb{R}^3 \setminus \Omega}, E_\Omega) \in H_{\mathbb{R}^3 \setminus \Omega} \times H_\Omega\) s.t.

- \(E_\Omega\) solves parabolic equation + init. cond.
- \(E_{\mathbb{R}^3 \setminus \Omega}\) solves elliptic equation
- interface conditions are satisfied

**Problem:** Theory (solution spaces, coercivity constants, etc.) depends on \(\Omega = \text{supp } \sigma\) and on lower bounds of \(\sigma|_\Omega\).

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Unified approach?

Parabolic-elliptic eddy current equation

\[ \partial_t (\sigma E) + \text{curl} \left( \frac{1}{\mu} \text{curl} E \right) = -\partial_t J \quad \text{in} \quad \mathbb{R}^3 \times [0, T[ \]

Inverse problem: Find \( \sigma \) (or \( \Omega = \text{supp } \sigma \)) from measurements of \( E \)

\[ \text{requires unified solution theory} \]

Test for unified theory: Can we linearize \( E \) w.r.t. \( \sigma \)?

How does the solution of an elliptic equation change

if the equation becomes a little bit parabolic?

(For scalar analogue: Frühauf/H./Scherzer 2007, H. 2007)
Rigorous formulation

Rigorous formulation: Let $\mu \in L_+^\infty$, $\sigma \in L^\infty$, $\sigma \geq 0$,

\[ J_t \in L^2(0, T, W(\text{curl}')) \quad \text{with} \quad \text{div} \, J_t = 0 \]
\[ E_0 \in L^2(\mathbb{R}^3)^3 \quad \text{with} \quad \text{div}(\sigma E_0) = 0. \]

For $E \in L^2(0, T, W(\text{curl}))$ the eddy current equations

\[ \partial_t (\sigma E) + \text{curl} \left( \frac{1}{\mu} \text{curl} \, E \right) = -J_t \quad \text{in} \quad \mathbb{R}^3 \times ]0, T[ \]
\[ \sqrt{\sigma} E(x, 0) = \sqrt{\sigma(x)} E_0(x) \quad \text{in} \quad \mathbb{R}^3 \]

are well-defined and (if solvable) uniquely determine $\text{curl} \, E$, $\sqrt{\sigma} E$. 

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Natural variational formulation

Natural unified variational formulation ($E_0 = 0$ for simplicity):

Find $E \in L^2(0, T, W(\text{curl}))$ that solves

$$\int_0^T \int_{\mathbb{R}^3} \left( \sigma E \cdot \partial_t \Phi - \frac{1}{\mu} \text{curl} E \cdot \text{curl} \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.$$

for all smooth $\Phi$ with $\Phi(\cdot, T) = 0$.

- equivalent to eddy current equation
- not coercive, does not yield existence results
Gauged formulation

Gauged unified variational formulation \((E_0 = 0\) for simplicity)

Find divergence-free \(E \in L^2(0, T, W^1(\mathbb{R}^3))\) that solves

\[
\int_0^T \int_{\mathbb{R}^3} \left( \sigma E \cdot \partial_t \Phi - \frac{1}{\mu} \text{curl} E \cdot \text{curl} \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.
\]

for all smooth divergence-free \(\Phi\) with \(\Phi(\cdot, T) = 0\).

- coercive, yields existence and continuity results
- not equivalent to eddy current equation
  \((\sigma \neq \text{const.} \implies \text{div } \sigma E \neq \sigma \text{ div } E)\)
- does not determine true solution up to gauge (curl-free) field

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Coercive unified formulation

How to obtain coercive + equivalent unified formulation?

- Ansatz \( E = A + \nabla \varphi \) with divergence-free \( A \).
  
  \emph{(almost the standard \((A, \varphi)\)-formulation with Coulomb gauge)}

- Consider \( \nabla \varphi = \nabla \varphi_A \) as function of \( A \) by solving
  \[ \text{div} \, \sigma \nabla \varphi_A = - \text{div} \, \sigma A. \]

  \( \Rightarrow \text{div} \, \sigma E = 0 \).

- Obtain coercive formulation for \( A \)
  
  \( \text{(Lions-Lax-Milgram Theorem} \Rightarrow \text{Solvability and continuity results)} \)

- \( A \) determines \( E \)
  
  \( \text{(more precisely: curl} \, E \text{ and} \sqrt{\sigma}E) \)
Unified variational formulation

Find divergence-free \( A \in L^2(0, T, W^1(\mathbb{R}^3)) \) that solves
\[
\int_0^T \int_{\mathbb{R}^3} \left( \sigma (A + \nabla \varphi_A) \cdot \partial_t \Phi - \frac{1}{\mu} \text{curl} A \cdot \text{curl} \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.
\]
for all smooth divergence-free \( \Phi \) with \( \Phi(\cdot, T) = 0 \).

- coercive, uniquely solvable
- \( E := A + \nabla \varphi_A \) is one solution of the eddy current equation
- \( \text{curl} E, \sqrt{\sigma E} \) depend continuously on \( J_t \) (uniformly w.r.t. \( \sigma \))
  (for all solutions of the eddy current equation)

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Asymptotic results

Unified variational formulation

- allows to rigorously linearize $E$ w.r.t. $\sigma$ around $\sigma_0 = 0$
  (elliptic equation becoming a little bit parabolic in some region...)
- easily extends from $\mathbb{R}^3$ to bounded domain $O$
  ($O$ simply conn. with Lipschitz-boundary, $\nu \wedge E|_{\partial O} = 0$)
- justifies parabolic regularization: If $E_\epsilon$ solves

  \[
  \partial_t (\sigma_\epsilon E_\epsilon) + \text{curl} \left( \frac{1}{\mu} \text{curl } E_\epsilon \right) = -\partial_t J \quad \text{in } O \times ]0, T[,
  \]

  with $\sigma_\epsilon(x) = \max\{\sigma(x), \epsilon\}$ then

  \[
  \sigma_\epsilon E_\epsilon \to \sigma E, \quad \text{curl } E_\epsilon \to \text{curl } E
  \]

  \[(Arnold/H., \text{ submitted to proceedings of IPDO 2013)}\]
Open problems

- Theory requires some regularity of $\Omega = \text{supp} \sigma$ and $\sigma \in L^\infty_+(\Omega)$ in order to determine $\varphi$ from $A$.

- Solution theory for
  \[
  \text{div} \sigma \nabla \varphi = - \text{div} \sigma A
  \]
  for general $\sigma \in L^\infty$, $\sigma \geq 0$?

- Elliptic regularization of the variational formulation 
  (i.e., adding $\epsilon \int_0^T \int_{\mathbb{R}^3} A \cdot \Phi \, \text{d}x$) is justified, but relation to elliptic regularization of the PDE
  \[
  \partial_t (\sigma E) + \text{curl} \left( \frac{1}{\mu} \text{curl} E \right) + \epsilon E = - \partial_t J \quad \text{in } \mathbb{R}^3 \times ]0, T[,
  \]
  is not clear.
The inverse problem
Detecting conductors:

- Apply surface currents $J$ on $S$ (divergence-free, no electrostatic effects)
- Measure electric field $E$ on $S$ (tangential component, up to grad. fields)
- Measurement operator

$$\Lambda_\sigma : J_t \mapsto \gamma_\tau E := (\nu \wedge E|_S) \wedge \nu$$

Locate $\Omega = \text{supp } \sigma$ in

$$\partial_t (\sigma E) + \text{curl } \left( \frac{1}{\mu} \text{curl } E \right) = -J_t \quad \text{in } \mathbb{R}^3 \times [0, T]$$

(+ zero IC) from all possible surface currents and measured values.
Measurement operator

\[ S \subset \mathbb{R}^3_0 \]

\[ TL^2 : = \{ u \in L^2(S)^3 \mid u \cdot \nu = 0 \} \]

\[ TL_\diamond^2 : = \{ u \in TL^2 \mid \int_S u \cdot \nabla \psi = 0 \quad \forall \text{ smooth } \psi \} \]

Measurement operator

\[ \Lambda_\sigma : L^2(0, T, TL_\diamond^2) \to L^2(0, T, TL'_\diamond), \quad J_t \mapsto \gamma_\tau E, \]

where \( E \) solves eddy current eq. with \([\nu \times \text{curl } E]_S = J_t\) on \( S \).

Remark

\[ TL'_{\diamond} \cong TL^2 / TL^2_{\diamond} \perp \leadsto E \text{ not unique, but } \Lambda_\sigma \text{ well-defined.} \]
Sampling methods

Non-iterative shape detection methods:

- **Linear Sampling Method (Colton/Kirsch 1996)**
  - characterizes subset of scatterer by range test
  - allows fast numerical implementation

- **Factorization Method (Kirsch 1998)**
  - characterizes scatterer by range test
  - yields uniqueness under definiteness assumptions
  - allows fast numerical implementation

- Beyond LSM/FM?
Sampling ingredients

Ingredients for LSM and FM:

- **Reference measurements:** \( \Lambda := \Lambda_\sigma - \Lambda_0 \),
  \( \Lambda_0 : J_t \mapsto \gamma_T F \), \( F \) solves \( \text{curl} \ \text{curl} \ F = -J_t \) in \( \mathbb{R}^3 \times ]0, T[ \).

- **Time-integration:** Consider \( I\Lambda \),
  with \( I : E(\cdot, \cdot) \mapsto \int_0^T E(\cdot, t) \, dt \)

- **Singular test functions**
  \[ G_{z,d}(x) := \text{curl} \left( \frac{d}{4\pi |x - z|} \right), \quad x \in \mathbb{R}^3 \setminus \{z\} \]
LSM and FM

Arnold/H. (submitted):
For every $z$ below $S$, $z \notin \Omega$ and direction $d \in \mathbb{R}^3$.

Theorem (LSM)

$$\gamma_{\tau} G_{z,d} \in \mathcal{R}(I\Lambda) \Rightarrow z \in \Omega$$

Theorem (FM)

If, additionally, $\sup \mu|_{\Omega} < 1$ (diamagnetic scatterer)

$$\gamma_{\tau} G_{z,d} \in \mathcal{R}(I(\Lambda + \Lambda')^{1/2}) \Leftrightarrow z \in \Omega$$
Beyond LSM/FM?

- Beyond LSM/FM?: Monotony methods
- For EIT: $\Lambda_\sigma$ NtD-operator for conductivity $\sigma = 1 + \chi_D$
  
  $$D = \text{Union of all balls } B \text{ where } \Lambda_{1+\chi_B} \leq \Lambda_\sigma (\text{H./Ullrich})$$
  
  (under the assumptions of the FM)
- stable test criterion (no infinity tests)
- allows fast numerical implementation
- allows extensions to indefinite cases
Conclusions

Inverse transient eddy current problems
- require unified parabolic-elliptic theory
- can be approached by sampling methods (LSM/FM)

Open problems
- Solution theory for
  \[ \text{div} \sigma \nabla \varphi = - \text{div} \sigma A \]
  for general \( \sigma \in L^\infty, \sigma \geq 0 \)?
- Monotony based methods beyond EIT?
  Monotony for parabolic-elliptic problems?