Monotony based imaging in EIT

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Mathematical Model

Forward operator of EIT:

$$\Lambda: \ \sigma \mapsto \Lambda(\sigma), \quad \text{"conductivity"} \mapsto \text{"measurements"}$$

- Conductivity: $\sigma \in L^\infty_+(\Omega)$
- Continuum model: $\Lambda(\sigma)$: Neumann-Dirichlet-operator

$$\Lambda(\sigma): \ g \mapsto u|_{\partial\Omega}, \quad \text{"applied current"} \mapsto \text{"measured voltage"}$$

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega, \quad \sigma \partial_\nu u|_{\partial\Omega} = g \quad \text{on } \partial\Omega. \tag{1}$$

- Linear elliptic PDE theory:

$$\Lambda(\sigma): L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega) \text{ linear, compact, self-adjoint}$$

Inverse problem

Non-linear forward operator of EIT

$$\Lambda : \ \sigma \mapsto \Lambda(\sigma), \quad L^\infty_+(\Omega) \to \mathcal{L}(L^2_\diamond(\partial\Omega))$$

Inverse problem of EIT:

$$\Lambda(\sigma) \mapsto \sigma?$$

- Uniqueness ("Calderón problem"): Is Λ injective?
- Convergent numerical methods to reconstruct σ?

Reconstruction

Convergent numerical methods to reconstruct σ?

- Newton iteration: almost no theory
  Dobson (1992): (Local) convergence for regularized EIT equation.
  Lechleiter/Rieder(2008): (Local) convergence for discretized setting.
- D-bar method: convergent 2D-implementation for $\sigma \in C^2$
  Knudsen, Lassas, Mueller, Siltanen (2008)

In practice:

- large jumps in conductivity
- large interest in detecting shapes / inclusions / anomalies

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Inclusion/shape detection problem:

$$\text{Reconstruct } \operatorname{supp}(\sigma - \sigma_0), \quad \sigma_0\text{: reference conductivity.}$$

Monotony

$$\int_\Omega (\sigma_1 - \sigma_2) |\nabla u_1|^2 \, \mathrm{d}x \leq (g, (\Lambda(\sigma_2) - \Lambda(\sigma_1)) g)$$

u_1 solution corresponding to σ_1 and boundary current g.

Simple consequence:

$$\sigma_1 \leq \sigma_2 \quad \Longrightarrow \quad \Lambda(\sigma_1) \geq \Lambda(\sigma_2)$$

Monotony based imaging

- True conductivity: $\sigma = 1 + \chi_D$, D: unknown inclusion

$\rightsquigarrow$ $\Lambda(\sigma)$: measured data

- Test conductivity: $\kappa = 1 + \chi_B$, B: small ball

$\rightsquigarrow$ $\Lambda(\kappa)$ can be simulated for different balls B

Monotony:

$$B \subseteq D \quad \Longrightarrow \quad \Lambda(\sigma) \geq \Lambda(\kappa)$$

Monotony based reconstruction algo. for EIT

Tamburrino/Rubinacci (2002)

- For all balls B, calculate $\Lambda(\kappa)$ and test whether $\Lambda(\sigma) \geq \Lambda(\kappa)$

$\rightsquigarrow$ Result: upper bound of D.

Monotony based imaging

Monotony based reconstruction *(up to now...)*

- Simple theory, simple implementation
- Regularization seems straight-forward
- Only reconstructs upper bound
- Expensive, requires one forward solution for each test ball
- Needs definiteness assumption

Comparison: Factorization Method *Kirsch 1998, Hanke/Brühl 2000*

- Complicated theory and implementation
- No known convergent regularization strategies
- Reconstructs exact shape (if $\Omega \setminus \overline{D}$ connected)
- Cheap, requires only one homogeneous forward solution
- Needs definiteness assumption

Converse montony relation

Theorem *(H./Ullrich, 2010)*
$\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$, $\kappa = 1 + \chi_B$.

$$B \not\subseteq D \quad \Longrightarrow \quad \Lambda(\kappa) \not\geq \Lambda(\sigma).$$

$\rightsquigarrow$ Monotony method detects exact shape.

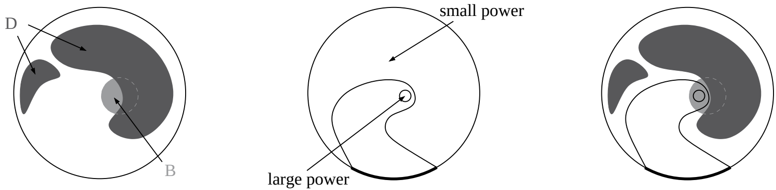
(Extensions possible for non-connected complement, inhomogeneous inclusions or background, continuous transitions between inclusion and background,. . .)

Converse montony relation

Proof $(\sigma = 1 + \chi_D,\ \kappa = 1 + \chi_B)$

$$\int_\Omega (\kappa - \sigma)|\nabla u_\kappa|^2 \, \mathrm{d}x \leq (g, (\Lambda(\sigma) - \Lambda(\kappa))g)$$

Apply localized potentials (H 2008) to control power term $|\nabla u_\kappa|^2$.



$\rightsquigarrow \exists g : \ (g, (\Lambda(\sigma) - \Lambda(\kappa))g) \geq 0 \quad \Longrightarrow \quad \Lambda(\sigma) \not\leq \Lambda(\kappa)$

Implementation/Definiteness

Computation costs:

- Using linear approx. of $\Lambda(\kappa)$ still fulfills monotony relation (still exact, no linearization error)

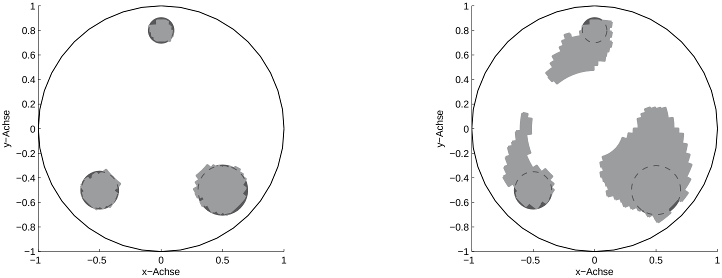
$\rightsquigarrow$ Fast implementation, requires only homogeneous forward solution

- Comp. cost equivalent to linearized methods or FM

Indefinite inclusions (larger and smaller than background conductivity)

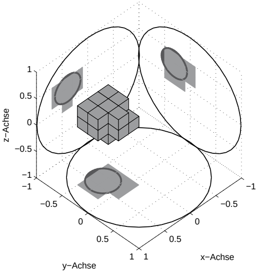
- can be treated by step-wise shrinking of larger test domains.

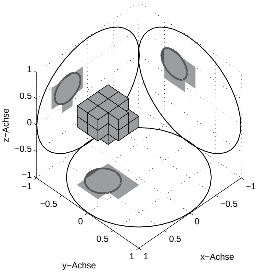
Numerical results



Reconstructions with exact data and with 0.1% noise.

Numerical results





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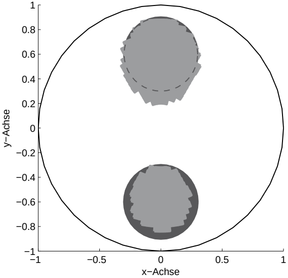

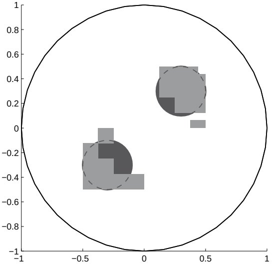

Reconstructions for smooth transitions between inclusion and background and for the indefinite case.

Summary

New results on the monotonicity method of Tamburrino and Rubinacci

- Method yields the exact shape not just an upper bound
- Method can be efficiently implemented by linearization (while still reconstructing the exact shape)

Possible advantages

- Rigorous treatment of indefinite inclusions seems possible
- Convergent implementation of testing criteria seems possible

Goal

- Enhance linearized/iterative methods by exact shape reconstruction *(H./Seo SIMA 2010, H./Seo/Woo IEEE TMI 2010)*