Novel tomography techniques and parameter identification problems

Bastian von Harrach

harrach@ma.tum.de

Department of Mathematics - M1, Technische Universität München

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Overview

- Parameter identification problems
  - in diffuse optical tomography (DOT)
- Linearized reconstruction algorithms
  - in electrical impedance tomography (EIT)

Parameter identification problems in DOT

Diffuse optical tomography

Diffuse optical tomography (DOT):

- Transilluminate biological tissue with visible or near-infrared light
- Goal: Reconstruct spatial image of interior physical properties.

  Relevant quantities (in diffusive regime):
  - Scattering
  - Absorption
- Applications:
  - Breast cancer detection
  - Bedside-imaging of neonatal brain function

Topical reviews:

Arridge & Schotland (2009), Gibson, Hebden & Arridge (2005), Arridge (1999)

Mathematical Model

- General Forward Model:
  Photon transport models (Boltzmann transport equation)
  Recent review: Bal, Inverse Problems 25, 053001 (48pp), 2009.
- For highly scattering media:
  - DC diffusion approximation for photon density u:

$$-\nabla \cdot (a \nabla u) + cu = 0 \quad \text{in } B \subset \mathbb{R}^n,$$

u :	$B \to \mathbb{R}$:	photon density
a :	$B \to \mathbb{R}$:	diffusion/scattering coefficient
c :	$B \to \mathbb{R}$:	absorption coefficient

  - Boundary measurements (idealized):

Neumann and Dirichlet data $u|_S$, $a\partial_\nu u|_S$ on $S \subseteq \partial B$.

Remaining boundary assumed to be insulated, $a\partial_\nu u|_{\partial B \setminus \overline{S}} = 0$.

Forward & inverse problem

DC diffuse optical tomography:

$$-\nabla \cdot (a\nabla u) + cu = 0 \quad \text{in } B \subset \mathbb{R}^n,\ n \geq 2,$$

B bounded with smooth boundary, $S \subseteq \partial B$ open part, $a, c \in L^\infty_+(B)$.

- $\forall g \in L^2(S)\ \exists!$ solution $u \in H^1(B)$: $a\partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{on } B \setminus \overline{S}. \end{cases}$
- (Local) Neumann-to-Dirichlet map

$$\Lambda_{a,c} : \ g \mapsto u|_S, \quad L^2(S) \to L^2(S)$$

is linear, compact and self-adjoint.

Inverse Problem: Can we reconstruct a and c from $\Lambda_{a,c}$?

Non-uniqueness

DC diffuse optical tomography:

$$-\nabla \cdot (a\nabla u) + cu = 0$$

Arridge/Lionheart (1998 Opt. Lett. 23 882–4):

- $v := \sqrt{a}u$ solves

$$-\Delta v + \eta v = 0, \quad \text{with} \quad \eta = \frac{\Delta\sqrt{a}}{\sqrt{a}} + \frac{c}{a}.$$

- $a = 1$ around $S \quad \rightsquigarrow \quad (u|_S, a\partial_\nu u|_S) = (v|_S, \partial_\nu v|_S)$.

$\rightsquigarrow$ $\Lambda_{a,c}$ only depends on effective absorption $\eta = \eta(a, c)$.

Absorption and scattering effects cannot be distinguished.

(Note: Argument requires smooth scattering coefficient a).

Experimental results

Theory: Absorption and scattering effects cannot be distinguished.

Practice:
Successful separate reconstructions of absorption and scattering (from phantom experiment using dc diffusion model!)

Pei et al. (2001), Jiang et al. (2002), Schmitz et al. (2002), Xu et al. (2002)

$\rightsquigarrow$ Practice contradicts theory!

Pei et al. (2001):

"As a matter of established methodological principle (...) empirical facts have the right-of-way; if a theoretical derivation yields a conclusion that is at odds with experimental results, the reconciliatory burden falls on the theorist, not on the experimentalist."

New uniqueness result

Theorem (**H.**, Inverse Problems 2009)

- $a_1, a_2 \in L^\infty_+(B)$ piecewise constant
- $c_1, c_2 \in L^\infty_+(B)$ piecewise analytic

If $\Lambda_{a_1,c_1} = \Lambda_{a_2,c_2}$ then $a_1 = a_2$ and $c_1 = c_2$.

- Piecewise constantness seems fulfilled for phantom experiments.

$\rightsquigarrow$ Result reconciles theory with practice.

- Measurements contain more than just the effective absorption!

Next slides: Idea of the proof using monotony and localized potentials.

Monotony result

Lemma

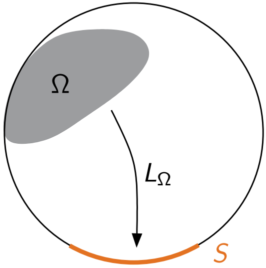
Let $a_1, a_2, c_1, c_2 \in L^\infty_+(B)$. Then for all $g \in L^2(S)$,

$$\begin{aligned}&\int_B \left((a_2 - a_1)|\nabla u_1|^2 + (c_2 - c_1)|u_1|^2\right) \, \mathrm{d}x \\ &\geq \langle (\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2})g, g\rangle \\ &\geq \int_B \left((a_2 - a_1)|\nabla u_2|^2 + (c_2 - c_1)|u_2|^2\right) \, \mathrm{d}x,\end{aligned}$$

$u_1, u_2 \in H^1(B)$: solutions for (a_1, c_1), resp., (a_2, c_2).

Can we control $|u_j|^2$ and $|\nabla u_j|^2$?

Virtual Measurements



$$L_\Omega: \ (H^1(\Omega))' \to L^2(S), \quad f \mapsto u|_S,$$

where $u \in H^1(B)$ solves

$$\int_B (a\nabla u \cdot \nabla v + cuv) \ \mathrm{d}x = \langle f, v \rangle$$

(essentially: $-\nabla \cdot (a \nabla u) + cu = f$)

Virtual Measurements

- Unique continuation:
  If $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ connected neighbourhood of S then
  $$\mathcal{R}(L_{\Omega_1}) \cap \mathcal{R}(L_{\Omega_2}) = 0.$$

- Dual operator:
  $L'_\Omega : \; L^2(S) \to H^1(\Omega), \quad g \mapsto u|_\Omega$, where u solves
  $$-\nabla \cdot (a \nabla u) + cu = 0, \qquad a\partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}$$

Some functional analysis

Lemma

Let X, Y be two reflexive Banach spaces, $A \in \mathcal{L}(X,Y)$, $y \in Y$. Then

$$y \in \mathcal{R}(A) \quad \text{iff} \quad |\langle y', y \rangle| \leq C \|A' y'\| \quad \forall y' \in Y'.$$

Corollary

If $\|L'_{\Omega_1} g\| \leq C \|L'_{\Omega_2} g\|$ $\forall$ fluxes g, i.e., if $\|u|_{\Omega_1}\|_{H^1} \leq C \|u|_{\Omega_2}\|_{H^1}$ for the corresponding densities u, then $\mathcal{R}(L_{\Omega_1}) \subseteq \mathcal{R}(L_{\Omega_2})$.

Contraposition

$\mathcal{R}(L_{\Omega_1}) \not\subseteq \mathcal{R}(L_{\Omega_2}) \rightsquigarrow \exists (g_k)$ such that the solutions (u_k) satisfy

$$\|u_k|_{\Omega_1}\|_{H^1(\Omega_1)} \to \infty \quad \text{and} \quad \|u_k|_{\Omega_2}\|_{H^1(\Omega_2)} \to 0.$$

Some functional analysis

Similarly
(by unique continuation and regularity)

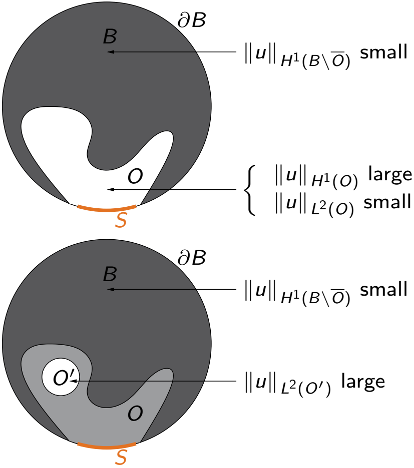
$$\mathcal{L}(L^2(\Omega)) \subsetneq \mathcal{L}(H^1(\Omega)').$$

$\leadsto \exists (g_k)$ such that the solutions (u_k) satisfy

$$\| u_k|_\Omega \|_{H^1(\Omega)} \to \infty \quad \text{and} \quad \| u_k|_\Omega \|_{L^2(\Omega)} \to 0.$$

Localized potentials

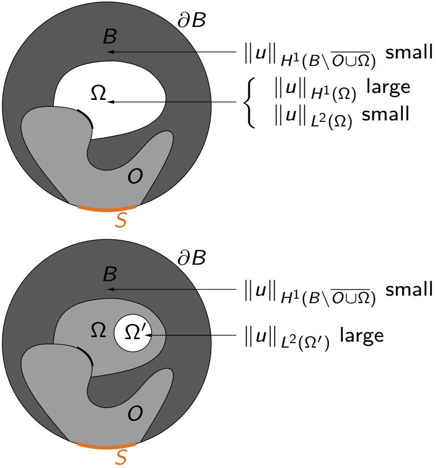
Lemma There exist solutions u with



Localized potentials

Technische Universität München

Lemma There exist solutions u with



Proof of uniqueness

Monotony

$$\begin{aligned}
&\int_B \left((a_2 - a_1)|\nabla u_1|^2 + (c_2 - c_1)|u_1|^2\right) \,\mathrm{d}x \\
&\geq \langle (\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2})g, g\rangle \\
&\geq \int_B \left((a_2 - a_1)|\nabla u_2|^2 + (c_2 - c_1)|u_2|^2\right) \,\mathrm{d}x,
\end{aligned}$$

Proof of the uniqueness result (*very sketchy ...*)

Start with region next to S

- Use loc. pot. with $|\nabla u|^2 \to \infty$ in that region $\rightsquigarrow a_1 = a_2$
- Then use loc. pot. with $|u|^2 \to \infty$ in that region $\rightsquigarrow c_1 = c_2$
- Repeat over all regions.

Uniqueness or not?

- Arridge/Lionheart (1998): Non-uniqueness for general smooth (a, c).
- **H.** (2009): Uniqueness for piecew. constant a, piecew. analytic c.

What information about (a, c) does $\Lambda_{a,c}$ really contain?

Formally(!), $\Lambda_{a,c}$ can only determine $\eta = \frac{\Delta\sqrt{a}}{\sqrt{a}} + \frac{c}{a}$.

Jumps in a or ∇a $\rightsquigarrow$ distributional singularities in $\Delta\sqrt{a}$.

Bold guess: Maybe $\Lambda_{a,c}$ determines

- η where a and c are smooth,
- jumps in a and ∇a.

(However, note that $\Delta\sqrt{a}/\sqrt{a}$ is not well-defined for non-smooth a ...)

Exact characterization

Theorem (**H.**, submitted for publication)

Let $a_1, a_2, c_1, c_2 \in L^\infty_+(B)$ piecewise analytic on joint partition

$$B = O_1 \cup \ldots \cup O_J \cup \Gamma, \qquad \partial O_1 \cup \ldots \cup \partial O_J = \partial B \cup \Gamma.$$

Then, $\Lambda_{a_1,c_1} = \Lambda_{a_2,c_2}$ if and only if

(a) $a_1|_S = a_2|_S$, and $\partial_\nu a_1|_S = \partial_\nu a_2|_S$ on S,

(b) $\frac{\partial_\nu a_1}{a_1}|_{\partial B \setminus \overline{S}} = \frac{\partial_\nu a_2}{a_2}|_{\partial B \setminus \overline{S}}$ on $\partial B \setminus \overline{S}$,

(c) $\eta_1 := \frac{\Delta\sqrt{a_1}}{\sqrt{a_1}} + \frac{c_1}{a_1} = \frac{\Delta\sqrt{a_2}}{\sqrt{a_2}} + \frac{c_2}{a_2} =: \eta_2$ on $B \setminus \Gamma$,

(d) $\frac{a_1^+|_\Gamma}{a_1^-|_\Gamma} = \frac{a_2^+|_\Gamma}{a_2^-|_\Gamma}$, and $\frac{[\partial_\nu a_2]_\Gamma}{a_2^-|_\Gamma} = \frac{[\partial_\nu a_1]_\Gamma}{a_1^-|_\Gamma}$ on Γ.

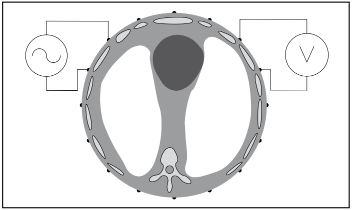
Proof

Proof relies on more general monotony results, e.g.,

$$\int_S g\left(\Lambda_{a_2,c_2} - \Lambda_{a_1,c_1}\right) g \, \mathrm{d}s$$
$$\leq \int_{B\setminus\Gamma} (\eta_1 - \eta_2)\, a_2 |u_2|^2 \, \mathrm{d}x$$
$$+ \int_S \left(1 - \frac{\sqrt{a_2}}{\sqrt{a_1}}\right) g u_2 \, \mathrm{d}s - \int_{\partial B} \left(\frac{\partial_\nu a_1}{2a_1} - \frac{\partial_\nu a_2}{2a_2}\right) a_2 |u_2|^2 \, \mathrm{d}s$$
$$+ \int_\Gamma \left\{ \frac{1}{2} \left([\partial_\nu a_2]_\Gamma - \left[\frac{a_2}{a_1}\partial_\nu a_1\right]_\Gamma \right) |u_2|^2 - 2 \left[\frac{\sqrt{a_2}}{\sqrt{a_1}}\right]_\Gamma a_1 \partial_\nu u_1 u_2 \right\} \, \mathrm{d}s$$

Then localized potentials are used to control $\|u\|_{H^1}$, $\|u\|_{L^2}$ on subsets and $\|u|_\Sigma\|_{L^2}$ on boundary parts.

Linearized reconstruction algorithms in EIT



Electrical impedance tomography (EIT):

- Apply currents $\sigma \partial_\nu u|_{\partial B}$ (Neumann boundary data)
  - $\rightsquigarrow$ Electric potential u solves

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } B$$

- Measure voltages $u|_{\partial B}$ (Dirichlet boundary data)

Current-Voltage-Measurements $\rightsquigarrow$ Neumann-to-Dirichlet map $\Lambda(\sigma)$

Inverse problem

Non-linear forward operator of EIT

$$\Lambda : \ \sigma \mapsto \Lambda(\sigma), \quad L^\infty_+(B) \to \mathcal{L}(L^2_\diamond(\partial B))$$

Inverse problem of EIT: $\Lambda(\sigma) \mapsto \sigma$?

Localized potentials $\rightsquigarrow$ Uniqueness for piecewise analytic conductivities

already known: Druskin (1982+85), Kohn/Vogelius (1984+85)

Linearization

Generic approach: Linearization

$$\Lambda(\sigma) - \Lambda(\sigma_0) \approx \Lambda'(\sigma_0)(\sigma - \sigma_0)$$

σ_0: known reference conductivity / initial guess / ...

$\Lambda'(\sigma_0)$: Fréchet-Derivative / sensitivity matrix.

$$\Lambda'(\sigma_0) : L^\infty_+(B) \to \mathcal{L}(L^2_\diamond(\partial B)).$$

$\rightsquigarrow$ Solve linearized equation for difference $\sigma - \sigma_0$.

Often: $\operatorname{supp}(\sigma - \sigma_0) \subset\subset B$ compact. *("shape" / "inclusion")*

Linearization

Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- Multiple possibilities to measure residual norm and to regularize.
- No rigorous theory for single linearization step.
- Almost no theory for Newton iteration:
  - Dobson (1992): (Local) convergence for regularized EIT equation.
  - Lechleiter/Rieder(2008): (Local) convergence for discretized setting.
  - No (local) convergence theory for non-discretized case!

Linearization

Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- Seemingly, no rigorous results possible for single linearization step.
- Seemingly, only justifiable for small $\sigma - \sigma_0$ (local results).

Here: Rigorous and global(!) result about the linearization error.

Exact Linearization

Theorem (**H.**/Seo, accepted to SIAM J. Math. Anal.)

Let κ, σ, σ_0 piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

(a) $\mathrm{supp}_{\partial B}\kappa = \mathrm{supp}_{\partial B}(\sigma - \sigma_0)$.

(b) $\frac{\sigma_0}{\sigma}(\sigma - \sigma_0) \leq \kappa \leq \sigma - \sigma_0$ on the bndry of $\mathrm{supp}_{\partial B}(\sigma - \sigma_0)$.

$\mathrm{supp}_{\partial B}$: outer support (= support, if support is compact and has conn. complement)

- Exact solution of lin. equation yields correct (outer) shape.
- No assumptions on $\sigma - \sigma_0$!

$\rightsquigarrow$ Linearization error does not lead to shape errors.

Proof: Combination of monotony and localized potentials.

Proof

- Exact linearization: $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$
- Monotony: For all "reference solutions" u_0:

$$\int_B (\sigma - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x$$

$$\geq \underbrace{\langle g, (\Lambda(\sigma) - \Lambda(\sigma_0))\, g\rangle}_{= \int_B \kappa |\nabla u_0|^2 \,\mathrm{d}x} \geq \int_B \frac{\sigma_0}{\sigma}(\sigma - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x.$$

- Use localized potentials to control $|\nabla u_0|^2$

$\rightsquigarrow$ $\mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$

Non-exact Linearization?

Theorem requires $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$.

- Existence of exact solution is unknown!
- In practice: finite-dimensional, noisy measurements

Proof only requires monotony estimate.

- Approximately solve linearized equ. s.t. estimate is still fulfilled

$\rightsquigarrow$ Globally convergent algorithm for non-exact linearization (not nicely implementable!)

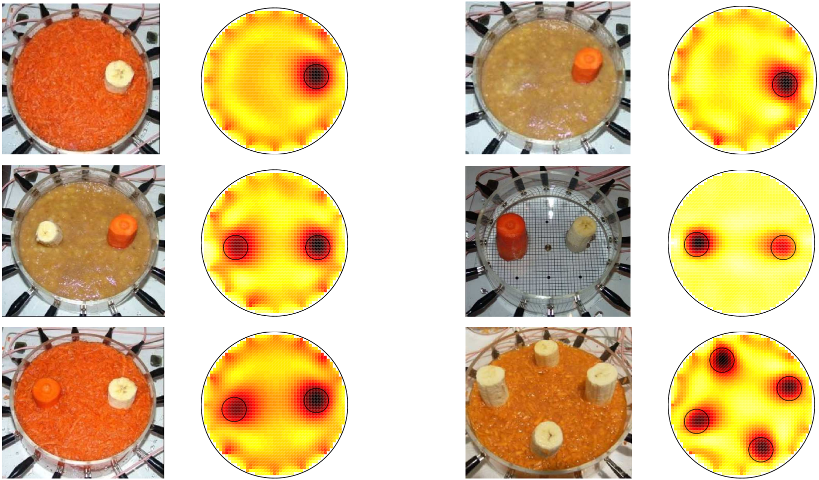
Open questions / ongoing work

- Obtain convergent algorithm in "nice form"
  (e.g., Tikhonov regularization with special penalty term)

Experimental result

- Similar results hold if reference measurement is taken at the same object but with another frequency
  ($\rightsquigarrow$ complex conductivites, weighted frequency differences)
- Next slide: Experimental tests of a *heuristic* combination of linearization and localized potentials.

Experimental result



(**H.**/Seo/Woo, to appear in IEEE Trans. Med. Imaging)

Conclusion

- Novel tomography techniques lead to mathematical parameter identification problems.
- Uniqueness questions may have non-trivial answers.

  In diffuse optical tomography:
  - Uniqueness for piecewise constant coefficients (even for piecw. linear/analytic)
  - No uniqueness for general smooth coefficients
- Close interplay between uniqueness arguments and convergent reconstruction algorithms.