Exact shape-reconstruction by one-step linearization in electrical impedance tomography

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Mathematical Model

Forward operator of EIT:

$$\Lambda : \ \sigma \mapsto \Lambda(\sigma), \quad \text{"conductivity"} \mapsto \text{"measurements"}$$

- Conductivity: $\sigma \in L^\infty_+(\Omega)$
- Continuum model: $\Lambda(\sigma)$: Neumann-Dirichlet-operator

$$\Lambda(\sigma) : \ g \mapsto u|_{\partial\Omega}, \quad \text{"applied current"} \mapsto \text{"measured voltage"}$$

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega, \quad \sigma \partial_\nu u|_{\partial\Omega} = g \quad \text{on } \partial\Omega. \qquad (1)$$

- Linear elliptic PDE theory:

$$\forall g \in L^2_\diamond(\partial\Omega) \quad \exists! u \in H^1_\diamond(\Omega) \text{ solving (1).}$$

$$\Lambda(\sigma) : L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega) \text{ linear, compact, self-adjoint}$$

Inverse problem

Non-linear forward operator of EIT

$$\Lambda : \ \sigma \mapsto \Lambda(\sigma), \quad L^\infty_+(\Omega) \to \mathcal{L}(L^2_\diamond(\partial\Omega))$$

Inverse problem of EIT: $\Lambda(\sigma) \mapsto \sigma?$

Generic approach: Linearization

$$\Lambda(\sigma) - \Lambda(\sigma_0) \approx \Lambda'(\sigma_0)(\sigma - \sigma_0)$$

σ_0: known reference conductivity

$\Lambda'(\sigma_0)$: Fréchet-Derivative / sensitivity matrix.

$$\Lambda'(\sigma_0) : L^\infty_+(\Omega) \to \mathcal{L}(L^2_\diamond(\partial\Omega)).$$

Often: $\operatorname{supp}(\sigma - \sigma_0) \subset\subset \Omega$ compact. *(”shape“ / ”inclusion“)*

Linearization

Linear reconstruction method

e.g. NOSER (Cheney et al., 1990), GREIT (Adler et al., 2009)

Solve $\Lambda'(\sigma_0)\kappa \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, then $\kappa \approx \sigma - \sigma_0$.

- Multiple possibilities to measure residual norm and to regularize.
- No rigorous theory for single linearization step.
- Very little theory for Newton iteration.
- Seemingly, no rigorous results possible for single linearization step.
- Seemingly, only justifiable for small $\sigma - \sigma_0$.

In this talk: Rigorous and global(!) result about the linearization error.

Exact Linearization

Theorem *(**H.**/Seo, 2009)*

Let κ, σ, σ_0 piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

(a) $\mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

(b) $\frac{\sigma_0}{\sigma}(\sigma - \sigma_0) \leq \kappa \leq \sigma - \sigma_0$ on the bndry of $\mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

$\mathrm{supp}_{\partial\Omega}$: outer support (= supp, if supp is compact and has conn. complement)

- Exact solution of lin. equation yields correct (outer) shape.
- No assumptions on $\sigma - \sigma_0$!

$\rightsquigarrow$ Single-step linearization error does not affect shape reconstrution.

Proof: Combination of monotony and localized potentials.

Monotony

Monotony *(in the sense of quadr. forms)*:

$$\Lambda'(\sigma_0)(\sigma-\sigma_0) \leq \underbrace{\Lambda(\sigma)-\Lambda(\sigma_0)}_{=\Lambda'(\sigma_0)\kappa} \leq \Lambda'(\sigma_0)\left(\frac{\sigma_0}{\sigma}(\sigma-\sigma_0)\right).$$

Quadratic forms / energy formulation:

$$\int_{\partial\Omega} g\Lambda(\sigma_0)g\,\mathrm{d}s = \int_\Omega \sigma_0|\nabla u_0|^2\,\mathrm{d}x$$

$$\int_{\partial\Omega} g\Lambda(\sigma)g\,\mathrm{d}s = \int_\Omega \sigma|\nabla u|^2\,\mathrm{d}x$$

$$\int_{\partial\Omega} g\left(\Lambda(\sigma_0)'\kappa\right)g\,\mathrm{d}s = -\int_\Omega \kappa|\nabla u_0|^2\,\mathrm{d}x$$

u_0 (resp. u): solution corresponding to σ_0 (resp. σ) and bndry current g.

Bounds on squares

Exact linearization yields

$$\int_\Omega (\sigma - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x \geq \int_\Omega \kappa |\nabla u_0|^2 \,\mathrm{d}x \geq \int_\Omega \frac{\sigma_0}{\sigma}(\sigma - \sigma_0)|\nabla u_0|^2 \,\mathrm{d}x.$$

for all ”reference solutions“ u_0.

Does this imply

$$\sigma - \sigma_0 \geq \kappa \geq \frac{\sigma_0}{\sigma}(\sigma - \sigma_0) \text{ ?}$$

- Famous concept of inverse problems for PDEs:

  ”Completeness of products“ (of solutions of a PDE)

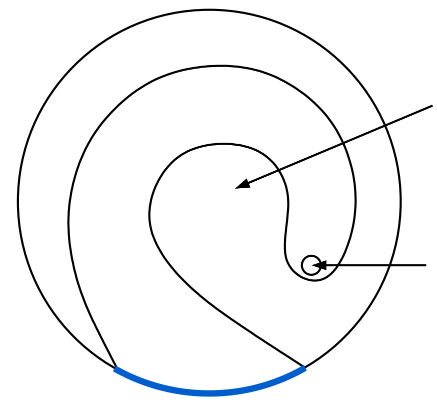
- Here: *”bounds on squares“* (of gradients of solutions of a PDE).

Localized potentials

$$\int_\Omega (\sigma - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x \geq \int_\Omega \kappa |\nabla u_0|^2 \, \mathrm{d}x \geq \int_\Omega \frac{\sigma_0}{\sigma}(\sigma - \sigma_0)|\nabla u_0|^2 \, \mathrm{d}x.$$

Localized potentials: ***(H.** 2008)*

Make $|\nabla u_0|^2$ arbitrarily large in a region connected to the boundary but keep it small outside the connecting domain.



$$\mathrm{supp}_{\partial\Omega} \frac{\sigma_0}{\sigma}(\sigma - \sigma_0) = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0) \quad \rightsquigarrow \quad \mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$$

Also: $\frac{\sigma_0}{\sigma}(\sigma - \sigma_0) \leq \kappa \leq \sigma - \sigma_0$ on the bndry of $\mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$

Non-exact Linearization?

Theorem ***(H./Seo, 2009)***
Let κ, σ, σ_0 piecewise analytic and $\Lambda'(\sigma_0)\kappa = \Lambda(\sigma) - \Lambda(\sigma_0)$. Then

(a) $\mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

(b) $\frac{\sigma_0}{\sigma}(\sigma - \sigma_0) \leq \kappa \leq \sigma - \sigma_0$ on the bndry of $\mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

- Existence of exact solution is unknown!
- In practice: finite-dimensional, noisy measurements.

Proof only requires

$$\Lambda'(\sigma_0)(\sigma - \sigma_0) \leq \Lambda'(\sigma_0)\kappa \leq \Lambda'(\sigma_0)\left(\frac{\sigma_0}{\sigma}(\sigma - \sigma_0)\right). \qquad (*)$$

$\leadsto$ Solve linearized equation s.t. (*) is fulfilled.

Non-exact Linearization

Additional definiteness assumption: $\sigma \geq \sigma_0$.

Assume we are given

- Noisy data $\tilde{\Lambda}_m(\sigma) - \tilde{\Lambda}_m(\sigma_0) \to \Lambda(\sigma) - \Lambda(\sigma_0)$
- Noisy sensitivity $\tilde{\Lambda}_m'(\sigma_0) \to \Lambda'(\sigma_0)$.
- Finite-dim. subspace $V_1 \subset V_2 \subset \ldots \subset L^2_\diamond(\partial\Omega)$ with dense union.

Equip V_k with norm

$$\|g\|^2_{(m)} := \langle (\tilde{\Lambda}_m(\sigma) - \tilde{\Lambda}_m(\sigma_0))g, g\rangle.$$

Minimize (Galerkin approx. of) linearization residual

$$\tilde{\Lambda}(\sigma) - \tilde{\Lambda}(\sigma_0) - \tilde{\Lambda}'(\sigma_0)\kappa_m$$

in the sense of quadratic forms on V_k.

Non-exact Linearization

Theorem *(**H.**/Seo, 2009)*
For appropriately chosen $\delta_1, \delta_2 > 0$, every V_k and suff. large m,

$$\exists \kappa_m : \quad -\delta_1 \leq \tilde{\Lambda}(\sigma) - \tilde{\Lambda}(\sigma_0) - \tilde{\Lambda}'(\sigma_0)\kappa_m \leq \delta_2.$$

(in the sense of quadr. forms on V_k, κ_m piecewise analytic)

Every piecewise analytic L^∞-limit κ of a converging subsequence fulfills

(a) $\quad \mathrm{supp}_{\partial\Omega}\kappa = \mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

(b) $\quad \left(\frac{\sigma_0}{\sigma} - \delta_1\right)(\sigma - \sigma_0) \leq \kappa \leq (\delta_2 + 1)(\sigma - \sigma_0)$ on bndry of $\mathrm{supp}_{\partial\Omega}(\sigma - \sigma_0)$.

Convergence guaranteed if $\sigma - \sigma_0$ belongs to fin-dim. ansatz space.

$\rightsquigarrow$ *Globally convergent shape reconstruction by one-step linearization.*

Summary and open questions

- The linearization error in EIT does not affect the shape.
- With additional definiteness assumption, we derived a

  local one-step linearization algorithm

  with *globally convergent* shape reconstruction properties.
- Additional definiteness property is typical for shape reconstruction.

Open questions

- Numerical implementation?
- Formulation as Tikhonov regularization with special norms?
- Definiteness only enters in V_k-norm. Can this be replaced by other oszillation-preventing regularization?