Fast shape-reconstruction in electrical impedance tomography

Bastian von Harrach
bastian.harrach@uni-wuerzburg.de

(joint work with Marcel Ullrich)

Institut für Mathematik - IX, Universität Würzburg

Chinese-German Symposium on Inverse Problems
Saarbrücken, Germany, October 07–11, 2012.
Mathematical Model

Forward operator of EIT:

\[ \Lambda : \sigma \mapsto \Lambda(\sigma), \quad \text{"conductivity" } \mapsto \text{"measurements"} \]

- Conductivity: \( \sigma \in L_+^{\infty}(\Omega) \)
- Continuum model: \( \Lambda(\sigma) \): Neumann-Dirichlet-operator
  \[
  \Lambda(\sigma) : g \mapsto u|_{\partial\Omega}, \quad \text{"applied current" } \mapsto \text{"measured voltage"}
  \]
  \[
  \nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega, \quad \sigma \partial_{\nu} u|_{\partial\Omega} = g \quad \text{on } \partial\Omega.
  \]
- Linear elliptic PDE theory:
  \[
  \Lambda(\sigma) : L^2_\circ(\partial\Omega) \to L^2_\circ(\partial\Omega) \quad \text{linear, compact, self-adjoint}
  \]

B. Harrach: Fast shape-reconstruction in EIT
Inverse problem

Non-linear forward operator of EIT

$$\Lambda : \sigma \mapsto \Lambda(\sigma), \quad L^\infty(\Omega) \to L(L^2(\partial\Omega))$$

Inverse problem of EIT:

$$\Lambda(\sigma) \mapsto \sigma?$$

- Uniqueness ("Calderón problem"): Is $\Lambda$ injective?
- Convergent numerical methods to reconstruct $\sigma$?
Reconstruction

Convergent numerical methods to reconstruct $\sigma$?

- Newton iteration: almost no theory
  - Dobson (1992): (Local) convergence for regularized EIT equation.

- D-bar method: convergent 2D-implementation for $\sigma \in C^2$

In practice:

- large jumps in conductivity
- large interest in detecting shapes / inclusions / anomalies

Inclusion/shape detection problem:

$$\Lambda(\sigma) \mapsto \text{supp}(\sigma - \sigma_0)?, \quad \sigma_0: \text{reference conductivity}.$$
Shape detection

Promising approach: Factorization method *(Kirsch 1998)*


Typical result:

\[
  z \notin \text{supp}(\sigma - \sigma_0) \quad \text{iff} \quad \lim_{\alpha \to 0} l_\alpha(z) = \infty.
\]

(*\(l_\alpha(z):\) indicator function*)

Unsolved problems since 1998:

- Convergent regularization strategies for ”infinity test”?
- Theory needs definiteness assumption, e.g., \(\sigma \geq \sigma_0\) everywhere

---

In this talk: A monotonicity based sampling method.

B. Harrach: Fast shape-reconstruction in EIT
Monotony

\[ \int_{\Omega} (\sigma_1 - \sigma_2) |\nabla u_1|^2 \, dx \leq (g, (\Lambda(\sigma_2) - \Lambda(\sigma_1))g) \]

\( u_1 \) solution corresponding to \( \sigma_1 \) and boundary current \( g \).

Simple consequence:

\( \sigma_1 \geq \sigma_2 \implies \Lambda(\sigma_1) \leq \Lambda(\sigma_2) \)
Monotony based imaging

- True conductivity: $\sigma = 1 + \chi_D$, $D$: unknown inclusion
- $\Lambda(\sigma)$: measured data
- Test conductivity: $1 + \chi_B$, $B$: small ball
- $\Lambda(1 + \chi_B)$ can be simulated for different balls $B$

Monotony:

$$B \subseteq D \implies 1 + \chi_B \leq 1 + \chi_D = \sigma \implies \Lambda(1 + \chi_B) \geq \Lambda(\sigma)$$

---

Monotony based reconstruction algo. for EIT \textit{(Tamburrino/Rubinacci 02)}

- For all $B$, calculate $\Lambda(1 + \chi_B)$ & test whether $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$
- Result: upper bound of $D$.

---

\textit{Only an upper bound? Converse monotony relation?}

B. Harrach: Fast shape-reconstruction in EIT
Sample result \((H./Ullrich)\)
\[ \Omega \setminus \overline{D} \text{ connected. } \sigma = 1 + \chi_D. \]

\[ B \not\subseteq D \implies \Lambda(1 + \chi_B) \not\geq \Lambda(\sigma). \]

\(~\implies\) Monotony method detects exact shape.

\((Extensions\ possible\ for\ non-connected\ complement,\ inhomogeneous\ inclusions\ or\ background,\ continuous\ transitions\ between\ inclusion\ and\ background,\ldots\)\)

B. Harrach: Fast shape-reconstruction in EIT
Converse monotony relation

**Proof** \((\sigma = 1 + \chi_D, \kappa = 1 + \chi_B)\)

\[
\int_{\Omega} (\kappa - \sigma)|\nabla u_\kappa|^2 \, dx \leq (g, (\Lambda(\sigma) - \Lambda(\kappa))g)
\]

Apply localized potentials (H 2008) to control power term \(|\nabla u_\kappa|^2\).

\[
\exists g : (g, (\Lambda(\sigma) - \Lambda(\kappa))g) \geq 0 \implies \Lambda(\sigma) \not\leq \Lambda(\kappa)
\]

B. Harrach: Fast shape-reconstruction in EIT
Fast implementation

- Testing $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$ is expensive. One forw. prob. per $B$.

---

**Theorem** ([H./Seo, SIAM J. Math. Anal. 2010])

Let $\kappa$, $\sigma$, $\sigma_0$ piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

$$\text{supp}_{\partial\Omega}\kappa = \text{supp}_{\partial\Omega}(\sigma - \sigma_0).$$

$\text{supp}_{\partial\Omega}$: outer support ( = support, if support is compact and has conn. complement)

$\rightsquigarrow$ Replacing $\Lambda(1 + \chi_B)$ by its linear approx. should still recover the exact shape (linearization error does not affect the shape!)
Fast implementation

Sample result *(H./Ullrich, submitted)*
\[ \Omega \setminus \overline{D} \text{ connected. } \sigma = 1 + \chi_D. \]

\[ B \subseteq D \iff \Lambda(1 + \chi_B) \geq \Lambda(\sigma) \]
\[ \iff \Lambda(1) + \frac{1}{2} \Lambda'(1) \chi_B \geq \Lambda(\sigma). \]

⇝ Fast, requires only homogeneous forward solution

➤ Comp. cost equivalent to linearized methods or FM

(Again, extensions possible for non-connected complement, inhomogeneous inclusions or background, continuous transitions between inclusion and background,...)
Sample result (H./Ullrich, submitted)

Indefinite inclusions: Let \( \sigma = 1 + \chi_{D^+} - \frac{1}{2} \chi_{D^-} \) where \( D^+, D^- \) are disjoint and their union has conn. complement.

\[
D^+ \cup D^- \subseteq B \quad \iff \quad \Lambda(1 + \chi_B) \geq \Lambda(\sigma) \geq \Lambda(1 - \frac{1}{2} \chi_B)
\]

\[
\iff \quad \Lambda(1) + k\Lambda'(1)\chi_B \leq \Lambda(\sigma) \leq \Lambda(1) - \Lambda'(1)\chi_B.
\]

- General (e.g., indefinite) cases can be treated by step-wise shrinking of larger test domains.
**Theorem** *(H./Ullrich, submitted)*

Let $\sigma \in L^\infty_+(\Omega)$ be piecewise analytic. Then the (outer) support of $\sigma - 1$ is the intersection of all sets $B \subseteq \Omega$ (with no holes) s.t.

$$
\exists \alpha > 0 : \quad \Lambda(1 + \alpha \chi_B) \leq \Lambda(\sigma) \leq \Lambda(1 - \frac{1}{\alpha} \chi_B).
$$

or, equivalently, such that

$$
\exists \alpha > 0 : \quad \Lambda(1) + \alpha \Lambda'(1) \chi_B \leq \Lambda(\sigma) \leq \Lambda(1) - \alpha \Lambda'(1) \chi_B.
$$
Reconstructions with exact data and with 0.1% noise.

B. Harrach: Fast shape-reconstruction in EIT
Reconstructions with exact data and with 0.1% noise.
Reconstructions for smooth transitions between inclusion and background and for the indefinite case.
Goal: Enhance linearized methods

Standard linearized method vs. heuristic combination with FM for frequency-difference EIT without ref. measurements


B. Harrach: Fast shape-reconstruction in EIT
More goals / collaborations?

- Monotony/linearization arguments recover the shape. Also the coefficient on the outer boundary? *(Detect flaws/corrosions of steel in concrete?)*

- Detect inclusions in unknown background? Use bounds on contrast and monotony?

- Sampling/Factorization Method origins from inverse scattering. Extend our results to inverse scattering? Positivity up to finite-dimensional subspaces?

- More applications: complex conductivities, diffuse optical tomography, eddy currents, ...