Inverse problems for the transient eddy current equation

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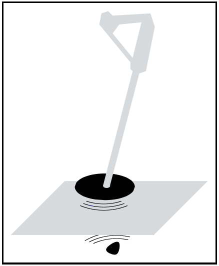
Motivation

Inverse Electromagnetics & Eddy currents

Inverse Electromagnetics

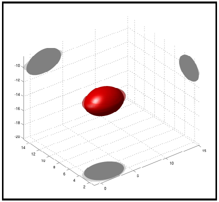
Inverse Electromagnetics:

- Generate EM field
  (drive excitation current through coil)
- Measure EM field
  (induced voltages in meas. coil)
- Gain information from measurements



Applications:

- Metal detection *(buried conductor)*
- Non-destructive testing
  (crack in metal, metal in concrete)
- ...



Maxwell's equations

Classical Electromagnetics: Maxwell's equations

$$\operatorname{curl} H = \epsilon \partial_t E + \sigma E + J \qquad \text{in } \mathbb{R}^3 \times]0, T[$$

$$\operatorname{curl} E = -\mu \partial_t H \qquad \text{in } \mathbb{R}^3 \times]0, T[$$

$E(x,t)$:	Electric field	$\epsilon(x)$:	Permittivity
$H(x,t)$:	Magnetic field	$\mu(x)$:	Permeability
$J(x,t)$:	Excitation current	$\sigma(x)$:	Conductivity

Knowing J, σ, μ, ϵ + init. cond. determines E and H.

Eddy currents

Maxwell's equations

$$\operatorname{curl} H = \epsilon \partial_t E + \sigma E + J \qquad \text{in } \mathbb{R}^3 \times]0,T[$$
$$\operatorname{curl} E = -\mu \partial_t H \qquad \text{in } \mathbb{R}^3 \times]0,T[$$

Eddy current approximation: Neglect displacement currents $\epsilon \partial_t E$

- Justified for low-frequency excitations
(Alonso 1999, Ammari/Buffa/Nédélec 2000)

$$\rightsquigarrow \qquad \partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times]0,T[$$

Where's Eddy?

- $\sigma = 0$: (Quasi-)Magnetostatics

$$\mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\,E\right) = -\partial_t J$$

Excitation $\partial_t J$ instantly generates magn. field $\frac{1}{\mu}\,\mathrm{curl}\,E = -\partial_t H$.

- $\sigma \neq 0$: Eddy currents

$$\partial_t(\sigma E) + \mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\,E\right) = -\partial_t J$$

$\partial_t J$ generates changing magn. field + currents inside conductor

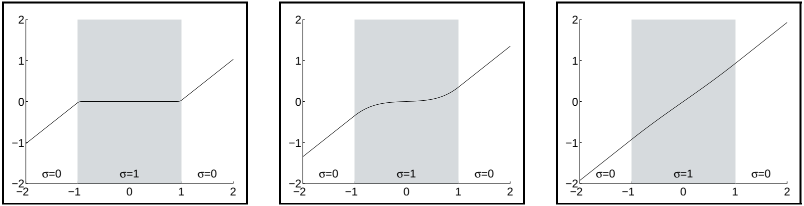
Induced currents oppose what created them (Lenz law)

Parabolic-elliptic equations

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times]0, T[$$

- parabolic inside conductor $\Omega = \operatorname{supp}(\sigma)$
- elliptic outside conductor

Scalar example: $(\sigma u)_t = u_{xx},\ u(\cdot, 0) = 0,\ u_x(-2, \cdot) = u_x(2, \cdot) = 1.$



The direct problem

Unified variational formulation
for the parabolic-elliptic eddy current problem

Standard approach

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times]0,T[$$

Standard approach: *Decouple elliptic and parabolic part (e.g. Bossavit 1999, Acevedo/Meddahi/Rodriguez 2009)*

Find $(E_{\mathbb{R}^3\setminus\Omega}, E_\Omega) \in H_{\mathbb{R}^3\setminus\Omega} \times H_\Omega$ s.t.

- E_Ω solves parabolic equation + init. cond.
- $E_{\mathbb{R}^3\setminus\Omega}$ solves elliptic equation
- interface conditions are satisfied

Problem: Theory (solution spaces, coercivity constants, etc.) depends on $\Omega = \operatorname{supp}\sigma$ and on lower bounds of $\sigma|_\Omega$.

Unified approach?

Parabolic-elliptic eddy current equation

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -\partial_t J \quad \text{in } \mathbb{R}^3 \times]0, T[$$

Inverse problem: Find σ (or $\Omega = \operatorname{supp} \sigma$) from measurements of E

- requires unified solution theory

Test for unified theory: Can we linearize E w.r.t. σ?

How does the solution of an elliptic equation change if the equation becomes a little bit parabolic?

(For scalar analogue: Frühauf/**H.**/Scherzer 2007, **H.** 2007)

Rigorous formulation

Rigorous formulation: Let $\mu \in L^\infty_+$, $\sigma \in L^\infty$, $\sigma \geq 0$,

$$J_t \in L^2(0,T,W(\mathrm{curl})') \qquad \text{with } \operatorname{div} J_t = 0$$
$$E_0 \in L^2(\mathbb{R}^3)^3 \qquad \text{with } \operatorname{div}(\sigma E_0) = 0.$$

For $E \in L^2(0,T,W(\mathrm{curl}))$ the eddy current equations

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -J_t \qquad \text{in } \mathbb{R}^3\times]0,T[$$
$$\sqrt{\sigma}E(x,0) = \sqrt{\sigma(x)}E_0(x) \quad \text{in } \mathbb{R}^3$$

are well-defined and (if solvable) uniquely determine $\operatorname{curl} E$, $\sqrt{\sigma}E$.

Natural variational formulation

Natural unified variational formulation ($E_0 = 0$ for simplicity):

Find $E \in L^2(0, T, W(\text{curl}))$ that solves

$$\int_0^T \int_{\mathbb{R}^3} \left(\sigma E \cdot \partial_t \Phi - \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.$$

for all smooth Φ with $\Phi(\cdot, T) = 0$.

- equivalent to eddy current equation
- not coercive, does not yield existence results

Gauged formulation

Gauged unified variational formulation ($E_0 = 0$ for simplicity)

Find divergence-free $E \in L^2(0, T, W^1(\mathbb{R}^3))$ that solves

$$\int_0^T \int_{\mathbb{R}^3} \left(\sigma E \cdot \partial_t \Phi - \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.$$

for all smooth divergence-free Φ with $\Phi(\cdot, T) = 0$.

- coercive, yields existence and continuity results
- not equivalent to eddy current equation ($\sigma \neq$ const. $\rightsquigarrow \operatorname{div} \sigma E \neq \sigma \operatorname{div} E$)
- does not determine true solution up to gauge (curl-free) field

Coercive unified formulation

How to obtain coercive + equivalent unified formulation?

- Ansatz $E = A + \nabla\varphi$ with divergence-free A.
  (almost the standard (A, φ)-formulation with Coulomb gauge)
- Consider $\nabla\varphi = \nabla\varphi_A$ as function of A by solving
  $$\operatorname{div}\sigma\nabla\varphi_A = -\operatorname{div}\sigma A.$$
  ($\rightsquigarrow \operatorname{div}\sigma E = 0$).
- Obtain coercive formulation for A
  (Lions-Lax-Milgram Theorem $\rightsquigarrow$ Solvability and continuity results)
- A determines E
  (more precisely: $\operatorname{curl} E$ and $\sqrt{\sigma}E$)

Unified variational formulation

Unified variational formulation *(Arnold/**H.**, SIAP, 2012)*

Find divergence-free $A \in L^2(0, T, W^1(\mathbb{R}^3))$ that solves

$$\int_0^T \int_{\mathbb{R}^3} \left(\sigma(A + \nabla \varphi_A) \cdot \partial_t \Phi - \frac{1}{\mu} \operatorname{curl} A \cdot \operatorname{curl} \Phi \right) = \int_0^T \int_{\mathbb{R}^3} J_t \cdot \Phi.$$

for all smooth divergence-free Φ with $\Phi(\cdot, T) = 0$.

- coercive, uniquely solvable
- $E := A + \nabla \varphi_A$ is one solution of the eddy current equation

$\rightsquigarrow$ $\operatorname{curl} E$, $\sqrt{\sigma} E$ depend continuously on J_t (uniformly w.r.t. σ) (for all solutions of the eddy current equation)

Asymptotic results

Unified variational formulation

- allows to rigorously linearize E w.r.t. σ around $\sigma_0 = 0$
  (elliptic equation becoming a little bit parabolic in some region...)
- easily extends from $\mathbb{R}^3$ to bounded domain O
  (O simply conn. with Lipschitz-boundary, $\nu \wedge E|_{\partial O} = 0$)
- justifies parabolic regularization: If E_ϵ solves

$$\partial_t(\sigma_\epsilon E_\epsilon) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E_\epsilon\right) = -\partial_t J \quad \text{in } O\times]0,T[,$$

  with $\sigma_\epsilon(x) = \max\{\sigma(x), \epsilon\}$ then

$$\sigma_\epsilon E_\epsilon \to \sigma E, \quad \operatorname{curl} E_\epsilon \to \operatorname{curl} E$$

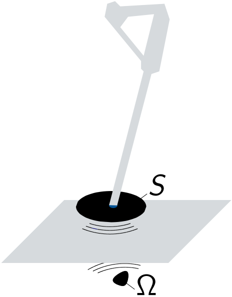
  *(Arnold/**H.**, submitted to proceedings of IPDO 2013)*

Open problems

- Theory requires some regularity of $\Omega = \operatorname{supp} \sigma$ and $\sigma \in L^\infty_+(\Omega)$ in order to determine φ from A.
- Solution theory for
$$\operatorname{div} \sigma \nabla \varphi = -\operatorname{div} \sigma A$$
for general $\sigma \in L^\infty$, $\sigma \geq 0$?
- Elliptic regularization of the variational formulation
(i.e., adding $\epsilon \int_0^T \int_{\mathbb{R}^3} A \cdot \Phi \, \mathrm{d}x$)
is justified, but relation to elliptic regularization of the PDE
$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu} \operatorname{curl} E\right) + \epsilon E = -\partial_t J \quad \text{in } \mathbb{R}^3 \times]0, T[,$$
is not clear.

The inverse problem

Setup



Detecting conductors:

- Apply surface currents J on S
  (divergence-free, no electrostatic effects)
- Measure electric field E on S
  (tangential component, up to grad. fields)
- Measurement operator

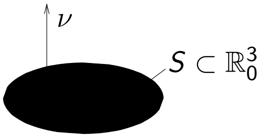
$$\Lambda_\sigma : J_t \mapsto \gamma_\tau E := (\nu \wedge E|_S) \wedge \nu$$

Locate $\Omega = \operatorname{supp} \sigma$ in

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -J_t \qquad \text{in } \mathbb{R}^3 \times]0, T[$$

(+ zero IC) from all possible surface currents and measured values.

Measurement operator



$$TL^2 := \{u \in L^2(S)^3 \mid u \cdot \nu = 0\}$$

$$TL^2_\diamond := \{u \in TL^2 \mid \int_S u \cdot \nabla\psi = 0 \quad \forall \text{ smooth } \psi\}$$

Measurement operator

$$\Lambda_\sigma : L^2(0,T,TL^2_\diamond) \to L^2(0,T,TL^{2\prime}_\diamond), \quad J_t \mapsto \gamma_\tau E,$$

where E solves eddy current eq. with $[\nu \times \operatorname{curl} E]_S = J_t$ on S.

Remark

$$TL^{2\prime}_\diamond \cong TL^2 / TL^{2\perp}_\diamond \;\rightsquigarrow\; E \text{ not unique, but } \Lambda_\sigma \text{ well-defined.}$$

Sampling methods

Non-iterative shape detection methods:

- Linear Sampling Method (Colton/Kirsch 1996)
  - characterizes subset of scatterer by range test
  - allows fast numerical implementation
- Factorization Method (Kirsch 1998)
  - characterizes scatterer by range test
  - yields uniqueness under definiteness assumptions
  - allows fast numerical implementation

- Beyond LSM/FM?

Sampling ingredients

Ingredients for LSM and FM:

- Reference measurements: $\Lambda := \Lambda_\sigma - \Lambda_0$,
  $\Lambda_0 : J_t \mapsto \gamma_\tau F, \quad F$ solves $\operatorname{curl}\operatorname{curl} F = -J_t \quad$ in $\mathbb{R}^3\times]0,T[$.
- Time-integration: Consider $I\Lambda$,
  with $I: \; E(\cdot,\cdot) \mapsto \int_0^T E(\cdot,t)\,\mathrm{d}t$
- Singular test functions

$$G_{z,d}(x) := \operatorname{curl} \frac{d}{4\pi|x-z|}, \qquad x \in \mathbb{R}^3 \setminus \{z\}$$

LSM and FM

*Arnold/***H.** *(submitted)*:
For every z below S, $z \notin \Omega$ and direction $d \in \mathbb{R}^3$.

Theorem (LSM)

$$\gamma_\tau G_{z,d} \in \mathcal{R}(I\Lambda) \quad \Rightarrow \quad z \in \Omega$$

Theorem (FM)
If, additionally, $\sup \mu|_\Omega < 1$ (diamagnetic scatterer)

$$\gamma_\tau G_{z,d} \in \mathcal{R}(I(\Lambda + \Lambda')^{1/2}) \quad \Leftrightarrow \quad z \in \Omega$$

Beyond LSM/FM?

- Beyond LSM/FM?: Monotony methods
- For EIT: Λ_σ NtD-operator for conductivity $\sigma = 1 + \chi_D$

  D = Union of all balls B where $\Lambda_{1+\chi_B} \leq \Lambda_\sigma$ (**H.**/Ullrich)

  (under the assumptions of the FM)
- stable test criterion (no infinity tests)
- allows fast numerical implementation
- allows extensions to indefinite cases

Conclusions

Inverse transient eddy current problems

- require unified parabolic-elliptic theory
- can be approached by sampling methods (LSM/FM)

Open problems

- Solution theory for
$$\operatorname{div} \sigma \nabla \varphi = -\operatorname{div} \sigma A$$
for general $\sigma \in L^\infty$, $\sigma \geq 0$?
- Monotony based methods beyond EIT?
Monotony for parabolic-elliptic problems?