Localized potentials for elliptic inverse coefficient problems

Bastian von Harrach

harrach@ma.tum.de

Fakultät für Mathematik, M1, Technische Universität München, Germany

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The Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n \tag{1}$$

from all possible Dirichlet and Neumann boundary values

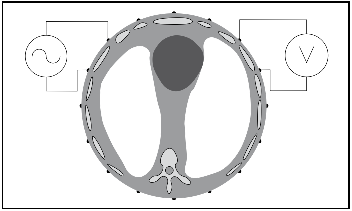
$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \ : \ u \text{ solves } (1)\}\,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator (NtD)**

$$\Lambda_\sigma : \ L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Application: EIT



Electrical impedance tomography (EIT):

- Apply currents $\sigma\partial_\nu u|_{\partial\Omega}$ (Neumann boundary data)
  - ⇝ Electric potential u in Ω (solution of $\nabla \cdot (\sigma \nabla u) = 0$)
- Measure voltages $u|_{\partial\Omega}$ (Dirichlet boundary data)

Current-Voltage-Measurements ⇝ Fin.-dim. approx. to Λ_σ

Partial data

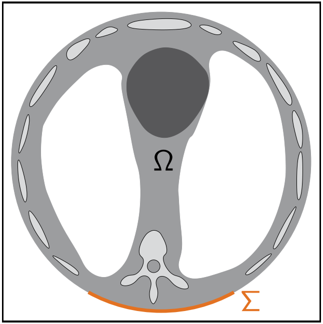
Measurements on open part of boundary $\Sigma \subset \partial\Omega$:
($\partial\Omega \setminus \Sigma$ is kept insulated.)

Recover σ from

$$\Lambda_\sigma : \; L^2_\diamond(\Sigma) \to L^2_\diamond(\Sigma), \quad g \mapsto u|_\Sigma,$$

where u solves $\nabla \cdot (\sigma \nabla u) = 0$ with

$$\sigma \partial_\nu u|_\Sigma = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$



Uniqueness results

- Measurements on complete boundary:

  Calderón (1980), Kohn/Vogelius (1984), Sylvester/Uhlmann (1987), Nachman (1996), Astala/Päivärinta (2006)

- Measurements on part of the boundary:

  Bukhgeim/Uhlmann (2002), Knudsen (2006), Isakov (2007), Kenig/Sjöstrand/Uhlmann (2007), **H**. *(2008), Imanuvilov/Uhlmann/Yamamoto (2009)*

In this talk:

A new method (*localized potentials*) to prove uniqueness results

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Monotony

For two conductivities $\sigma_0, \sigma_1 \in L^\infty(\Omega)$:

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x$$
$$\geq \int_\Sigma g\,(\Lambda_{\sigma_0} - \Lambda_{\sigma_1})\,g \,\mathrm{d}s \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x$$

for all solutions u_0 of

$$\nabla \cdot (\sigma_0 \nabla u_0) = 0, \quad \sigma_0 \partial_\nu u_0|_\Sigma = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$

(e.g., Kang/Seo/Sheen 1997, Kirsch 2005, Ide et al. 2007, ***H.****/Seo 2009+10)*

Can we control $|\nabla u_0|^2$?

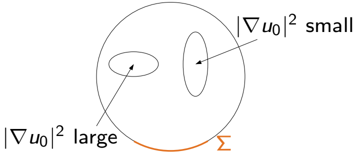
Theorem *(**H.**, 2008)*

Let σ_0 fulfill unique continuation principle (UCP),

$$\overline{D_1} \cap \overline{D_2} = \emptyset, \quad \text{and} \quad \Omega \setminus (\overline{D}_1 \cup \overline{D}_2) \text{ be connected with } \Sigma.$$

Then there exist solutions $u_0^{(k)}$, $k \in \mathbb{N}$ with

$$\int_{D_1} \left| \nabla u_0^{(k)} \right|^2 \, \mathrm{d}x \to \infty \quad \text{and} \quad \int_{D_2} \left| \nabla u_0^{(k)} \right|^2 \, \mathrm{d}x \to 0.$$

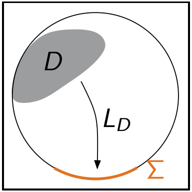


Proof 1/3

Virtual measurements:

$L_D: \; H^1_\diamond(D)' \to L^2_\diamond(\Sigma), \quad f \mapsto u|_\Sigma,$ with

$$\int_\Omega \sigma \nabla u \cdot \nabla v \, \mathrm{d}x = \langle f, v|_D \rangle \quad \forall v \in H^1_\diamond(D).$$



By (UCP): If $\overline{D_1} \cap \overline{D_2} = \emptyset$ and $\Omega \setminus (\overline{D}_1 \cup \overline{D}_2)$ is connected with Σ, then $\mathcal{R}(L_{D_1}) \cap \mathcal{R}(L_{D_2}) = 0$.

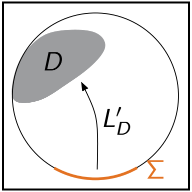
Sources on different domains yield different virtual measurements.

Proof 2/3

Dual operator:

$$L_D' : \; L^2_\diamond(\Sigma) \to H^1_\diamond(D), \quad g \mapsto u|_D, , \text{ with}$$

$$\nabla \cdot (\sigma \nabla u) = 0, \quad \sigma \partial_\nu u|_\Sigma = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$



Evaluating solutions on D is dual operation to virtual measurements.

Proof 3/3

Functional analysis:

X, Y_1, Y_2 reflexive Banach spaces, $L_1 \in \mathcal{L}(Y_1, X)$, $L_2 \in \mathcal{L}(Y_1, X)$.

$$\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2) \quad \Longleftrightarrow \quad \|L_1' x\| \lesssim \|L_2' x\| \ \forall x \in X'.$$

Here: $\mathcal{R}(L_{D_1}) \not\subseteq \mathcal{R}(L_{D_2}) \quad \Longrightarrow \quad \|u_0|_{D_1}\|_{H^1_\diamond} \not\lesssim \|u_0|_{D_2}\|_{H^1_\diamond}.$

If two sources do not generate the same data, then the respective evaluations are not bounded by each other.

Note: $H^1_\diamond(D)'$-source $\quad \longleftrightarrow \quad$ $H^1_\diamond(D)$-evaluation.

Consequences

- Back to Calderón problem: Let $\Lambda_{\sigma_0} = \Lambda_{\sigma_1}$, σ_0 fulfills (UCP).
- By monotony,

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x \geq 0 \geq \int_\Omega \frac{\sigma_0}{\sigma_1}(\sigma_1 - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x \quad \forall u_0$$

- Assume: $\exists$ neighbourhood U of Σ in which $\sigma_1 \geq \sigma_0$ but $\sigma_1 \not\equiv \sigma_0$

$\rightsquigarrow$ Potential with localized energy in U contradicts monotony.

Higher conductivity reachable by the bndry cannot be balanced out.

Corollary *(Kohn/Vogelius, 1984/85)*

Calderón problem is uniquely solvable for piecw.-anal. conductivities.

Two coefficients

Can we recover two coefficients $a, c \in L^\infty_+(\Omega)$ in

$$-\nabla \cdot (a \nabla u) + cu = 0 \quad \text{in } \Omega \tag{2}$$

from the NtD (with partial data)

$$\Lambda_{(a,c)}: \; L^2(\Sigma) \to L^2(\Sigma), \quad g \mapsto u|_\Sigma,$$

where u solves (2) with

$$\sigma \partial_\nu u|_\Sigma = \begin{cases} g & \text{on } \Sigma, \\ 0 & \text{else.} \end{cases}$$

Application: Diffuse optical tomography (DOT).

Monotony

$$\int_\Omega \left((a_2 - a_1)|\nabla u_1|^2 + (c_2 - c_1)|u_1|^2\right) \,\mathrm{d}x$$
$$\geq \int_\Sigma g\left(\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2}\right) g \,\mathrm{d}s$$
$$\geq \int_\Omega \left((a_2 - a_1)|\nabla u_2|^2 + (c_2 - c_1)|u_2|^2\right) \,\mathrm{d}x,$$

Method of localized potentials:

- Again, sources on different regions produce different data.
- $(H^1)'$-sources produce different data than L^2-sources

$$\Longrightarrow \quad \|u\|_{H^1(D)} \not\lesssim \|u\|_{L^2(D)}.$$

We can control $|\nabla u_1|^2$ and $|u_1|^2$ separately.

Uniqueness

Theorem *(**H.**, 2009)*

Let

- $a_1, a_2 \in L^\infty_+(\Omega)$ piecewise constant,
- $c_1, c_2 \in L^\infty_+(\Omega)$ piecewise analytic.

Then

$$\Lambda_{a_1,c_1} = \Lambda_{a_2,c_2} \quad \Longleftrightarrow \quad a_1 = a_2, \quad c_1 = c_2.$$

Note that $v := \sqrt{a}u$ transforms $-\nabla \cdot (a \nabla u) + cu = 0$ into

$$-\Delta v + \eta v = 0, \quad \eta := \frac{\Delta \sqrt{a}}{\sqrt{a}} + \frac{c}{a}$$

(when the coefficients are smooth).

Uniqueness

Theorem *(**H.**, submitted)*
Let $a_1, a_2, c_1, c_2 \in L^\infty_+(\Omega)$ be piecew. anal. Then $\Lambda_{(a_1,c_1)} = \Lambda_{(a_2,c_2)}$ if and only if

(a)	$a_1\|_\Sigma = a_2\|_\Sigma, \quad \partial_\nu a_1\|_\Sigma = \partial_\nu a_2\|_\Sigma$	on Σ,
(b)	$\frac{\partial_\nu a_1}{a_1}\|_{\partial B\setminus\overline{S}} = \frac{\partial_\nu a_2}{a_2}\|_{\partial B\setminus\overline{S}}$	on $\partial\Omega \setminus \Sigma$,
(c)	$\eta_1 = \eta_2$	in smooth regions,
(d)	$\frac{a_1^+\|_\Gamma}{a_1^-\|_\Gamma} = \frac{a_2^+\|_\Gamma}{a_2^-\|_\Gamma}, \quad \frac{[\partial_\nu a_2]_\Gamma}{a_2^-\|_\Gamma} = \frac{[\partial_\nu a_1]_\Gamma}{a_1^-\|_\Gamma}$	on inner boundaries Γ.

NtD $\Lambda_{(a,c)}$ determines $\eta = \frac{\Delta\sqrt{a}}{\sqrt{a}} + \frac{c}{a}$ and the jumps of a and ∇a.

Conclusions

Method of localized potentials

- relies on (UCP) and simple functional analytic duality principles,
- extends known uniqueness results on Calderón problem,
- yields uniqueness results for determination of two coefficients,
- requires local definiteness of the coefficients, e.g., piecw. anal.

Method is non-constructive, but can be used for

- local convergence of Newton algo. *(Lechleiter/Rieder, 2008)*
- shape reconstruction by single linearization step *(**H.**/Seo, 2010)*
- monotonicity based reconstruction algo. *(**H.**/Ullrich, in progress)*