Fast shape-reconstruction in electrical impedance tomography

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Calderón problem

Calderón problem: Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n \tag{1}$$

from all possible Dirichlet and Neumann boundary values

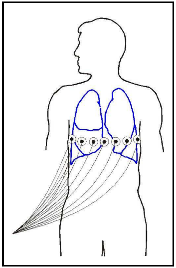
$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \ : \ u \text{ solves } (1)\}\,?$$

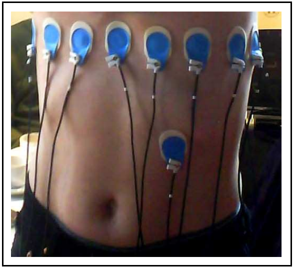
Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator (NtD)**

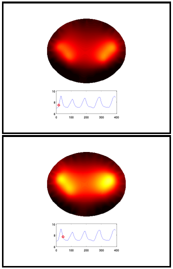
$$\Lambda(\sigma): \ L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u = g$ on $\partial\Omega$.

EIT







Electrical impedance tomography (EIT):

- Apply currents $\sigma \partial_\nu u|_{\partial\Omega}$ (Neumann boundary data)
  - $\leadsto$ Electric potential u in Ω (solution of $\nabla \cdot (\sigma \nabla u) = 0$)
- Measure voltages $u|_{\partial\Omega}$ (Dirichlet boundary data)

Current-Voltage-Measurements $\leadsto$ Fin.-dim. approx. to $\Lambda(\sigma)$

Inverse problem

Non-linear forward operator of EIT

$$\Lambda : \ \sigma \mapsto \Lambda(\sigma), \quad L^\infty_+(\Omega) \to \mathcal{L}(L^2_\diamond(\partial\Omega))$$

Inverse problem of EIT:

$$\Lambda(\sigma) \mapsto \sigma?$$

Uniqueness ("Calderón problem"):

- Measurements on complete boundary:

  Calderón (1980), Druskin (1982+85), Kohn/Vogelius (1984+85), Sylvester/Uhlmann (1987), Nachman (1996), Astala/Päivärinta (2006)

- Measurements on part of the boundary:

  Bukhgeim/Uhlmann ('02), Knudsen ('06), Isakov ('07), Kenig/Sjöstrand/Uhlmann ('07), **H***. ('08), Imanuvilov/Uhlmann/Yamamoto ('09)*

Linearization

Generic approach: Linearization

$$\Lambda(\sigma) - \Lambda(\sigma_0) \approx \Lambda'(\sigma_0)(\sigma - \sigma_0)$$

σ_0: known reference conductivity / initial guess / ...

$\Lambda'(\sigma_0)$: Fréchet-Derivative / sensitivity matrix.

$$\Lambda'(\sigma_0) : L^\infty_+(\Omega) \to \mathcal{L}(L^2_\diamond(\partial\Omega)).$$

$\rightsquigarrow$ Solve linearized equation for difference $\sigma - \sigma_0$.

Often: $\mathrm{supp}(\sigma - \sigma_0) \subset\subset \Omega$ compact. *("shape" / "inclusion")*

Linearization

Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

$$\text{Solve } \Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0), \text{ then } \kappa \approx \sigma - \sigma_0.$$

- Multiple possibilities to measure residual norm and to regularize.
- No rigorous theory for single linearization step.
- Almost no theory for Newton iteration:

  Dobson (1992): (Local) convergence for regularized EIT equation.
  Lechleiter/Rieder(2008): (Local) convergence for discretized setting.
  No (local) convergence theory for non-discretized case!

Linearization

Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

It seems that

- EIT is a non-linear problem, many Newton-iterations required.
- No rigorous results possible for single linearization step.
- Linearization only justifiable for small $\sigma - \sigma_0$ (local results).

In this talk:

- Shape detection in EIT is essentially a linear problem!
- Fast shape detection algorithms are possible.

Exact Linearization

Theorem (**H.**/Seo, SIAM J. Math. Anal. 2010)

Let κ, σ, σ_0 piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

(a) $\mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

(b) $\frac{\sigma_0}{\sigma}(\sigma - \sigma_0) \leq \kappa \leq \sigma - \sigma_0$ on the bndry of $\mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

$\mathrm{supp}_{\partial\Omega}$: outer support (= support, if support is compact and has conn. complement)

- Exact solution of lin. equation yields correct (outer) shape.
- No assumptions on $\sigma - \sigma_0$!

$\rightsquigarrow$ Linearization error does not lead to shape errors.

Proof

- Exact linearization: $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$
- Monotony: For all "reference solutions" u_0:

$$\int_\Omega (\sigma - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x \geq \underbrace{\langle g, (\Lambda(\sigma) - \Lambda(\sigma_0))\, g\rangle}_{= \int_\Omega \kappa |\nabla u_0|^2 \,\mathrm{d}x} \geq \int_\Omega \frac{\sigma_0}{\sigma}(\sigma - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x.$$

- Use localized potentials (**H** 2008) to control $|\nabla u_0|^2$

$\rightsquigarrow$ $\mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$

- Similarly, $\frac{\sigma_0}{\sigma}(\sigma - \sigma_0) \leq \kappa \leq \sigma - \sigma_0$ on bndry of $\mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$

Non-exact Linearization?

Theorem requires $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$.

- Existence of exact solution is unknown!
- In practice: finite-dimensional, noisy measurements

Ongoing research:

- How to use this result for fast shape detection

(Fast = based on linearized equation, i.e., only one forward solution)

Promising approach:

- Reconstruction algorithm based on monotony arguments

Monotony

$$\int_\Omega (\sigma_1 - \sigma_2)|\nabla u_1|^2 \, \mathrm{d}x \leq (g, (\Lambda(\sigma_2) - \Lambda(\sigma_1))g)$$

u_1 solution corresponding to σ_1 and boundary current g.

Simple consequence:

$$\sigma_1 \leq \sigma_2 \quad \Longrightarrow \quad \Lambda(\sigma_1) \geq \Lambda(\sigma_2)$$

Monotony based imaging

- True conductivity: $\sigma = 1 + \chi_D$, D: unknown inclusion

$\rightsquigarrow$ $\Lambda(\sigma)$: measured data

- Test conductivity: $\kappa = 1 + \chi_B$, B: small ball

$\rightsquigarrow$ $\Lambda(\kappa)$ can be simulated for different balls B

Monotony:

$$B \subseteq D \quad \Longrightarrow \quad \Lambda(\sigma) \geq \Lambda(\kappa)$$

Monotony based reconstruction algo. for EIT *(Tamburrino/Rubinacci 02)*

- For all balls B, calculate $\Lambda(\kappa)$ and test whether $\Lambda(\sigma) \geq \Lambda(\kappa)$

$\rightsquigarrow$ Result: upper bound of D.

Only an upper bound? Converse montony relation?

Converse montony relation

Theorem *(H./Ullrich)*
$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$, $\kappa = 1 + \chi_B$.

$$B \not\subseteq D \quad \Longrightarrow \quad \Lambda(\kappa) \not\geq \Lambda(\sigma).$$

$\rightsquigarrow$ Monotony method detects exact shape.

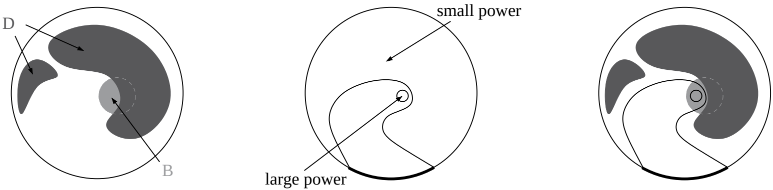
(Extensions possible for non-connected complement, inhomogeneous inclusions or background, continuous transitions between inclusion and background,...)

Converse montony relation

Proof $\quad (\sigma = 1 + \chi_D,\ \kappa = 1 + \chi_B)$

$$\int_\Omega (\kappa - \sigma)|\nabla u_\kappa|^2 \, \mathrm{d}x \leq (g, (\Lambda(\sigma) - \Lambda(\kappa))g)$$

Apply localized potentials (H 2008) to control power term $|\nabla u_\kappa|^2$.



$\rightsquigarrow \exists g : \ (g, (\Lambda(\sigma) - \Lambda(\kappa))g) \geq 0 \quad \Longrightarrow \quad \Lambda(\sigma) \not\leq \Lambda(\kappa)$

Fast implementation

- Testing $\Lambda(\sigma) \geq \Lambda(\kappa)$ is expensive. One forward problem per κ.
- Using linear approx. of $\Lambda(\kappa)$ still fulfills monotony relation (still exact, no linearization error!)

Theorem *(H./Ullrich)*

$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$, $\kappa = 1 + k\chi_B$ (here: $0 < k \leq 1/2$)

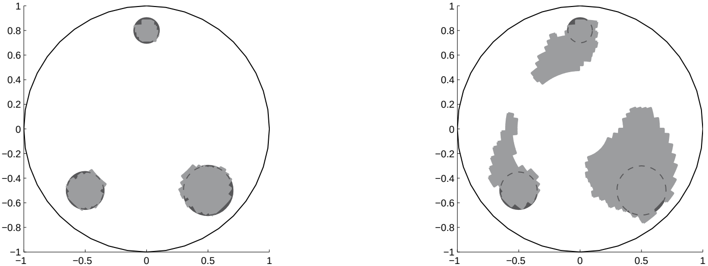
$$B \subseteq D \quad \Longleftrightarrow \quad \Lambda(\mathbb{1}) + k\Lambda'(\mathbb{1})\chi_B \geq \Lambda(\sigma).$$

$\leadsto$ Fast implementation, requires only homogeneous forward solution

- Comp. cost equivalent to standard linearized methods

(Again, extensions possible for non-connected complement, inhomogeneous inclusions or background, continuous transitions between inclusion and background,...)

Numerical results



Reconstructions with exact data and with 0.1% noise.

Conclusions

- Electrical impedance tomography is a non-linear problem
- For shape detection it can be replaced by a linear problem without losing information
- Designing fast, convergent shape detection algorithms is possible but non-trivial.
- Promising approach: monotony-based methods.