Inverse coefficient problems and shape reconstruction

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Electrical impedance tomography (EIT)

Apply electric currents on subject’s boundary
Measure necessary voltages
Reconstruct conductivity inside subject.

Images from BMBF-project on EIT
(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)

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Mathematical Model

Electrical potential $u(x)$ solves

$$\nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega$$

$\Omega \subset \mathbb{R}^n$: imaged body, $n \geq 2$

$\sigma(x)$: conductivity

$u(x)$: electrical potential

Idealistic model for boundary measurements (continuum model):

$\sigma \partial_{\nu} u(x)|_{\partial \Omega}$: applied electric current

$u(x)|_{\partial \Omega}$: measured boundary voltage (potential)
Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \quad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial \Omega}, \sigma \partial_{\nu} u|_{\partial \Omega}) : u \text{ solves } (1)\} ?$$

Equivalent: Recover $\sigma$ from Neumann-to-Dirichlet-Operator

$$\Lambda(\sigma) : L^2_\diamond(\partial \Omega) \rightarrow L^2_\diamond(\partial \Omega), \quad g \mapsto u|_{\partial \Omega},$$

where $u$ solves (1) with $\sigma \partial_{\nu} u|_{\partial \Omega} = g.$
Partial/local data

Measurements on open part of boundary $\Sigma \subset \partial \Omega$: ($\partial \Omega \setminus \Sigma$ is kept insulated.)

Recover $\sigma$ from

$$\Lambda(\sigma) : L^2(\Sigma) \rightarrow L^2(\Sigma), \quad g \mapsto u|_{\Sigma},$$

where $u$ solves $\nabla \cdot (\sigma \nabla u) = 0$ with

$$\sigma \partial_{\nu} u|_{\Sigma} = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else}. \end{cases}$$
Challenges

Challenges in inverse coefficient problems such as EIT:

- **Uniqueness**
  - Is $\sigma$ uniquely determined from the NtD $\Lambda(\sigma)$?

- **Non-linearity and ill-posedness**
  - Reconstruction algorithms to determine $\sigma$ from $\Lambda(\sigma)$?
  - Local/global convergence results?

- **Realistic data**
  - What can we recover from real measurements? (Finite number of electrodes, realistic electrode models, . . .)
  - Measurement and modelling errors? Resolution?

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**In this talk:** A simple strategy (monotonicity + localized potentials) to attack these challenges.
Uniqueness
Uniqueness results

- Measurements on complete boundary (full data):

- Measurements on part of the boundary (local data):

- \( L^\infty \) coefficients are uniquely determined from full data in 2D.
- In all cases, piecew.-anal. coefficients are uniquely determined.
- Sophisticated research on uniqueness for \( \approx C^2 \)-coefficients (based on CGO-solutions for Schrödinger eq. \(-\Delta u + qu = 0\), \( q = \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}} \)).
Monotonicity

For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\sigma_0 \leq \sigma_1 \implies \Lambda(\sigma_0) \geq \Lambda(\sigma_1)$$

This follows from

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \geq \int_\Sigma g(\Lambda(\sigma_0) - \Lambda(\sigma_1)) g \geq \int_\Omega \frac{\sigma_0}{\sigma_1} (\sigma_1 - \sigma_0)|\nabla u_0|^2$$

for all solutions $u_0$ of

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_{\nu} u_0|_{\Sigma} = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$

(e.g., Kang/Seo/Sheen 1997, Ikehata 1998)

Can we prove uniqueness by controlling $|\nabla u_0|^2$?

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Localized potentials

**Theorem (H., 2008)**

Let $\sigma_0$ fulfill unique continuation principle (UCP),

$$\overline{D_1} \cap \overline{D_2} = \emptyset, \quad \text{and} \quad \Omega \setminus (\overline{D_1} \cup \overline{D_2}) \text{ be connected with } \Sigma.$$

Then there exist solutions $u_0^{(k)}$, $k \in \mathbb{N}$ with

$$\int_{D_1} |\nabla u_0^{(k)}|^2 \, dx \to \infty \quad \text{and} \quad \int_{D_2} |\nabla u_0^{(k)}|^2 \, dx \to 0.$$

$|\nabla u_0|^2$ small

$|\nabla u_0|^2$ large

$\Sigma$

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Proof 1/3

Virtual measurements:

\[ L_D : H^1(D)' \to L^2(\Sigma), \quad f \mapsto u|_{\Sigma}, \text{ with} \]
\[ \int_{\Omega} \sigma \nabla u \cdot \nabla v \, dx = \langle f, v|_{D} \rangle \quad \forall v \in H^1(D). \]

By (UCP): If \( \overline{D_1} \cap \overline{D_2} = \emptyset \) and \( \Omega \setminus (\overline{D_1} \cup \overline{D_2}) \) is connected with \( \Sigma \), then \( \mathcal{R}(L_{D_1}) \cap \mathcal{R}(L_{D_2}) = 0. \)

Sources on different domains yield different virtual measurements.

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Proof 2/3

Dual operator:

\[ L'_D : L^2_\diamond(\Sigma) \rightarrow H^1_\diamond(D), \quad g \mapsto u|_D, \] with

\[ \nabla \cdot (\sigma \nabla u) = 0, \quad \sigma \partial_\nu u|_\Sigma = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases} \]

Evaluating solutions on \( D \) is dual operation to virtual measurements.

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Proof 3/3

Functional analysis:
$X, Y_1, Y_2$ reflexive Banach spaces, $L_1 \in \mathcal{L}(Y_1, X)$, $L_2 \in \mathcal{L}(Y_1, X)$.

$$\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2) \iff \|L'_1 x\| \lesssim \|L'_2 x\| \ \forall x \in X'.$$

Here: $\mathcal{R}(L_{D_1}) \not\subseteq \mathcal{R}(L_{D_2}) \implies \|u_0|_{D_1}\|_{H^1_\diamond} \not\lesssim \|u_0|_{D_2}\|_{H^1_\diamond}$.

If two sources do not generate the same data, then the respective evaluations are not bounded by each other.

Note: $H^1_\diamond(D)'$-source $\iff H^1_\diamond(D)$-evaluation.
Consequences

- Back to Calderón: Let $\Lambda(\sigma_0) = \Lambda(\sigma_1)$, $\sigma_0$ fulfills (UCP).
- By monotonicity,

\[
\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \, dx \geq 0 \geq \int_\Omega \frac{\sigma_0}{\sigma_1} (\sigma_1 - \sigma_0)|\nabla u_0|^2 \, dx \quad \forall u_0
\]

- Assume: $\exists$ neighbourhood $U$ of $\Sigma$ where $\sigma_1 \geq \sigma_0$ but $\sigma_1 \neq \sigma_0$

$\Rightarrow$ Potential with localized energy in $U$ contradicts monotonicity.

Higher conductivity reachable by the bndry cannot be balanced out.

Corollary (Druskin 1982±85, Kohn/Vogelius, 1984±85)
Calderón problem is uniquely solvable for piecw.-anal. conductivities.

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Two coefficients

Can we recover two coefficients $a(x), c(x) \in L^\infty_+(\Omega)$ in

$$- \nabla \cdot (a \nabla u) + cu = 0 \quad \text{in } \Omega$$  \hspace{1cm} (1)

from the NtD (with partial data)

$$\Lambda(a, c) : L^2(\Sigma) \to L^2(\Sigma), \quad g \mapsto u|\Sigma,$$

where $u$ solves (1) with

$$\sigma \partial_\nu u|\Sigma = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else}. \end{cases}$$

Application: Diffuse optical tomography (DOT).

Quasilinear case $a(u), c(x)$: Egger, Pietschmann, Schlottbom 2013

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Monotonicity

\[
\int_{\Omega} \left( (a_2 - a_1)|\nabla u_1|^2 + (c_2 - c_1)|u_1|^2 \right) \, dx \\
\geq \int_{\Sigma} g \left( \Lambda(a_1, c_1) - \Lambda(a_2, c_2) \right) g \, ds \\
\geq \int_{\Omega} \left( (a_2 - a_1)|\nabla u_2|^2 + (c_2 - c_1)|u_2|^2 \right) \, dx,
\]

Method of localized potentials:

▷ Again, sources on different regions produce different data.
▷ \((H^1)'\)-sources produce different data than \(L^2\)-sources

\[\Rightarrow \|u\|_{H^1(D)} \not\lesssim \|u\|_{L^2(D)} \cdot\]

We can control \(|\nabla u_1|^2\) and \(|u_1|^2\) separately.

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Uniqueness

Theorem (H., 2009)

Let

- $a_1, a_2 \in L^\infty_+(\Omega)$ piecewise constant,
- $c_1, c_2 \in L^\infty_+(\Omega)$ piecewise analytic.

Then

$$\Lambda(a_1, c_1) = \Lambda(a_2, c_2) \iff a_1 = a_2, \quad c_1 = c_2.$$

Note that $v := \sqrt{au}$ transforms $-\nabla \cdot (a \nabla u) + cu = 0$ into

$$-\Delta v + \eta v = 0, \quad \eta := \frac{\Delta \sqrt{a}}{\sqrt{a}} + \frac{c}{a}$$

(when the coefficients are smooth).
Uniqueness

Theorem (H., 2012)
Let $a_1, a_2, c_1, c_2 \in L^\infty_+(\Omega)$ be piecew. analytic. Then $\Lambda(a_1, c_1) = \Lambda(a_2, c_2)$ if and only if

(a) $a_1|\Sigma = a_2|\Sigma$, $\partial_\nu a_1|\Sigma = \partial_\nu a_2|\Sigma$ on $\Sigma$, 
(b) $\frac{\partial_\nu a_1}{a_1}|_{\partial B \setminus \overline{\Sigma}} = \frac{\partial_\nu a_2}{a_2}|_{\partial B \setminus \overline{\Sigma}}$ on $\partial \Omega \setminus \Sigma$, 
(c) $\eta_1 = \eta_2$ in smooth regions, 
(d) $\frac{a_1^+|\Gamma}{a_1^-|\Gamma} = \frac{a_2^+|\Gamma}{a_2^-|\Gamma}$, $\frac{[\partial_\nu a_2]|\Gamma}{a_2^-|\Gamma} = \frac{[\partial_\nu a_1]|\Gamma}{a_1^-|\Gamma}$ on inner boundaries $\Gamma$.

NtD $\Lambda(a, c)$ determines $\eta = \frac{\Delta \sqrt{a}}{\sqrt{a}} + \frac{c}{a}$ and the jumps of $a$ and $\nabla a$. 

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Non-linearity
Non-linearity

Back to the non-linear forward operator of EIT

\[ \Lambda : \sigma \mapsto \Lambda(\sigma), \quad L^\infty_{+}(\Omega) \to \mathcal{L}(L^2_{\circ}(\Sigma)) \]

Generic approach for inverting \( \Lambda \): **Linearization**

\[ \Lambda(\sigma) - \Lambda(\sigma_0) \approx \Lambda'(\sigma_0)(\sigma - \sigma_0) \]

\( \sigma_0 \): known reference conductivity / initial guess / \ldots

\( \Lambda'(\sigma_0) \): Fréchet-Derivative / sensitivity matrix.

\[ \Lambda'(\sigma_0) : L^\infty_{+}(\Omega) \to \mathcal{L}(L^2_{\circ}(\Sigma)). \]

\( \leadsto \) Solve linearized equation for difference \( \sigma - \sigma_0 \).

**Often:** \( \text{supp}(\sigma - \sigma_0) \subset \Omega \) ("shape" / "inclusion")

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Linearization

Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda' (\sigma_0) \kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- Multiple possibilities to measure residual norm and to regularize.
- No rigorous theory for single linearization step.
- Almost no theory for Newton iteration:
  - Dobson (1992): (Local) convergence for regularized EIT equation.
  - No (local) convergence theory for non-discretized case!
    Non-linearity condition (Scherzer / tangential cone cond.) still open problem
- D-bar method: convergent 2D-implementation for $\sigma \in C^2$ and full bndry data (Knudsen, Lassas, Mueller, Siltanen 2008)
Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- **Seemingly**, no rigorous results possible for single lineariz. step.
- **Seemingly**, only justifiable for small $\sigma - \sigma_0$ (local results).

**Here:** Rigorous and global(!) result about the linearization error.
Linearization and shape reconstruction

Theorem (H./Seo 2010)

Let $\kappa$, $\sigma$, $\sigma_0$ piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

$$\text{supp}_\Sigma \kappa = \text{supp}_\Sigma (\sigma - \sigma_0)$$

$\text{supp}_\Sigma$: outer support ( = support, if support is compact and has conn. complement)

- Solution of lin. equation yields correct (outer) shape.
- No assumptions on $\sigma - \sigma_0$!
- Linearization error does not lead to shape errors.

Taking the (wrong) reference current paths for reconstruction still yields the correct shape information!
Proof

▷ Linearization: \( \Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \)

▷ Monotonicity: For all ”reference solutions“ \( u_0 \):

\[
\int_{\Omega} (\sigma - \sigma_0)|\nabla u_0|^2 \, dx \\
\geq \int_{\Sigma} g \left( \Lambda(\sigma_0) - \Lambda(\sigma) \right) g \\
\geq \int_{\Omega} \frac{\sigma_0}{\sigma} (\sigma - \sigma_0)|\nabla u_0|^2 \, dx.
\]

\[
= \int_{\Sigma} g \left( \Lambda'(\sigma_0)\kappa \right) g = \int_{\Omega} \kappa|\nabla u_0|^2 \, dx
\]

▷ Use localized potentials to control \( |\nabla u_0|^2 \)

\[\supp_{\Sigma}\kappa = \supp_{\Sigma}(\sigma - \sigma_0)\]

In shape reconstruction problems we can avoid non-linearity.

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Reconstruction from realistic data
Monotonicity based imaging

- Monotonicity:
  \[ \tau \leq \sigma \implies \Lambda(\tau) \geq \Lambda(\sigma) \]

- Idea: Simulate \( \Lambda(\tau) \) for test cond. \( \tau \) and compare with \( \Lambda(\sigma) \).
  \( \text{(Tamburrino/Rubinacci 02, Lionheart, Soleimani, Ventre, …)} \)

- Inclusion detection: For \( \sigma = 1 + \chi_D \) with unknown \( D \), use \( \tau = 1 + \chi_B \), with small ball \( B \).
  \[ B \subseteq D \implies \tau \leq \sigma \implies \Lambda(\tau) \geq \Lambda(\sigma) \]

- Algorithm: Mark all balls \( B \) with \( \Lambda(1 + \chi_B) \geq \Lambda(\sigma) \)

- Result: upper bound of \( D \).

Only an upper bound? Converse monotonicity relation?

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Converse monotonicity relation


$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

$$B \subseteq D \iff \Lambda(1 + \chi_B) \geq \Lambda(\sigma).$$

$\Rightarrow$ Monotonicity method detects exact shape.

For faster implementation:

$$B \subseteq D \iff \Lambda(1) + \frac{1}{2} \Lambda'(1) \chi_B \geq \Lambda(\sigma).$$

$\Rightarrow$ Linearized monotonicity method detects exact shape.

**Proof:** Monotonicity + localized potentials

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General case


Let \( \sigma \in L_1^\infty(\Omega) \) be piecewise analytic. The intersection of all *hole-free* \( C \subseteq \overline{\Omega} \) with

\[
\exists \alpha > 1 : \Lambda(1 + \alpha \chi_C) \leq \Lambda(\sigma) \leq \Lambda(1 - \chi_C/\alpha)
\]

is identical to the (outer) support of \( \sigma - 1 \).

- Result also holds with linearized condition

\[
\exists \alpha > 1 : \Lambda(1) + \alpha \Lambda'(1) \chi_C \leq \Lambda(\sigma) \leq \Lambda(1) - \alpha \Lambda'(1) \chi_C.
\]

- Result covers indefinite case,
  e.g., \( \sigma = 1 + \chi_{D_1} - \frac{1}{2} \chi_{D_2} \)

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Monotonicity based shape reconstruction

Monotonicity based reconstruction
- is intuitive, yet rigorous
- is stable (no infinity or range tests)
- works for pcw. anal. conductivities (no definiteness conditions)
- requires only the reference solution

Approach is closely related to (and heavily inspired by)
- Factorization Method of Kirsch and Hanke
  \((in\ EIT: \text{Brühl, Hakula, H.}, \text{Hyvönen, Lechleiter, Nachman, Päivärinta, Pursiainen, Schappel, Schmitt, Seo, Teirilä, Woo, \ldots })\)
- Ikehata’s Enclosure Method and probing with Sylvester-Uhlmann-CGOs \((\text{Ide, Isozaki, Nakata, Siltanen, Wang, \ldots })\)
- Classic inclusion detection results \((\text{Friedmann, Isakov, \ldots })\)
Realistic data & Uncertainties

- Finite number of electrodes, CEM, noisy data $\Lambda^\delta(\sigma)$
- Unknown background, e.g., $1 - \epsilon \leq \sigma_0(x) \leq 1 + \epsilon$
- Anomaly with some minimal contrast to background, e.g.,
  $$\sigma(x) = \sigma_0(x) + \kappa(x) \chi_D, \quad \kappa(x) \geq 1$$
- Can we rigorously guarantee to find inclusion $D$?

H./Ullrich: Monotonicity-based
Rigorous Resolution Guarantee

- If $D = \emptyset$, methods return $\emptyset$.
- If $D \supset \omega_i$ then it is detected.

(Here: 32 electrodes, $\epsilon = 1\%$, $\delta = 1.4\%$)
Conclusions

Using monotonicity and localized potentials we showed that

- Uniqueness results for piecewise smooth parameters may significantly differ from that for globally smooth ones.
- In shape reconstruction problems we can avoid non-linearity.
- Resolution guarantees for locating anomalies in unknown backgrounds with realistic finite precision data are possible.

Major limitations / open problems for our approach

- Piecewise analyticity required to prevent infinite oscillations.
- Voltage has to be measured on current-driven electrodes.