



Simultaneous identification of multiple coefficients in an elliptic PDE

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Calderón problem: Can we recover $\sigma \in L_+^\infty(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n \quad (1)$$

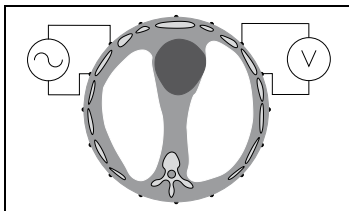
from all possible Dirichlet and Neumann boundary values

$$\left\{ (u|_\Sigma, \sigma \partial_\nu u|_\Sigma) : u \text{ solves (1) with } u|_{\partial\Omega \setminus \bar{\Sigma}} = 0 \right\} ?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator (NtD)**

$$\Lambda_\sigma : L_\diamond^2(\Sigma) \rightarrow L_\diamond^2(\Sigma), \quad g \mapsto u|_\Sigma,$$

where u solves (1) with $\sigma \partial_\nu u = \begin{cases} g & \text{on } \Sigma \\ 0 & \text{on } \partial\Omega \setminus \bar{\Sigma} \end{cases}$



Electrical impedance tomography (EIT):

- ▶ Apply currents $\sigma \partial_\nu u|_{\partial\Omega}$ (Neumann boundary data)
 \rightsquigarrow Electric potential u in Ω (solution of $\nabla \cdot (\sigma \nabla u) = 0$)
- ▶ Measure voltages $u|_{\partial\Omega}$ (Dirichlet boundary data)

Current-Voltage-Measurements \rightsquigarrow Fin.-dim. approx. to Λ_σ

Does Λ_σ determine σ ?

- ▶ Measurements on complete boundary:

*Calderón (1980), Druskin (1982+85), Kohn/Vogelius (1984+85),
Sylvester/Uhlmann (1987), Nachman (1996), Astala/Päiväranta (2006)*

- ▶ Measurements on part of the boundary:

*Bukhgeim/Uhlmann (2002), Knudsen (2006), Isakov (2007),
Kenig/Sjöstrand/Uhlmann (2007), H. (2008),
Imanuvilov/Uhlmann/Yamamoto (2009)*

In this talk:

A new method (*localized potentials*) to prove identifiability results
(for this and other elliptic inverse multi-coefficient problems with partial data)

For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\begin{aligned} & \int_{\Omega} (\sigma_1 - \sigma_0) |\nabla u_0|^2 \, dx \\ & \geq \int_{\Sigma} g (\Lambda_{\sigma_0} - \Lambda_{\sigma_1}) g \, ds \geq \int_{\Omega} \frac{\sigma_0}{\sigma_1} (\sigma_1 - \sigma_0) |\nabla u_0|^2 \, dx \end{aligned}$$

for all solutions u_0 of

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_\nu u_0|_{\Sigma} = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$

(e.g., Kang/Seo/Sheen 1997, Kirsch 2005, Ide et al. 2007, H./Seo 2009+10)

Can we control $|\nabla u_0|^2$?

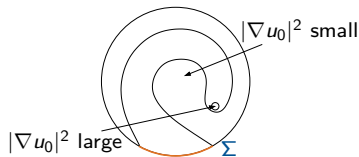
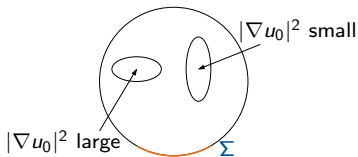
Theorem (H., 2008)

Let σ_0 fulfill unique continuation principle (UCP),

$$\overline{D_1} \cap \overline{D_2} = \emptyset, \quad \text{and} \quad \Omega \setminus (\overline{D_1} \cup \overline{D_2}) \text{ be connected with } \Sigma.$$

Then there exist solutions $u_0^{(k)}$, $k \in \mathbb{N}$ with

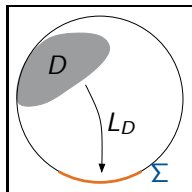
$$\int_{D_1} |\nabla u_0^{(k)}|^2 dx \rightarrow \infty \quad \text{and} \quad \int_{D_2} |\nabla u_0^{(k)}|^2 dx \rightarrow 0.$$



Virtual measurements:

$L_D : H_{\diamond}^1(D)' \rightarrow L_{\diamond}^2(\Sigma), \quad f \mapsto u|_{\Sigma},$ with

$$\int_{\Omega} \sigma \nabla u \cdot \nabla v \, dx = \langle f, v|_D \rangle \quad \forall v \in H_{\diamond}^1(D).$$



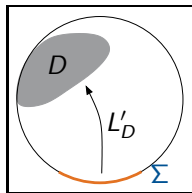
By (UCP): If $\overline{D_1} \cap \overline{D_2} = \emptyset$ and $\Omega \setminus (\overline{D_1} \cup \overline{D_2})$ is connected with Σ , then $\mathcal{R}(L_{D_1}) \cap \mathcal{R}(L_{D_2}) = 0$.

Sources on different domains yield different virtual measurements.

Dual operator:

$$L'_D : L^2_\diamond(\Sigma) \rightarrow H^1_\diamond(D), \quad g \mapsto u|_D, \text{ with}$$

$$\nabla \cdot (\sigma \nabla u) = 0, \quad \sigma \partial_\nu u|_\Sigma = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$



Evaluating solutions on D is dual operation to virtual measurements.

Functional analysis:

X, Y_1, Y_2 reflexive Banach spaces, $L_1 \in \mathcal{L}(Y_1, X)$, $L_2 \in \mathcal{L}(Y_1, X)$.

$$\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2) \iff \|L_1'x\| \lesssim \|L_2'x\| \quad \forall x \in X'.$$

Here: $\mathcal{R}(L_{D_1}) \not\subseteq \mathcal{R}(L_{D_2}) \implies \|u_0|_{D_1}\|_{H_\diamond^1} \not\lesssim \|u_0|_{D_2}\|_{H_\diamond^1}$.

If two sources do not generate the same data, then the respective evaluations are not bounded by each other.

Note: $H_\diamond^1(D)'$ -source $\longleftrightarrow H_\diamond^1(D)$ -evaluation.

- ▶ Back to identifiability problem: Let $\Lambda_{\sigma_0} = \Lambda_{\sigma_1}$, σ_0 fulfills (UCP).
- ▶ By monotony,

$$\int_{\Omega} (\sigma_1 - \sigma_0) |\nabla u_0|^2 dx \geq 0 \geq \int_{\Omega} \frac{\sigma_0}{\sigma_1} (\sigma_1 - \sigma_0) |\nabla u_0|^2 dx \quad \forall u_0$$

- ▶ Assume: \exists neighbourhood U of Σ in which $\sigma_1 \geq \sigma_0$ but $\sigma_1 \neq \sigma_0$
 \rightsquigarrow Potential with localized energy in U contradicts monotony.

Higher conductivity reachable by the bndry cannot be balanced out.

Corollary (*Kohn/Vogelius, 1984/85*)

Calderón problem is uniquely solvable for piecw.-anal. conductivities.

Can we recover two coefficients $a, c \in L^{\infty}_+(\Omega)$ in

$$-\nabla \cdot (a\nabla u) + cu = 0 \quad \text{in } \Omega \quad (2)$$

from the NtD (with partial data)

$$\Lambda_{(a,c)} : L^2(\Sigma) \rightarrow L^2(\Sigma), \quad g \mapsto u|_{\Sigma},$$

where u solves (2) with

$$\sigma \partial_{\nu} u|_{\Sigma} = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$

Application: Diffuse optical tomography (DOT).

$$\begin{aligned}
 & \int_{\Omega} ((a_2 - a_1)|\nabla u_1|^2 + (c_2 - c_1)|u_1|^2) \, dx \\
 & \geq \int_{\Sigma} g (\Lambda_{a_1, c_1} - \Lambda_{a_2, c_2}) g \, ds \\
 & \geq \int_{\Omega} ((a_2 - a_1)|\nabla u_2|^2 + (c_2 - c_1)|u_2|^2) \, dx,
 \end{aligned}$$

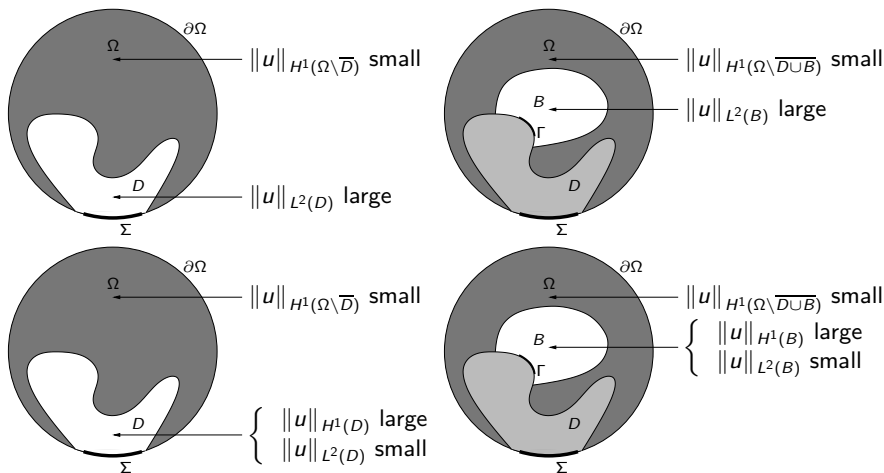
Method of localized potentials:

- ▶ Again, sources on different regions produce different data.
- ▶ $(H^1)'$ -sources produce different data than L^2 -sources

$$\implies \|u\|_{H^1(D)} \not\lesssim \|u\|_{L^2(D)}.$$

We can control $|\nabla u_1|^2$ and $|u_1|^2$ separately.

Lemma (H., 2009) There exist solutions u with



Theorem (H., 2009)

Let

- ▶ $a_1, a_2 \in L_+^\infty(\Omega)$ piecewise constant,
- ▶ $c_1, c_2 \in L_+^\infty(\Omega)$ piecewise constant.

Then

$$\Lambda_{a_1, c_1} = \Lambda_{a_2, c_2} \iff a_1 = a_2, \quad c_1 = c_2.$$

Note that $v := \sqrt{a}u$ transforms $-\nabla \cdot (a\nabla u) + cu = 0$ into

$$-\Delta v + \eta v = 0, \quad \eta := \frac{\Delta \sqrt{a}}{\sqrt{a}} + \frac{c}{a}$$

(when the coefficients are smooth).

\rightsquigarrow **No uniqueness for smooth a and c !**

Now let a_1, c_1, a_2, c_2 be piecewise analytic (anal. on $\Omega \setminus \Gamma$)

$$\begin{aligned} & \int_{\Sigma} g (\Lambda_{a_2, c_2} - \Lambda_{a_1, c_1}) g \, ds \\ & \leq \int_{\Omega \setminus \Gamma} (\eta_1 - \eta_2) a_2 |u_2|^2 \, dx \\ & \quad + \int_{\Sigma} \left(1 - \frac{\sqrt{a_2}}{\sqrt{a_1}} \right) g u_2 \, ds - \int_{\partial\Omega} \left(\frac{\partial_\nu a_1}{2a_1} - \frac{\partial_\nu a_2}{2a_2} \right) a_2 |u_2|^2 \, ds \\ & \quad + \int_{\Gamma} \left\{ \frac{1}{2} \left([\partial_\nu a_2]_{\Gamma} - \left[\frac{a_2}{a_1} \partial_\nu a_1 \right]_{\Gamma} \right) |u_2|^2 - 2 \left[\frac{\sqrt{a_2}}{\sqrt{a_1}} \right]_{\Gamma} a_1 \partial_\nu u_1 u_2 \right\} \, ds, \end{aligned}$$

Control $\|u\|_{H^1}$, $\|u\|_{L^2}$ (on subsets), and $\|u|_{\Gamma}\|_{L^2}$ (on bndry parts)!

Theorem (H., 2010)

Let $a_1, a_2, c_1, c_2 \in L_+^\infty(\Omega)$ be piecew. anal. Then $\Lambda_{(a_1, c_1)} = \Lambda_{(a_2, c_2)}$ if and only if

$$(a) \quad a_1|_\Sigma = a_2|_\Sigma, \quad \partial_\nu a_1|_\Sigma = \partial_\nu a_2|_\Sigma \quad \text{on } \Sigma,$$

$$(b) \quad \frac{\partial_\nu a_1}{a_1}|_{\partial\Omega \setminus \bar{\Sigma}} = \frac{\partial_\nu a_2}{a_2}|_{\partial\Omega \setminus \bar{\Sigma}} \quad \text{on } \partial\Omega \setminus \Sigma,$$

$$(c) \quad \eta_1 = \eta_2 \quad \text{in smooth regions,}$$

$$(d) \quad \frac{a_1^+|_\Gamma}{a_1^-|_\Gamma} = \frac{a_2^+|_\Gamma}{a_2^-|_\Gamma}, \quad \frac{[\partial_\nu a_2]_\Gamma}{a_2^-|_\Gamma} = \frac{[\partial_\nu a_1]_\Gamma}{a_1^-|_\Gamma} \quad \text{on inner boundaries } \Gamma.$$

NtD $\Lambda_{(a,c)}$ determines $\eta = \frac{\Delta\sqrt{a}}{\sqrt{a}} + \frac{c}{a}$ and the jumps of a and ∇a .

Monotony + Localized potentials \implies Unique identifiability

- ▶ flexible, conceptually simple method to prove uniqueness results
- ▶ seems to be applicable to very general, multi-coeff. problems
- ▶ loc. potentials require only unique continuation results
- ▶ monotony arguments require piecewise analyticity
(even C^∞ might still oscillate too much)

Method is non-constructive, but can be used for

- ▶ local convergence of Newton algo. (*Lechleiter/Rieder, 2008*)
- ▶ shape reconstruction by single linearization step (*H./Seo, 2010*)
- ▶ monotonicity based reconstruction algo. (*H./Ullrich, in progress*)