Novel tomography techniques and parameter identification problems

Bastian von Harrach
harrach@ma.tum.de

Department of Mathematics - M1, Technische Universität München

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Overview

- Parameter identification problems
  - in diffuse optical tomography (DOT)

- Linearized reconstruction algorithms
  - in electrical impedance tomography (EIT)
Parameter identification problems in DOT
Diffuse optical tomography (DOT):

- Transilluminate biological tissue with visible or near-infrared light
- **Goal:** Reconstruct spatial image of interior physical properties.

  Relevant quantities (in diffusive regime):
  - Scattering
  - Absorption

- **Applications:**
  - Breast cancer detection
  - Bedside-imaging of neonatal brain function

Topical reviews:
Mathematical Model

- **General Forward Model:**
  Photon transport models (Boltzmann transport equation)

- **For highly scattering media:**
  - DC diffusion approximation for photon density $u$:
    \[-\nabla \cdot (a \nabla u) + cu = 0 \quad \text{in } B \subset \mathbb{R}^n,\]
    \[u : B \rightarrow \mathbb{R}: \text{photon density}\]
    \[a : B \rightarrow \mathbb{R}: \text{diffusion/scattering coefficient}\]
    \[c : B \rightarrow \mathbb{R}: \text{absorption coefficient}\]

- **Boundary measurements (idealized):**
  Neumann and Dirichlet data $u|_S$, $a \partial \nu u|_S$ on $S \subseteq \partial B$.
  Remaining boundary assumed to be insulated, $a \partial \nu u|_{\partial B \backslash S} = 0$. 

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Forward & inverse problem

DC diffuse optical tomography:

\[-\nabla \cdot (a \nabla u) + cu = 0 \quad \text{in } B \subset \mathbb{R}^n, \quad n \geq 2,\]

\(B\) bounded with smooth boundary, \(S \subseteq \partial B\) open part, \(a, c \in L^\infty(B)\).

\(\forall g \in L^2(S) \exists! \text{ solution } u \in H^1(B) : \quad \left. a \partial_\nu u \right|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{on } B \setminus \overline{S}. \end{cases}\)

\((\text{Local})\) Neumann-to-Dirichlet map

\[
\Lambda_{a,c} : g \mapsto u|_S, \quad L^2(S) \to L^2(S)
\]

is linear, compact and self-adjoint.

Inverse Problem: Can we reconstruct \(a\) and \(c\) from \(\Lambda_{a,c}\)?
Non-uniqueness

DC diffuse optical tomography:

\[-\nabla \cdot (a \nabla u) + cu = 0\]


- \( v := \sqrt{a}u \) solves
  \[-\Delta v + \eta v = 0, \quad \text{with} \quad \eta = \frac{\Delta \sqrt{a}}{\sqrt{a}} + \frac{c}{a}. \]

- \( a = 1 \) around \( S \)
  \( \leadsto \quad (u|_S, a \partial_\nu u|_S) = (v|_S, \partial_\nu v|_S). \)

\( \leadsto \quad \Lambda_{a,c} \) only depends on effective absorption \( \eta = \eta(a, c). \)

Absorption and scattering effects cannot be distinguished.

(Note: Argument requires smooth scattering coefficient \( a \)).
Experimental results

Theory: Absorption and scattering effects cannot be distinguished.

Practice:
Successful separate reconstructions of absorption and scattering (from phantom experiment using dc diffusion model!)
Pei et al. (2001), Jiang et al. (2002), Schmitz et al. (2002), Xu et al. (2002)

⇝ Practice contradicts theory!

Pei et al. (2001):

"As a matter of established methodological principle (...) empirical facts have the right-of-way; if a theoretical derivation yields a conclusion that is at odds with experimental results, the reconciliatory burden falls on the theorist, not on the experimentalist."
Theorem (H., Inverse Problems 2009)

- $a_1, a_2 \in L^\infty_+(B)$ piecewise constant
- $c_1, c_2 \in L^\infty_+(B)$ piecewise analytic

If $\Lambda_{a_1, c_1} = \Lambda_{a_2, c_2}$ then $a_1 = a_2$ and $c_1 = c_2$.

- Piecewise constantness seems fulfilled for phantom experiments.
- Result reconciles theory with practice.
- Measurements contain more than just the effective absorption!

Next slides: Idea of the proof using monotony and localized potentials.
Lemma
Let $a_1, a_2, c_1, c_2 \in L^\infty_+(B)$. Then for all $g \in L^2(S)$,

$$
\int_B \left( (a_2 - a_1)|\nabla u_1|^2 + (c_2 - c_1)|u_1|^2 \right) \, dx \\
\geq \langle (\Lambda a_1, c_1 - \Lambda a_2, c_2) g, g \rangle \\
\geq \int_B \left( (a_2 - a_1)|\nabla u_2|^2 + (c_2 - c_1)|u_2|^2 \right) \, dx,
$$

$u_1, u_2 \in H^1(B)$: solutions for $(a_1, c_1)$, resp., $(a_2, c_2)$.

Can we control $|u_j|^2$ and $|\nabla u_j|^2$?
Virtual Measurements

\[ L_\Omega : (H^1(\Omega))' \to L^2(S), \quad f \mapsto u|_S, \]

where \( u \in H^1(B) \) solves

\[
\int_B (a \nabla u \cdot \nabla v + cuv) \, dx = \langle f, v \rangle
\]

(essentially: \(-\nabla \cdot (a \nabla u) + cu = f\))
Virtual Measurements

- **Unique continuation:**
  If $\Omega_1 \cap \Omega_2 = \emptyset$, $B \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$ connected neighbourhood of $S$ then
  \[
  \mathcal{R}(L_{\Omega_1}) \cap \mathcal{R}(L_{\Omega_2}) = 0.
  \]

- **Dual operator:**
  \[
  L'_\Omega : L^2(S) \to H^1(\Omega), \quad g \mapsto u|_\Omega,
  \]
  where $u$ solves
  \[
  -\nabla \cdot (a \nabla u) + cu = 0, \quad a \partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}
  \]
Some functional analysis

**Lemma**
Let $X, Y$ be two reflexive Banach spaces, $A \in \mathcal{L}(X, Y)$, $y \in Y$. Then

$$y \in \mathcal{R}(A) \iff |\langle y', y \rangle| \leq C \|A'y'\| \quad \forall y' \in Y'.$$

**Corollary**
If $\|L'_{\Omega_1}g\| \leq C \|L'_{\Omega_2}g\|$ ∀ fluxes $g$, i.e., if $\|u|_{\Omega_1}\|_{H^1} \leq C \|u|_{\Omega_2}\|_{H^1}$ for the corresponding densities $u$, then $\mathcal{R}(L_{\Omega_1}) \subseteq \mathcal{R}(L_{\Omega_2})$.

**Contraposition**
$\mathcal{R}(L_{\Omega_1}) \not\subseteq \mathcal{R}(L_{\Omega_2}) \Rightarrow \exists (g_k)$ such that the solutions $(u_k)$ satisfy

$$\|u_k|_{\Omega_1}\|_{H^1(\Omega_1)} \to \infty \quad \text{and} \quad \|u_k|_{\Omega_2}\|_{H^1(\Omega_2)} \to 0.$$
Similarly
(by unique continuation and regularity)

\[ L(L^2(\Omega)) \subsetneq L(H^1(\Omega)^\prime). \]

\[ \exists (g_k) \text{ such that the solutions } (u_k) \text{ satisfy} \]

\[ \|u_k\|_{H^1(\Omega)} \to \infty \quad \text{and} \quad \|u_k\|_{L^2(\Omega)} \to 0. \]
Lemma There exist solutions \( u \) with

\[
\|u\|_{H^1(B \setminus \overline{O})} \text{ small}
\]

\[
\begin{cases}
\|u\|_{H^1(O)} \text{ large} \\
\|u\|_{L^2(O)} \text{ small}
\end{cases}
\]

\[
\|u\|_{L^2(O')} \text{ large}
\]
Localized potentials

**Lemma** There exist solutions \( u \) with

\[
\|u\|_{H^1(B \setminus \partial \Omega)} \text{ small} \quad \begin{cases} 
\|u\|_{H^1(\Omega)} \text{ large} \\
\|u\|_{L^2(\Omega)} \text{ small}
\end{cases}
\]

\[
\|u\|_{H^1(B \setminus \partial \Omega)} \text{ small} \quad \|u\|_{L^2(\Omega')} \text{ large}
\]
Proof of uniqueness

Monotony

\[
\int_B \left((a_2 - a_1)|\nabla u_1|^2 + (c_2 - c_1)|u_1|^2 \right) \, dx \\
\geq \langle (\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2})g, g \rangle \\
\geq \int_B \left((a_2 - a_1)|\nabla u_2|^2 + (c_2 - c_1)|u_2|^2 \right) \, dx,
\]

Proof of the uniqueness result (very sketchy . . .)

Start with region next to \( S \)

- Use loc. pot. with \(|\nabla u|^2 \to \infty\) in that region \( \rightsquigarrow a_1 = a_2 \)
- Then use loc. pot. with \(|u|^2 \to \infty\) in that region \( \rightsquigarrow c_1 = c_2 \)
- Repeat over all regions.
Uniqueness or not?

▶ Arridge/Lionheart (1998): Non-uniqueness for general smooth \((a, c)\).
▶ H. (2009): Uniqueness for piecew. constant \(a\), piecew. analytic \(c\).

What information about \((a, c)\) does \(\Lambda_{a,c}\) really contain?

Formally(!), \(\Lambda_{a,c}\) can only determine
\[
\eta = \frac{\Delta \sqrt{a}}{\sqrt{a}} + \frac{c}{a}.
\]

Jumps in \(a\) or \(\nabla a\) \(\mapsto\) distributional singularities in \(\Delta \sqrt{a}\).

**Bold guess:** Maybe \(\Lambda_{a,c}\) determines

▶ \(\eta\) where \(a\) and \(c\) are smooth,
▶ jumps in \(a\) and \(\nabla a\).

(However, note that \(\Delta \sqrt{a}/\sqrt{a}\) is not well-defined for non-smooth \(a\) . . . )
Theorem (H., submitted for publication)

Let \( a_1, a_2, c_1, c_2 \in \mathcal{L}_\infty^+(B) \) piecewise analytic on joint partition

\[
B = O_1 \cup \ldots \cup O_J \cup \Gamma, \quad \partial O_1 \cup \ldots \cup \partial O_J = \partial B \cup \Gamma.
\]

Then, \( \Lambda_{a_1, c_1} = \Lambda_{a_2, c_2} \) if and only if

(a) \( a_1|_S = a_2|_S \), and \( \partial_\nu a_1|_S = \partial_\nu a_2|_S \) on \( S \),

(b) \( \frac{\partial_\nu a_1}{a_1} \bigg|_{\partial B \setminus \bar{\sigma}} = \frac{\partial_\nu a_2}{a_2} \bigg|_{\partial B \setminus \bar{\sigma}} \) on \( \partial B \setminus \bar{\sigma} \),

(c) \( \eta_1 := \frac{\Delta \sqrt{a_1}}{a_1} + \frac{c_1}{a_1} = \frac{\Delta \sqrt{a_2}}{a_2} + \frac{c_2}{a_2} =: \eta_2 \) on \( B \setminus \Gamma \),

(d) \( \frac{a_1^+|_{\Gamma}}{a_1^-|_{\Gamma}} = \frac{a_2^+|_{\Gamma}}{a_2^-|_{\Gamma}} \), and \( \frac{[\partial_\nu a_2]|_{\Gamma}}{a_2^-|_{\Gamma}} = \frac{[\partial_\nu a_1]|_{\Gamma}}{a_1^-|_{\Gamma}} \) on \( \Gamma \).
Proof

Proof relies on more general monotony results, e.g.,

\[ \int_S g \left( \Lambda_{a_2, c_2} - \Lambda_{a_1, c_1} \right) g \, ds \]
\[ \leq \int_{B \setminus \Gamma} (\eta_1 - \eta_2) a_2 |u_2|^2 \, dx \]
\[ + \int_S \left( 1 - \frac{\sqrt{a_2}}{\sqrt{a_1}} \right) g u_2 \, ds - \int_{\partial B} \left( \frac{\partial_{\nu} a_1}{2a_1} - \frac{\partial_{\nu} a_2}{2a_2} \right) a_2 |u_2|^2 \, ds \]
\[ + \int_{\Gamma} \left\{ \frac{1}{2} \left( [\partial_{\nu} a_2]_{\Gamma} - \left[ \frac{a_2}{a_1} \partial_{\nu} a_1 \right]_{\Gamma} \right) |u_2|^2 - 2 \left[ \frac{\sqrt{a_2}}{\sqrt{a_1}} \right]_{\Gamma} a_1 \partial_{\nu} u_1 u_2 \right\} \, ds \]

Then localized potentials are used to control \( \|u\|_{H^1}, \|u\|_{L^2} \) on subsets and \( \|u|_\Sigma\|_{L^2} \) on boundary parts.
Linearized reconstruction algorithms in EIT
Electrical impedance tomography (EIT):

- **Apply currents** \( \sigma \partial_{\nu} u|_{\partial B} \) (Neumann boundary data)
  
  \( \rightsquigarrow \) Electric potential \( u \) solves
  \[
  \nabla \cdot (\sigma \nabla u) = 0 \quad \text{in} \; B
  \]

- **Measure voltages** \( u|_{\partial B} \) (Dirichlet boundary data)

**Current-Voltage-Measurements** \( \rightsquigarrow \) Neumann-to-Dirichlet map \( \Lambda(\sigma) \)
Inverse problem

Non-linear forward operator of EIT

\[ \Lambda : \sigma \mapsto \Lambda(\sigma), \quad L^\infty(B) \to \mathcal{L}(L^2_\diamond(\partial B)) \]

Inverse problem of EIT: \[ \Lambda(\sigma) \mapsto \sigma? \]

Localized potentials \( \rightsimeq \) Uniqueness for piecewise analytic conductivities

already known: Druskin (1982+85), Kohn/Vogelius (1984+85)

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Generic approach: Linearization

\[ \Lambda(\sigma) - \Lambda(\sigma_0) \approx \Lambda'(\sigma_0)(\sigma - \sigma_0) \]

\(\sigma_0\): known reference conductivity / initial guess / . . .

\(\Lambda'(\sigma_0)\): Fréchet-Derivative / sensitivity matrix.

\[ \Lambda'(\sigma_0) : L^\infty_+ (B) \to \mathcal{L}(L^2_0(\partial B)). \]

\(\leadsto\) Solve linearized equation for difference \(\sigma - \sigma_0\).

**Often:** \(\text{supp}(\sigma - \sigma_0) \subset\subset B\) compact. ("shape" / "inclusion")
Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- Multiple possibilities to measure residual norm and to regularize.
- No rigorous theory for single linearization step.
- Almost no theory for Newton iteration:
  - Dobson (1992): (Local) convergence for regularized EIT equation.
  - No (local) convergence theory for non-discretized case!
Linear reconstruction method

E.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- Seemingly, no rigorous results possible for single linearization step.
- Seemingly, only justifiable for small $\sigma - \sigma_0$ (local results).

**Here:** Rigorous and global(!) result about the linearization error.
Exact Linearization

**Theorem** (H./Seo, accepted to SIAM J. Math. Anal.)

Let $\kappa$, $\sigma$, $\sigma_0$ piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

(a) $\text{supp}_{\partial B} \kappa = \text{supp}_{\partial B}(\sigma - \sigma_0)$.

(b) $\frac{\sigma_0}{\sigma}(\sigma - \sigma_0) \leq \kappa \leq \sigma - \sigma_0$ on the boundary of $\text{supp}_{\partial B}(\sigma - \sigma_0)$.

$\text{supp}_{\partial B}$: outer support ( = support, if support is compact and has conn. complement)

- **Exact solution** of lin. equation yields correct (outer) shape.
- **No assumptions on** $\sigma - \sigma_0$!
- $\leadsto$ Linearization error does not lead to shape errors.

**Proof**: Combination of monotony and localized potentials.

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Proof

- Exact linearization: $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$

- Monotony: For all "reference solutions" $u_0$:

$$\int_B (\sigma - \sigma_0)|\nabla u_0|^2 \, dx \geq \langle g, (\Lambda(\sigma) - \Lambda(\sigma_0)) g \rangle \geq \int_B \frac{\sigma_0}{\sigma} (\sigma - \sigma_0)|\nabla u_0|^2 \, dx.$$

$$= \int_B \kappa |\nabla u_0|^2 \, dx$$

- Use localized potentials to control $|\nabla u_0|^2$

$\sim \text{supp}_{\partial \Omega} \kappa = \text{supp}_{\partial \Omega}(\sigma - \sigma_0)$
Non-exact Linearization?

Theorem requires $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$.

- Existence of exact solution is unknown!
- In practice: finite-dimensional, noisy measurements

Proof only requires monotony estimate.

- Approximately solve linearized equ. s.t. estimate is still fulfilled

$\sim\sim$ Globally convergent algorithm for non-exact linearization (not nicely implementable!)

Open questions / ongoing work

- Obtain convergent algorithm in ”nice form“ (e.g., Tikhonov regularization with special penalty term)
Similar results hold if reference measurement is taken at the same object but with another frequency (complex conductivities, weighted frequency differences).

Next slide: Experimental tests of a heuristic combination of linearization and localized potentials.
Experimental result


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Conclusion

- Novel tomography techniques lead to mathematical parameter identification problems.
- Uniqueness questions may have non-trivial answers.
  In diffuse optical tomography:
  - Uniqueness for piecewise constant coefficients (even for piecw. linear/analytic)
  - No uniqueness for general smooth coefficients
- Close interplay between uniqueness arguments and convergent reconstruction algorithms.