Fast shape-reconstruction in electrical impedance tomography

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Mathematical Model

Forward operator of EIT:

\[ \Lambda : \sigma \mapsto \Lambda(\sigma), \quad "\text{conductivity}" \mapsto "\text{measurements}" \]

- Conductivity: \( \sigma \in L_+^\infty(\Omega) \)
- Continuum model: \( \Lambda(\sigma) \): Neumann-Dirichlet-operator

\[ \Lambda(\sigma) : g \mapsto u|_{\partial \Omega}, \quad "\text{applied current}" \mapsto "\text{measured voltage}" \]
\[ \nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega, \quad \sigma \partial_\nu u|_{\partial \Omega} = g \quad \text{on } \partial \Omega. \]

- Linear elliptic PDE theory:

\[ \Lambda(\sigma) : L_+^2(\partial \Omega) \to L_+^2(\partial \Omega) \text{ linear, compact, self-adjoint} \]

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Inverse problem

Non-linear forward operator of EIT
\[ \Lambda : \sigma \mapsto \Lambda(\sigma), \quad L^\infty_+(\Omega) \rightarrow \mathcal{L}(L^2_\partial(\partial\Omega)) \]

Inverse problem of EIT:
\[ \Lambda(\sigma) \mapsto \sigma? \]

- Uniqueness ("Calderón problem"): Is \( \Lambda \) injective?
- Convergent numerical methods to reconstruct \( \sigma \)?
Convergent numerical methods to reconstruct $\sigma$?

- Newton iteration: almost no theory
  - Dobson (1992): (Local) convergence for regularized EIT equation.

- D-bar method: convergent 2D-implementation for $\sigma \in C^2$

In practice:

- large jumps in conductivity
- large interest in detecting shapes / inclusions / anomalies

Inclusion/shape detection problem:

$$\Lambda(\sigma) \mapsto \text{supp}(\sigma - \sigma_0)?, \quad \sigma_0: \text{reference conductivity}.$$
Promising approach: Factorization method \((Kirsch 1998)\)

- *FM for EIT (1999–): Brühl, Hakula, Hanke, H., Hyvönen, Kirsch, Lechleiter, Nachman, Päivärinta, Pursiainen, Schappel, Schmitt, Seo, Teirilä*

Typical result:

\[
z \notin \text{supp}(\sigma - \sigma_0) \quad \text{iff} \quad \lim_{\alpha \to 0} I_\alpha(z) = \infty.
\]

\((I_\alpha(z): \text{indicator function})\)

Unsolved problems since 1998:

- Convergent regularization strategies for "infinity test"?
- Theory needs definiteness assumption, e.g., \(\sigma \geq \sigma_0\) everywhere

In this talk: A monotonicity based sampling method.

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\[ \int_{\Omega} (\sigma_1 - \sigma_2) |\nabla u_1|^2 \, dx \leq (g, (\Lambda(\sigma_2) - \Lambda(\sigma_1))g) \]

\(u_1\) solution corresponding to \(\sigma_1\) and boundary current \(g\).

Simple consequence:

\[
\sigma_1 \geq \sigma_2 \implies \Lambda(\sigma_1) \leq \Lambda(\sigma_2)
\]
Monotony based imaging

- True conductivity: \( \sigma = 1 + \chi_D \), \( D \): unknown inclusion
- \( \Lambda(\sigma) \): measured data
- Test conductivity: \( 1 + \chi_B \), \( B \): small ball
- \( \Lambda(1 + \chi_B) \) can be simulated for different balls \( B \)

Monotony:

\[
B \subseteq D \implies 1 + \chi_B \leq 1 + \chi_D = \sigma \implies \Lambda(1 + \chi_B) \geq \Lambda(\sigma)
\]

Monotony based reconstruction algo. for EIT (Tamburrino/Rubinacci 02)

- For all \( B \), calculate \( \Lambda(1 + \chi_B) \) & test whether \( \Lambda(1 + \chi_B) \geq \Lambda(\sigma) \)
- Result: upper bound of \( D \).

Only an upper bound? Converse monotony relation?

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Theorem \((H./Ullrich)\)
\[\Omega \setminus \overline{D} \text{ connected. } \sigma = 1 + \chi_D.\]

\[B \nsubseteq D \implies \Lambda(1 + \chi_B) \nsubseteq \Lambda(\sigma).\]

\[\leadsto \text{ Monotony method detects exact shape.}\]

\(^\text{(Extensions possible for non-connected complement, inhomogeneous inclusions or background, continuous transitions between inclusion and background,\ldots)}\)
Converse monotony relation

Proof \( (\sigma = 1 + \chi_D, \kappa = 1 + \chi_B) \)

\[
\int_{\Omega} (\kappa - \sigma) |\nabla u_\kappa|^2 \, dx \leq (g, (\Lambda(\sigma) - \Lambda(\kappa))g)
\]

Apply localized potentials (H 2008) to control power term \(|\nabla u_\kappa|^2\).

\( \sim \exists g : (g, (\Lambda(\sigma) - \Lambda(\kappa))g) \geq 0 \implies \Lambda(\sigma) \nleq \Lambda(\kappa) \)
Fast implementation

▶ Testing $\Lambda(1 + \chi_B) \geq \Lambda(\sigma)$ is expensive. One forw. prob. per $B$.

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Let $\kappa, \sigma, \sigma_0$ piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

$$\text{supp}_{\partial \Omega} \kappa = \text{supp}_{\partial \Omega}(\sigma - \sigma_0).$$

$\text{supp}_{\partial \Omega}$: outer support ( = support, if support is compact and has conn. complement)

~~ Replacing $\Lambda(1 + \chi_B)$ by its linear approx. should still recover the exact shape (linearization error does not affect the shape!)
Theorem (H./Ullrich)
Let $\Omega \setminus \overline{D}$ connected, $0 < k \leq 1/2$.

$$B \subseteq D \iff \Lambda(\mathbb{1}) + k\Lambda'(\mathbb{1})\chi_B \geq \Lambda(\sigma).$$

Fast, requires only homogeneous forward solution

- Comp. cost equivalent to linearized methods or FM

(Again, extensions possible for non-connected complement, inhomogeneous inclusions or background, continuous transitions between inclusion and background, . . . )
Indefinite inclusions: $\sigma = 2 + \chi_{D^+} - \chi_{D^-}$.

With $\kappa^+ = 2 + \chi_B$. $\kappa^- = 2 - \chi_B$:

\[
D^+ \cup D^- \subseteq B \iff \Lambda(\kappa^-) \geq \Lambda(\sigma) \geq \Lambda(\kappa^+).
\]

- Indefinite inclusions can be treated by step-wise shrinking of larger test domains.
- Result can be linearized. Linearized version yields exact shape (no linearization error!)
- Result extends to general pcw. anal. conductivities (with some technical effort)

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Reconstructions with exact data and with 0.1% noise.
Reconstructions with exact data and with 0.1% noise.
Numerical results

Reconstructions for smooth transitions between inclusion and background and for the indefinite case.

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Goal: Enhance linearized methods

Standard linearized method vs. heuristic combination with FM for frequency-difference EIT without ref. measurements

Summary

New monotony based shape reconstruction method
- yields the exact shape not just an upper bound
- can be efficiently implemented by linearization (while still reconstructing the exact shape)

Advantages
- Rigorous treatment of indefinite inclusions
- Convergent regularization implementation of testing criteria seems possible

For practical applications:
- Enhance linearized/iterative methods by exact shape reconstruction *(H./Seo SIMA 2010, H./Seo/Woo IEEE TMI 2010)*