Fast shape-reconstruction in electrical impedance tomography

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(joint work with Marcel Ullrich)

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Mathematical Model

Forward operator of EIT:

$$\Lambda: \ \sigma \mapsto \Lambda(\sigma), \quad \text{"conductivity"} \mapsto \text{"measurements"}$$

- Conductivity: $\sigma \in L^\infty_+(\Omega)$
- Continuum model: $\Lambda(\sigma)$: Neumann-Dirichlet-operator

$$\Lambda(\sigma): \ g \mapsto u|_{\partial\Omega}, \quad \text{"applied current"} \mapsto \text{"measured voltage"}$$

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega, \quad \sigma \partial_\nu u|_{\partial\Omega} = g \quad \text{on } \partial\Omega.$$

- Linear elliptic PDE theory:

$$\Lambda(\sigma): L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega) \text{ linear, compact, self-adjoint}$$

Inverse problem

Non-linear forward operator of EIT

$$\Lambda: \ \sigma \mapsto \Lambda(\sigma), \quad L^\infty_+(\Omega) \to \mathcal{L}(L^2_\diamond(\partial\Omega))$$

Inverse problem of EIT:

$$\Lambda(\sigma) \mapsto \sigma?$$

- Uniqueness ("Calderón problem"): Is Λ injective?
- Convergent numerical methods to reconstruct σ?

Reconstruction

Convergent numerical methods to reconstruct σ?

- Newton iteration: almost no theory

  Dobson (1992): (Local) convergence for regularized EIT equation.

  Lechleiter/Rieder(2008): (Local) convergence for discretized setting.

- D-bar method: convergent 2D-implementation for $\sigma \in C^2$

  Knudsen, Lassas, Mueller, Siltanen (2008)

In practice:

- large jumps in conductivity
- large interest in detecting shapes / inclusions / anomalies

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Inclusion/shape detection problem:

$$\Lambda(\sigma) \mapsto \mathrm{supp}(\sigma - \sigma_0)?, \quad \sigma_0\text{: reference conductivity.}$$

Shape detection

Promising approach: Factorization method *(Kirsch 1998)*

- *FM for EIT (1999–): Brühl, Hakula, Hanke, H., Hyvönen, Kirsch, Lechleiter, Nachman, Päivärinta, Pursiainen, Schappel, Schmitt, Seo, Teirilä*

Typical result:

$$z \notin \mathrm{supp}(\sigma - \sigma_0) \quad \text{iff} \quad \lim_{\alpha \to 0} I_\alpha(z) = \infty.$$

($I_\alpha(z)$: indicator function)

Unsolved problems since 1998:

- Convergent regularization strategies for "infinity test"?
- Theory needs definiteness assumption, e.g., $\sigma \geq \sigma_0$ everywhere

In this talk: A monotonicity based sampling method.

Monotony

$$\int_\Omega (\sigma_1 - \sigma_2)|\nabla u_1|^2 \,\mathrm{d}x \leq (g, (\Lambda(\sigma_2) - \Lambda(\sigma_1))g)$$

u_1 solution corresponding to σ_1 and boundary current g.

Simple consequence:

$$\sigma_1 \geq \sigma_2 \quad \Longrightarrow \quad \Lambda(\sigma_1) \leq \Lambda(\sigma_2)$$

Monotony based imaging

- True conductivity: $\sigma = 1 + \chi_D$, D: unknown inclusion

$\rightsquigarrow$ $\Lambda(\sigma)$: measured data

- Test conductivity: $1 + \chi_B$, B: small ball

$\rightsquigarrow$ $\Lambda(1 + \chi_B)$ can be simulated for different balls B

Monotony:

$$B \subseteq D \quad \Longrightarrow \quad 1 + \chi_B \leq 1 + \chi_D = \sigma \quad \Longrightarrow \quad \Lambda(1 + \chi_B) \geq \Lambda(\sigma)$$

Monotony based reconstruction algo. for EIT *(Tamburrino/Rubinacci 02)*

- For all B, calculate $\Lambda(1 + \chi_B)$ & test whether $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$

$\rightsquigarrow$ Result: upper bound of D.

Only an upper bound? Converse monotony relation?

Converse montony relation

Sample result *(H./Ullrich)*
$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

$$B \not\subseteq D \quad \Longrightarrow \quad \Lambda(1+\chi_B) \not\geq \Lambda(\sigma).$$

$\rightsquigarrow$ Monotony method detects exact shape.

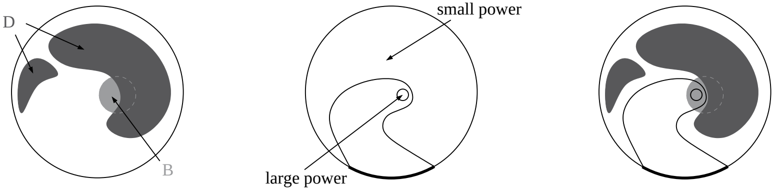
(Extensions possible for non-connected complement, inhomogeneous inclusions or background, continuous transitions between inclusion and background,...)

Converse montony relation

Proof $\quad (\sigma = 1 + \chi_D,\ \kappa = 1 + \chi_B)$

$$\int_\Omega (\kappa - \sigma) |\nabla u_\kappa|^2 \, \mathrm{d}x \leq (g, (\Lambda(\sigma) - \Lambda(\kappa)) g)$$

Apply localized potentials (H 2008) to control power term $|\nabla u_\kappa|^2$.



$$\rightsquigarrow \exists g : \ (g, (\Lambda(\sigma) - \Lambda(\kappa)) g) \geq 0 \quad \Longrightarrow \quad \Lambda(\sigma) \not\leq \Lambda(\kappa)$$

Fast implementation

- Testing $\Lambda(1+\chi_B) \geq \Lambda(\sigma)$ is expensive. One forw. prob. per B.

Theorem (**H.**/Seo, SIAM J. Math. Anal. 2010)

Let κ, σ, σ_0 piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

$$\mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0).$$

$\mathrm{supp}_{\partial\Omega}$: outer support (= support, if support is compact and has conn. complement)

$\rightsquigarrow$ Replacing $\Lambda(1+\chi_B)$ by its linear approx. should still recover the exact shape (linearization error does not affect the shape!)

Fast implementation

Sample result *(H./Ullrich, submitted)*
$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(1 + \chi_B) \geq \Lambda(\sigma)$$
$$\Longleftrightarrow \quad \Lambda(1) + \tfrac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

$\rightsquigarrow$ Fast, requires only homogeneous forward solution

- Comp. cost equivalent to linearized methods or FM

(Again, extensions possible for non-connected complement, inhomogeneous inclusions or background, continuous transitions between inclusion and background,...)

Indefinite inclusions

Sample result *(H./Ullrich, submitted)*
Indefinite inclusions: Let $\sigma = 1 + \chi_{D^+} - \frac{1}{2}\chi_{D^-}$ where D^+, D^- are disjoint and their union has conn. complement

$$D^+ \cup D^- \subseteq B \iff \Lambda(1+\chi_B) \geq \Lambda(\sigma) \geq \Lambda(1 - \tfrac{1}{2}\chi_B)$$
$$\iff \Lambda(1) + k\Lambda'(1)\chi_B \leq \Lambda(\sigma) \leq \Lambda(1) - \Lambda'(1)\chi_B.$$

- General (e.g., indefinite) cases can be treated by step-wise shrinking of larger test domains.

General result

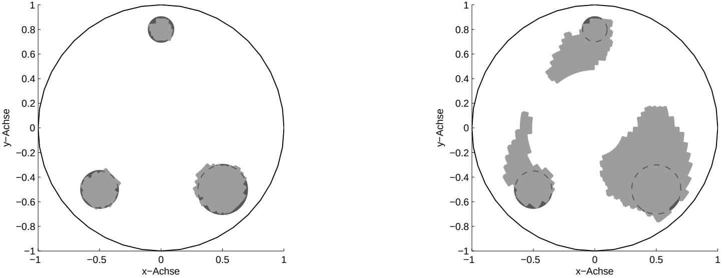
Theorem *(H./Ullrich, submitted)*
Let $\sigma \in L^\infty_+(\Omega)$ be piecewise analytic. Then the (outer) support of $\sigma - 1$ is the intersection of all sets $B \subseteq \Omega$ (with no holes) s.t.

$$\exists \alpha > 0: \quad \Lambda(1 + \alpha\chi_B) \leq \Lambda(\sigma) \leq \Lambda(1 - \tfrac{1}{\alpha}\chi_B).$$

or, equivalently, such that

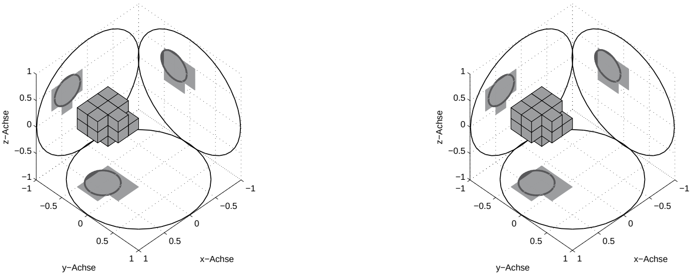
$$\exists \alpha > 0: \quad \Lambda(1) + \alpha\Lambda'(1)\chi_B \leq \Lambda(\sigma) \leq \Lambda(1) - \alpha\Lambda'(1)\chi_B.$$

Numerical results



Reconstructions with exact data and with 0.1% noise.

Numerical results



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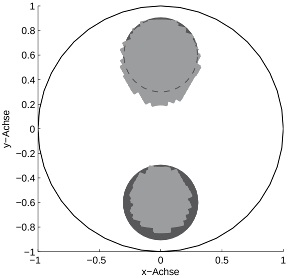
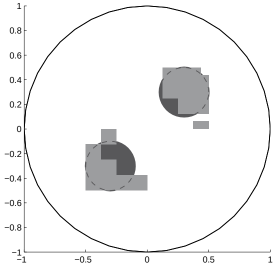
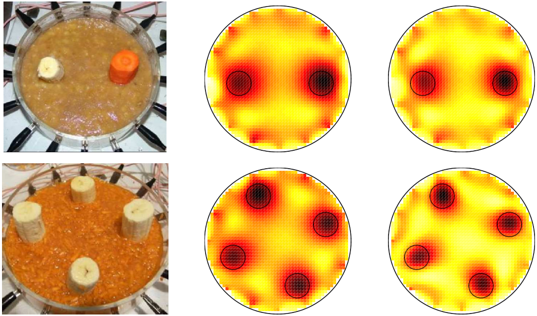




Reconstructions for smooth transitions between inclusion and background and for the indefinite case.

Outlook

Goal: Enhance linearized methods



Standard linearized method vs. heuristic combination with FM
for frequency-difference EIT without ref. measurements

(**H.**/Seo/Woo, IEEE Trans. Med. Imaging 2010)

More goals / collaborations?

- Monotony/linearization arguments recover the shape.
  Also the coefficient on the outer boundary?
  (Detect flaws/corrosions of steel in concrete?)

- Detect inclusions in unknown background?
  Use bounds on contrast and monotony?

- Sampling/Factorization Method origins from inverse scattering.
  Extend our results to inverse scattering?
  Positivity up to finite-dimensional subspaces?

- More applications: complex conductivities, diffuse optical tomography, eddy currents, ...