Uniqueness in optical tomography

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Yonsei University, Seoul, South Korea, 22 September 2009.

Diffuse optical tomography

Diffuse optical tomography (DOT):

- Transilluminate biological tissue with visible or near-infrared light
- Goal: Reconstruct spatial image of interior physical properties.

  Relevant quantities (in diffusive regime):
  - Scattering
  - Absorption
- Applications:
  - Breast cancer detection
  - Bedside-imaging of neonatal brain function

Topical reviews

- Gibson, Hebden and Arridge (2005), Arridge (1999)

Mathematical Model

- General Forward Model:

  Photon transport models (Boltzmann transport equation)

  Recent review: Bal, Inverse Problems 25, 053001 (48pp), 2009.

- For highly scattering media:
  - DC diffusion approximation for photon density u:

    $$-\nabla \cdot (a\nabla u) + cu = 0 \quad \text{in } B \subset \mathbb{R}^n,$$

    $u: \quad B \to \mathbb{R}$: photon density

    $a: \quad B \to \mathbb{R}$: diffusion/scattering coefficient

    $c: \quad B \to \mathbb{R}$: absorption coefficient

  - Boundary measurements (idealized):

    Neumann and Dirichlet data $u|_S$, $a\partial_\nu u|_S$ on $S \subseteq \partial B$.

    Remaining boundary assumed to be insulated, $a\partial_\nu u|_{\partial B \setminus \overline{S}} = 0$.

Forward and inverse problem

DC diffuse optical tomography:

$$-\nabla \cdot (a\nabla u) + cu = 0 \quad \text{in } B \subset \mathbb{R}^n,\ n \geq 2,$$

B bounded with smooth boundary, $S \subseteq \partial B$ open part, $a, c \in L^\infty_+(B)$.

- $\forall g \in L^2(S) \quad \exists!$ solution $u \in H^1(B):\ a\partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{on } B \setminus \overline{S}. \end{cases}$
- (Local) Neumann-to-Dirichlet map

$$\Lambda_{a,c}:\ g \mapsto u|_S, \quad L^2(S) \to L^2(S)$$

is linear, compact and self-adjoint.

Inverse Problem: Can we reconstruct a and c from $\Lambda_{a,c}$?

Non-uniqueness

DC diffuse optical tomography:

$$-\nabla \cdot (a\nabla u) + cu = 0$$

Arridge/Lionheart (1998 Opt. Lett. 23 882–4):

- $v := \sqrt{a}u$ solves

$$-\Delta v + \eta v = 0, \quad \text{with} \quad \eta = \frac{\Delta\sqrt{a}}{\sqrt{a}} + \frac{c}{a}.$$

- $a = 1$ around S $\quad\rightsquigarrow\quad$ $(u|_S, a\partial_\nu u|_S) = (v|_S, \partial_\nu v|_S)$.

$\rightsquigarrow$ $\Lambda_{a,c}$ only depends on effective absorption $\eta = \eta(a, c)$.

Absorption and scattering effects cannot be distinguished.

(Note: Argument requires smooth scattering coefficient a).

Experimental results

Theory:

Absorption and scattering effects cannot be distinguished.

Practice:

Pei et al. (2001), Jiang et al. (2002), Schmitz et al. (2002), Xu et al. (2002) successfully reconstructed separate images of absorption and scattering (from phantom experiment using dc diffusion model!)

$\rightsquigarrow$ Practice contradicts theory!

Pei et al. (2001):

"As a matter of established methodological principle (...) empirical facts have the right-of-way; if a theoretical derivation yields a conclusion that is at odds with experimental results, the reconciliatory burden falls on the theorist, not on the experimentalist."

New uniqueness result

Theorem *(H., Inverse Problems 25, 055010 (14pp), 2009)*

- $a_1, a_2 \in L^\infty_+(B)$ piecewise constant
- $c_1, c_2 \in L^\infty_+(B)$ piecewise analytic

If $\Lambda_{a_1,c_1} = \Lambda_{a_2,c_2}$ then $a_1 = a_2$ and $c_1 = c_2$.

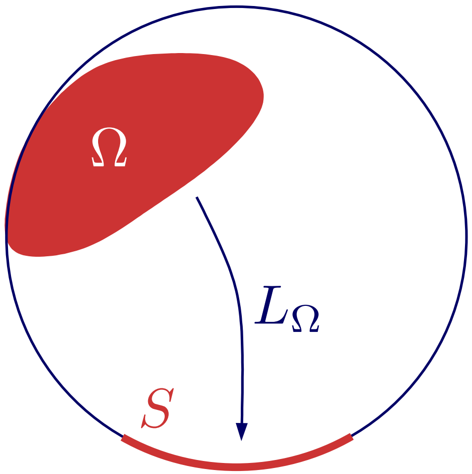
- Piecewise constantness seems fulfilled for phantom experiments.

$\rightsquigarrow$ Result reconciles theory with practice.

- Measurements contain more than just the effective absorption!

Next slides: Idea of the proof using localized potentials and monotony.

Virtual Measurements



$$L_\Omega : \ (H^1(\Omega))' \to L^2(S), \quad f \mapsto u|_S,$$

where $u \in H^1(B)$ solves

$$\int_B (a\nabla u \cdot \nabla v + cuv) \, \mathrm{d}x = \langle f, v \rangle$$

(essentially: $-\nabla \cdot (a\nabla u) + cu = f$)

- Unique continuation:
  If $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ connected neighbourhood of S then
  $$\mathcal{R}(L_{\Omega_1}) \cap \mathcal{R}(L_{\Omega_2}) = 0.$$
- Dual operator $L'_\Omega : \ L^2(S) \to H^1(\Omega), \quad g \mapsto u|_\Omega$, where u solves
  $$-\nabla \cdot (a\nabla u) + cu = 0, \qquad a\partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}$$

Some functional analysis

Lemma

Let X, Y be two reflexive Banach spaces, $A \in \mathcal{L}(X, Y)$, $y \in Y$. Then

$$y \in \mathcal{R}(A) \quad \text{iff} \quad |\langle y', y\rangle| \leq C \|A'y'\| \quad \forall y' \in Y'.$$

Corollary

If $\|L_{\Omega_1}' g\| \leq C \|L_{\Omega_2}' g\|$ for all applied fluxes g, i.e., $\|u|_{\Omega_1}\|_{H^1} \leq C \|u|_{\Omega_2}\|_{H^1}$ for the corresponding densities u, then $\mathcal{R}(L_{\Omega_1}) \subseteq \mathcal{R}(L_{\Omega_2})$.

Contraposition

$\mathcal{R}(L_{\Omega_1}) \not\subseteq \mathcal{R}(L_{\Omega_2}) \rightsquigarrow \exists (g_k)$ such that the solutions (u_k) satisfy

$$\|u_k|_{\Omega_1}\|_{H^1(\Omega_1)} \to \infty \quad \text{and} \quad \|u_k|_{\Omega_2}\|_{H^1(\Omega_2)} \to 0.$$

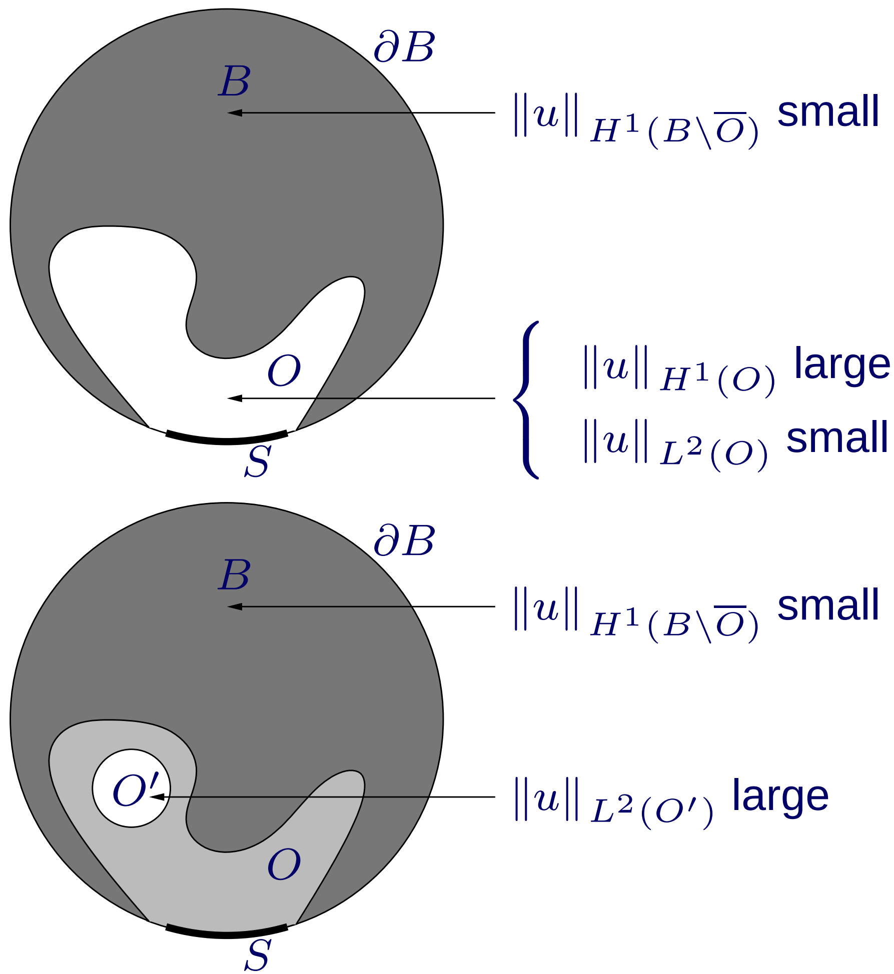
Similarly (by unique continuation and regularity)

$L(L^2(\Omega)) \subsetneq L(H^1(\Omega)') \rightsquigarrow \exists (g_k)$ such that the solutions (u_k) satisfy

$$\|u_k|_{\Omega}\|_{H^1(\Omega)} \to \infty \quad \text{and} \quad \|u_k|_{\Omega}\|_{L^2(\Omega)} \to 0.$$

Localized potentials

Lemma There exist solutions u with



Localized potentials

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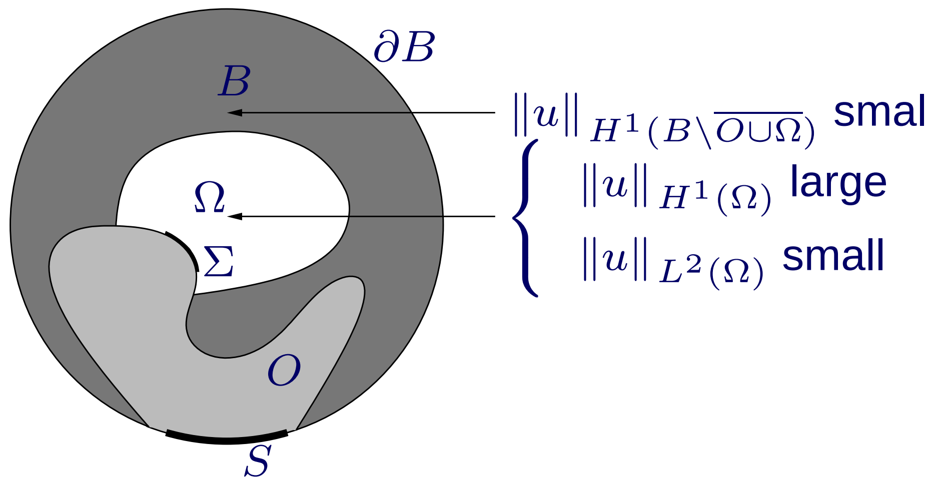

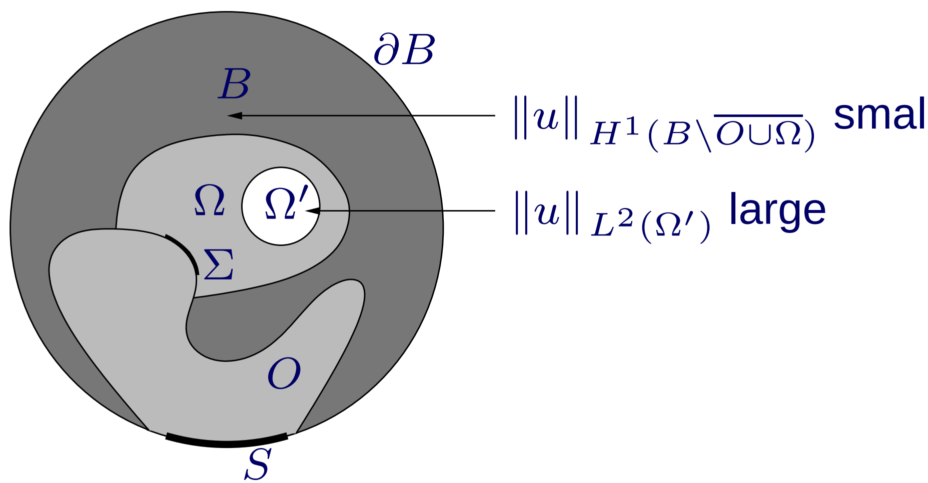

Monotony result

Lemma

Let $a_1, a_2, c_1, c_2 \in L^\infty_+(B)$. Then for all $g \in L^2(S)$,

$$\int_B \left((a_2 - a_1)|\nabla u_1|^2 + (c_2 - c_1)|u_1|^2\right) \,\mathrm{d}x$$

$$\geq \langle (\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2})g, g\rangle \geq \int_B \left((a_2 - a_1)|\nabla u_2|^2 + (c_2 - c_1)|u_2|^2\right) \,\mathrm{d}x,$$

$u_1, u_2 \in H^1(B)$: solutions for (a_1, c_1), resp., (a_2, c_2).

Proof of the uniqueness result (*very sketchy . . .*)

Start with region next to S

- Use loc. pot. with $|\nabla u|^2 \to \infty$ in that region $\rightsquigarrow a_1 = a_2$
- Then use loc. pot. with $|u|^2 \to \infty$ in that region $\rightsquigarrow c_1 = c_2$
- Repeat over all regions.

Uniqueness vs non-uniqueness

- Arridge/Lionheart (1998): Non-uniqueness for general smooth (a, c).
- H. (2009): Uniqueness for piecewise constant a, piecewise analytic c.

What information about (a, c) does $\Lambda_{a,c}$ really contain?

Formally(!), $\Lambda_{a,c}$ determines

$$\eta = \frac{\Delta\sqrt{a}}{\sqrt{a}} + \frac{c}{a}.$$

Jumps in a or ∇a $\rightsquigarrow$ distributional singularities in $\Delta\sqrt{a}$.

Bold guess: Maybe $\Lambda_{a,c}$ determines

- η where a and c are smooth,
- jumps in a and ∇a.

(However, note that $\Delta\sqrt{a}/\sqrt{a}$ is not well-defined for non-smooth a ...)

Exact characterization

Theorem *(H., submitted for publication)*

Let $a_1, a_2, c_1, c_2 \in L^\infty_+(B)$ piecewise analytic on joint partition

$$B = O_1 \cup \ldots \cup O_J \cup \Gamma, \qquad \partial O_1 \cup \ldots \cup \partial O_J = \partial B \cup \Gamma.$$

Then, $\Lambda_{a_1,c_1} = \Lambda_{a_2,c_2}$ if and only if

(a) $a_1|_S = a_2|_S$, and $\partial_\nu a_1|_S = \partial_\nu a_2|_S$ on S,

(b) $$\frac{\partial_\nu a_1}{a_1}|_{\partial B\setminus\overline{S}} = \frac{\partial_\nu a_2}{a_2}|_{\partial B\setminus\overline{S}}$$ on $\partial B \setminus \overline{S}$,

(c) $$\eta_1 := \frac{\Delta\sqrt{a_1}}{\sqrt{a_1}} + \frac{c_1}{a_1} = \frac{\Delta\sqrt{a_2}}{\sqrt{a_2}} + \frac{c_2}{a_2} =: \eta_2$$ on $B \setminus \Gamma$,

(d) $$\frac{a_1^+|_\Gamma}{a_1^-|_\Gamma} = \frac{a_2^+|_\Gamma}{a_2^-|_\Gamma},$$ and $$\frac{[\partial_\nu a_2]_\Gamma}{a_2^-|_\Gamma} = \frac{[\partial_\nu a_1]_\Gamma}{a_1^-|_\Gamma}$$ on Γ.

Proof

- Proof relies on more general monotony results, e.g.,

$$\begin{aligned}
&\int_S g\,(\Lambda_{a_2,c_2} - \Lambda_{a_1,c_1})\, g \,\mathrm{d}s \\
&\leq \int_{B\setminus\Gamma} (\eta_1 - \eta_2)\, a_2 |u_2|^2 \,\mathrm{d}x \\
&\quad + \int_S \left(1 - \frac{\sqrt{a_2}}{\sqrt{a_1}}\right) g u_2 \,\mathrm{d}s - \int_{\partial B} \left(\frac{\partial_\nu a_1}{2a_1} - \frac{\partial_\nu a_2}{2a_2}\right) a_2 |u_2|^2 \,\mathrm{d}s \\
&\quad + \int_\Gamma \left\{ \frac{1}{2} \left([\partial_\nu a_2]_\Gamma - \left[\frac{a_2}{a_1}\partial_\nu a_1\right]_\Gamma \right) |u_2|^2 - 2 \left[\frac{\sqrt{a_2}}{\sqrt{a_1}}\right]_\Gamma a_1 \partial_\nu u_1 u_2 \right\} \mathrm{d}s,
\end{aligned}$$

- Then localized potentials are used to control $\|u\|_{H^1}$, $\|u\|_{L^2}$ on subsets and $\|u|_\Sigma\|_{L^2}$ on boundary parts.

Conclusions

- DC intensity measurements in diffuse optical tomography determine *(for piecewise analytic coefficients)*
  - a combination of scattering and absorption coefficient (the ”effective absorption“),
  - jumps in the scattering coefficient and its derivative,
  - ⇝ Uniqueness for e.g. piecewise harmonic a and pcw. analytic c.
- Proof using localized potentials holds for
  - any dimension $n \geq 2$,
  - measurements on arbitrarily small part S of the boundary,
  - non-simply-connected domains (B may contain insulated holes).
- However, monotony arguments need ”local definiteness“
  - ⇝ no obvious extensions to C^∞-coefficients.