Recent progress on the factorization method for electrical impedance tomography

Bastian von Harrach

harrach@math.uni-stuttgart.de

Chair of Optimization and Inverse Problems, University of Stuttgart, Germany

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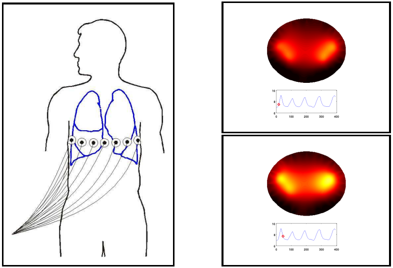
Electrical impedance tomography (EIT)



- Apply electric currents on subject's boundary
- Measure necessary voltages

⇝ Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)

Mathematical Model

Forward operator of EIT:

$$\Lambda : \ \sigma \mapsto \Lambda(\sigma), \quad \text{"conductivity"} \mapsto \text{"measurements"}$$

- Conductivity: $\sigma \in L^\infty_+(\Omega)$
- Continuum model: $\Lambda(\sigma)$: Neumann-Dirichlet-operator

$$\Lambda(\sigma) : \ g \mapsto u|_{\partial\Omega}, \quad \text{"applied current"} \mapsto \text{"measured voltage"}$$

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega, \quad \sigma \partial_\nu u|_{\partial\Omega} = g \quad \text{on } \partial\Omega.$$

- Linear elliptic PDE theory:

$$\Lambda(\sigma) : L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega) \text{ linear, compact, self-adjoint}$$

Inverse problem

Non-linear forward operator of EIT

$$\Lambda: \ \sigma \mapsto \Lambda(\sigma), \quad L^\infty_+(\Omega) \to \mathcal{L}(L^2_\diamond(\partial\Omega))$$

Inverse problem of EIT:

$$\Lambda(\sigma) \mapsto \sigma?$$

Mathematical challenges:

- Uniqueness ("Calderón problem"): Is Λ injective?
- Ill-posedness: Convergent numerical methods to reconstruct σ?
- Non-linearity: Non-linearity conditions for convergence results? Global vs. local convergence?

Shape detection

In practice:

- large jumps in conductivity
- large interest in detecting shapes / inclusions / anomalies

Inclusion/shape detection problem:

$$\Lambda(\sigma) \mapsto \mathrm{supp}(\sigma - \sigma_0)?, \quad \sigma_0\text{: reference conductivity.}$$

Advantages:

- Still contains the relevant information for most applications
- Simpler problem, more a-priori information
- Less affected by non-linearity *(**H.**/Seo 2010)*

Factorization method

Factorization method *(Inverse Scattering: Kirsch 1998, EIT: Hanke/Brühl 1999)*

$$z \in \operatorname{supp}(\sigma - \sigma_0) \quad \Longleftrightarrow \quad \Phi_z \in \mathcal{R}(|\Lambda(\sigma) - \Lambda(\sigma_0)|^{1/2}).$$

Φ_z*: dipole function with singularity in point z (and arbitrary direction)*

Progress on FM for EIT since 1998/99:

(Brühl, Hakula, Hanke, H., Hyvönen, Kirsch, Lechleiter, Nachman, Päivärinta, Pursiainen, Schappel, Schmitt, Seo, Teirilä, Woo)

- realistic electrode models, real data (not in this talk)
- simplified proofs, weakened assumptions, ...

In this talk: Formulation and proof of FM *(for continuous data in EIT)* from *(my personal)* today's standpoint

*(**H.**, to appear in Computational and Mathematical Methods in Medicine)*

Virtual measurement operators

Let $D \subseteq \Omega$ be open and $\overline{D} \subseteq \Omega$ have connected complement.

$$L_D: \; F \mapsto u|_{\partial\Omega}, \quad \text{"source term on } D\text{"} \mapsto \text{"measured voltage"}$$
$$\Delta u = \nabla \cdot F \quad \text{in } \Omega, \quad \partial_\nu u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega.$$

Properties of $L_D: \; L^2(D)^n \to L^2(\partial\Omega)$:

- For all $z \notin \partial D$ and associated dipole functions Φ_z:
$$z \in D \quad \Longleftrightarrow \quad \Phi_z \in \mathcal{R}(L_D).$$
- Adjoint $L_D^*: \; L^2(\partial\Omega) \to L^2(D)^n$:
$$L_D^*: \; g \mapsto \nabla u|_D, \quad \text{"current"} \mapsto \text{"(hom.) solution on } D\text{"}$$
$$\Delta u = 0 \quad \text{in } \Omega, \quad \partial_\nu u|_{\partial\Omega} = g \quad \text{on } \partial\Omega.$$

FM for simple case

Theorem

Let $\sigma = 1 + \chi_D$, $D \subseteq \Omega$ open, $\overline{D} \subseteq \Omega$ have connected complement.
For all $z \notin \partial D$ and associated dipole functions Φ_z

$$z \in D \quad \Longleftrightarrow \quad \Phi_z \in \mathcal{R}(|\Lambda(\sigma) - \Lambda(1)|^{1/2}).$$

Proof (traditional).

I. Introduce virtual measurement operators L_D
II. Prove Factorization $\Lambda(1) - \Lambda(\sigma) = LFL^*$
III. Study properties of F to show that $\mathcal{R}(|\Lambda(\sigma) - \Lambda(1)|^{1/2}) = \mathcal{R}(L)$

Here: replace II.+III. by monotony and range inclusions

Monotony and range inclusions

- Let $\sigma_1, \sigma_0 \in L^\infty_+(\Omega)$. Then, for all $g \in L^2_\diamond(\partial\Omega)$,

$$\int_\Omega (\sigma_0 - \sigma_1)|\nabla u_0|^2 \,\mathrm{d}x \leq \int_{\partial\Omega} g\left(\Lambda(\sigma_1) - \Lambda(\sigma_0)\right) g \,\mathrm{d}x$$
$$\leq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_0 - \sigma_1)|\nabla u_0|^2 \,\mathrm{d}x.$$

(Kang/Seo/Sheen 1997, Ikehata 1998)

- $A, B : \; H_1 \to H_2$ bnd. linear operators between Hilbert spaces

$$\|Ax\| \leq C\,\|Bx\| \;\forall x \quad \Longrightarrow \quad \mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$$

(Corollary of the Bourbaki's "14th important property of Banach spaces")

FM for simple case

Theorem

Let $\sigma = 1 + \chi_D$, $D \subseteq \Omega$ open, $\overline{D} \subseteq \Omega$ have connected complement.
For all $z \notin \partial D$ and associated dipole functions Φ_z

$$z \in D \quad \Longleftrightarrow \quad \Phi_z \in \mathcal{R}(|\Lambda(\sigma) - \Lambda(1)|^{1/2}).$$

Proof.

- Monotony:

$$\underbrace{\frac{1}{2}\int_D |\nabla u_0|^2 \,\mathrm{d}x}_{=\|L_D^* g\|^2} \leq \int_{\partial\Omega} g\,(\Lambda(1) - \Lambda(\sigma))\,g \,\mathrm{d}x \leq \underbrace{\int_D |\nabla u_0|^2 \,\mathrm{d}x}_{=\|L_D^* g\|^2}$$

$\rightsquigarrow$ $$\frac{1}{2}\|L_D^* g\|^2 \leq \|\,(\Lambda(1) - \Lambda(\sigma))^{1/2}\,g\|^2 \leq \|L_D^* g\|^2$$

$\rightsquigarrow$ $$\mathcal{R}(L_D) = \mathcal{R}\left(|\Lambda(1) - \Lambda(\sigma)|^{1/2}\right)$$

Advantages

Advantages of this factorization-free aproach:

- Monotony estimates simpler than studying "middle-operator" F
- Upper and lower range bound can be treated separately

Two applications where this helps:

- Dealing with a-priori separated, indefinite inclusions
  (originally treated by Grinberg/Kirsch 2004, Schmitt 2009)
- Dealing with non-connected complements, less regular conductivities
  (originally treated by **H.***/Hyvönen 2007)*

FM for indefinite case

Indefinite inclusions: $\sigma = 1 + \chi_{D^+} - 1/2\chi_{D^-}$

Open question: $z \in D^+ \cup D^- \quad \Longleftrightarrow \quad \Phi_z \in \mathcal{R}(|\Lambda(\sigma) - \Lambda(1)|^{1/2})$?

Monotony:

$$\frac{1}{2}\int_{D^-} |\nabla u_0|^2 \,\mathrm{d}x - \int_{D^+} |\nabla u_0|^2 \,\mathrm{d}x \leq \int_{\partial\Omega} g\,(\Lambda(\sigma) - \Lambda(1))\,g \,\mathrm{d}x$$
$$\leq \int_{D^-} |\nabla u_0|^2 \,\mathrm{d}x - \frac{1}{2}\int_{D^+} |\nabla u_0|^2 \,\mathrm{d}x$$

How to identify a-priori separated inclusions:

- Adding $\int_E |\nabla u_0|^2 \,\mathrm{d}x = \|L_E^* g\|^2$ with $E \supseteq D^+$ excludes $D+$

$\rightsquigarrow$ $\mathcal{R}(L_{E\cup D^-}) = \mathcal{R}(\Lambda(\sigma) - \Lambda(1) + 2L_E L_E^*)$

- D^- can be reconstructed after excluding $E \supset D^+$.

More general conductivities

For measurable $\kappa : \ \Omega \to \mathbb{R}$ we define *(slightly simplified)*

- the support $\mathrm{supp}\kappa$:
  complement of all open U with $\kappa|_U = 0$
- the outer support $\mathrm{out}_{\partial\Omega}\mathrm{supp}\kappa$:
  complement of all open U connected to $\partial\Omega$ with $\kappa|_U = 0$
- the inner support $\mathrm{innsupp}\kappa$:
  union of all open U with $\inf \kappa|_U > 0$

FM for general definite case

Theorem

Let

- $\sigma_0 \in L^\infty_+(\Omega)$ pcw. anal, $\sigma \in L^\infty_+(\Omega)$, either $\sigma \geq \sigma_0$, or $\sigma \leq \sigma_0$,
- $z \notin \partial D$ have a neighborhood in which σ_0 is analytic.

Then

$$z \in \mathrm{innsupp}(\sigma - \sigma_0) \Longrightarrow \Phi_z \in \mathcal{R}(|\Lambda(\sigma) - \Lambda(\sigma_0)|^{1/2}),$$
$$z \in \mathrm{out}_{\partial\Omega}\mathrm{supp}(\sigma - \sigma_0) \Longleftarrow \Phi_z \in \mathcal{R}(|\Lambda(\sigma) - \Lambda(\sigma_0)|^{1/2}).$$

Proof. Monotony + Properties of L_D for general $D \subseteq \Omega$.

FM detects inclusions up to difference between inner and outer supp.

FM for general indefinite case

Theorem

Let

- $\sigma_0 \in L^\infty_+(\Omega)$ pcw. anal, $\sigma \in L^\infty_+(\Omega)$,
- $E \subseteq \Omega$ measurable, $\sigma_0 \geq \sigma$ on $\Omega \setminus E$, $\alpha > \|\sigma_0 - \sigma\|_{L^\infty}$,
- $z \notin \partial D$ have a neighbourhood in which σ_0 is analytic.

Then

$$z \in \operatorname{innsupp}(\sigma - \sigma_0) \cup E \Longrightarrow \Phi_z \in \mathcal{R}(|\Lambda(\sigma) - \Lambda(\sigma_0) + \alpha L_E L_E^*|^{\frac{1}{2}}),$$

$$z \in \operatorname{out}_{\partial\Omega}(\operatorname{supp}(\sigma - \sigma_0) \cup E) \Longleftarrow \Phi_z \in \mathcal{R}(|\Lambda(\sigma) - \Lambda(\sigma_0) + \alpha L_E L_E^*|^{\frac{1}{2}}).$$

Analogous result holds for $\sigma_0 \leq \sigma$ on $\Omega \setminus E$

Indefinite inclusions can be detected by excluding domains.

Remarks and conclusions

Generalizations: Everything stays valid for

- measurements taken on open subset of boundary $\partial\Omega$,
- $\sigma_0 \in L^\infty_+$ if UCP and existence of dipoles is guaranteed.

Conclusions:

- FM detects inclusions up to diff. between inner and outer supp.
- FM requires definiteness condition on whole domain or after excluding an a-priori known part.
- Monotony and range inclusions yield simpler *factorization-free* proofs that seem easier to generalize.

Open problem:

- Monotony for inverse scattering? Up to fin.-dim. spaces?