EIT Lung Monitoring using the Factorization Method

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Mathematical model

EIT model using point electrodes and ”adjacent-adjacent” current patterns

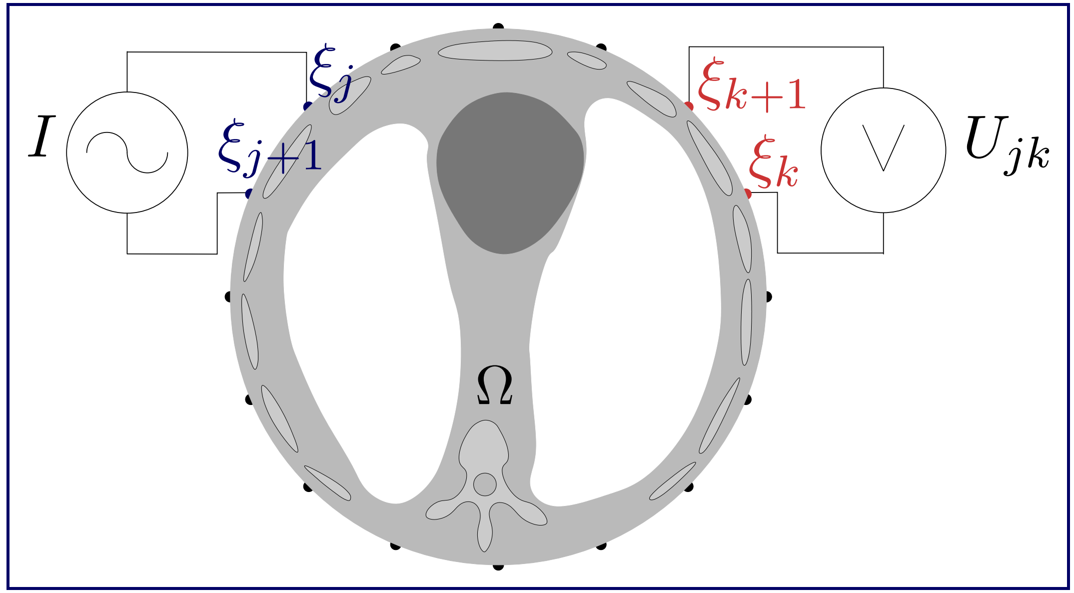
$$\nabla \cdot (\sigma(x) \nabla u_j(x)) = 0 \quad \text{in } \Omega$$

$$\sigma(x) \partial_\nu u_j(x)|_{\partial\Omega}$$

$$= I\delta(x - \xi_j) - I\delta(x - \xi_{j+1})$$

Measured voltage:

$$U_{jk} := u_j(\xi_{k+1}) - u_j(\xi_k)$$



I : applied current between electrodes ξ_j und ξ_{j+1} (here: fix $I = 1$),

$\sigma(x)$: conductivity,

$u_j(x)$: resulting electrical potential.

Measurements

U_{jk}: Voltage between k and $k+1$-th electrode needed to maintain current of $I = 1$ mA between j and $j+1$-th electrode.

$$\mathbb{U} = \begin{pmatrix} U_{1,1} & U_{1,2} & \cdots & U_{1,N} \\ U_{2,1} & U_{2,2} & \cdots & U_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ U_{N,1} & U_{N,2} & \cdots & U_{N,N} \end{pmatrix}$$

No measurements at current carrying electrodes

$\rightsquigarrow$ $U_{j,j-1}$, $U_{j,j}$, $U_{j,j+1}$ missing ($j = 1, \ldots, N$).

Reciprocity principle: $U_{j,k} = U_{k,j}$

$\rightsquigarrow$ only $16 \cdot (16-3)/2 = 104$ non-redundant entries.

tdEIT

In practice:

- Inevitable modelling errors (body shape, electrode position,...)

$\rightsquigarrow$ Reduction of model dependance by using a reference measurements with the same systematic errors

$\rightsquigarrow$ Time-difference EIT: Reconstruct $\sigma_1 - \sigma_0$ from measurements $\mathbb{U}^{(1)} - \mathbb{U}^{(0)}$ at different times, (e.g. $\mathbb{U}^{(0)}$: exhaled state).

Factorization method *(Kirsch 1998 for inverse scattering)*:

- reconstructs $\mathrm{supp}(\sigma_1 - \sigma_0)$ from Neumann-Dirichlet-maps, $\Lambda_1 - \Lambda_0$, i.e. from infinite-dimensional analogons of $\mathbb{U}^{(1)} - \mathbb{U}^{(0)}$.

FM for EIT (1999–2009):

Brühl, Hakula, Hanke, H., Hyvönen, Kirsch, Lechleiter, Nachman, Päivärinta, Pursiainen, Schappel, Schmitt, Seo, Teirilä

FM for real data

FM relies on infinite-dimensional NtDs

- $\mathbb{U}^{(1)}$, $\mathbb{U}^{(0)}$ approximate NtDs for large number of electrodes
- Some approximation results for the FM available.

(Theory: Lechleiter, Hyvönen, Hakula)

However,

- Practitioners keep electrode number small due to ill-posedness (”regularization by discretization”).
- Practitioners do not like infinite-dimensional arguments.
- No convergence theory for the FM („threshold choosing problem”)!

In this talk:

- Physical justificiation of the FM in a realistic (discrete) setting
- First results of the FM for human lung data (myself)

Two-phase lung model

For this talk: constant lung conductivity: $1+\sigma$, $(\sigma > -1)$,

constant background conductivity: 1.

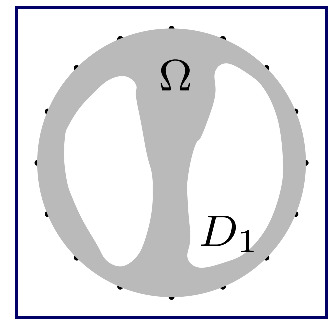
Analogous results for inhomogenous (but known!) background or $\sigma = \sigma(x,t)$.

(As long as lung is always less conductive than background.)

Measurements at current state $\mathbb{U}^{(1)}$:

Conductivity $\sigma_1 = 1 + \sigma\chi_{D_1}$,

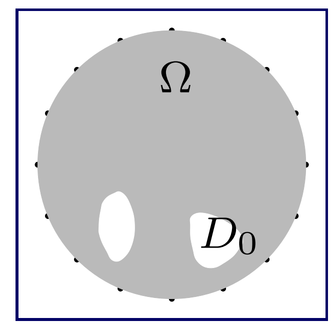
Current pattern $g \in \mathbb{R}^{16}$ generates voltage $u_g^{(1)}$.



Measurements at exhaled state $\mathbb{U}^{(0)}$:

Conductivity $\sigma_0 = 1 + \sigma\chi_{D_0}$, $\overline{D_0} \subset D_1$,

Current pattern $g \in \mathbb{R}^{16}$ generates voltage $u_g^{(0)}$.



Bilinear form

- Useful identity:

$$g \cdot (\mathbb{U}^{(0)} - \mathbb{U}^{(1)})h = \int_{D_1 \setminus D_0} \sigma \nabla u_g^{(1)} \cdot \nabla u_h^{(0)} \, \mathrm{d}x$$

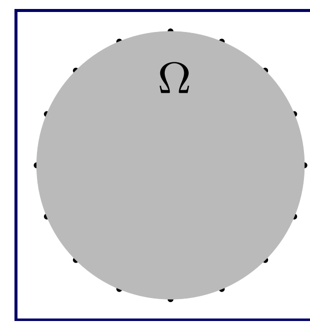
- Linearisation: $u_g^{(1)} \approx u_g^{(0)} \approx u_g^{(\mathrm{hom})}$

$$g \cdot (\mathbb{U}^{(0)} - \mathbb{U}^{(1)})h \approx \int_{D_1 \setminus D_0} \sigma \nabla u_g^{(\mathrm{hom})} \cdot \nabla u_h^{(\mathrm{hom})} \, \mathrm{d}x$$

(Virtual) background measurements:

Conductivity $\sigma_{\mathrm{hom}} = 1$,

Current pattern $g \in \mathbb{R}^{16}$ generates voltage $u_g^{(\mathrm{hom})}$.



Dipole Function

- Dipole measurements

$$\Phi_{z,d} = (\varphi_{z,d}(\xi_{k+1}) - \varphi_{z,d}(\xi_k))_{k=1}^{16}, \text{ where } \begin{cases} \Delta\varphi_{z,d} &= d\cdot\nabla\delta_z, \\ \partial_\nu\varphi_{z,d}|_{\partial\Omega} &= 0. \end{cases}$$

- Scalar products

$$g\cdot\Phi_{z,d} = d\cdot\nabla u_g^{(\mathrm{hom})}(z).$$

- Dipole preimage $h_{z,d} = (\mathbb{U}^{(0)} - \mathbb{U}^{(1)})^{-1}\Phi_{z,d}$:

$$d\cdot\nabla u_g^{(\mathrm{hom})}(z) = g\cdot\Phi_{z,d} = g\cdot(\mathbb{U}^{(0)} - \mathbb{U}^{(1)})h_{z,d}$$

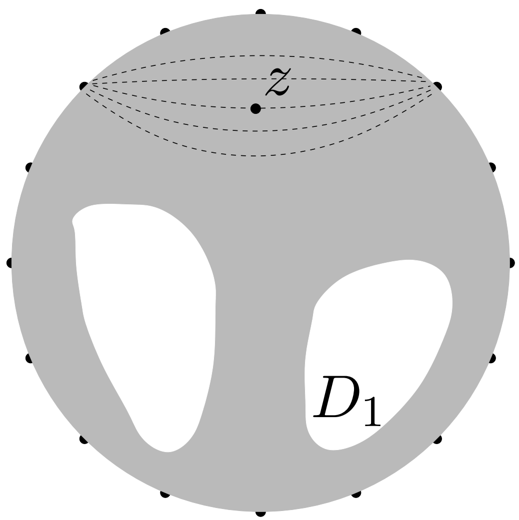
$$\approx \int_{D_1\setminus D_0} \sigma\nabla u_g^{(\mathrm{hom})}(x)\cdot\nabla u_{h_{z,d}}^{(\mathrm{hom})}(x)\,\mathrm{d}x$$

must hold for all applied current patterns $g\in\mathbb{R}^{16}$.

Localization

Dipole preimage $h_{z,d} = (\mathbb{U}^{(0)} - \mathbb{U}^{(1)})^{-1}\Phi_{z,d}$:

$$d \cdot \nabla u_g^{(\mathrm{hom})}(z) \approx \int_{D_1 \setminus D_0} \sigma \nabla u_g^{(\mathrm{hom})}(x) \cdot \nabla u_{h_{z,d}}^{(\mathrm{hom})}(x)\, \mathrm{d}x \qquad \forall g$$



$z \notin D_2$ "well-separated" from D_1:
(large current in z, little current through D_1)

$$\rightsquigarrow \|\nabla u_{h_{z,d}}^{(\mathrm{hom})}(x)\|_{L^2(D_1 \setminus D_0)} \text{ very large.}$$

For $z \in D_1$ one can show (in $\mathbb{R}^2$)

$$\|\nabla u_{h_{z,d}}^{(\mathrm{hom})}(x)\|_{L^2(D_1 \setminus D_0)} \lesssim \frac{1}{\mathrm{dist}(z, \partial D_1)}$$

Plotting $z \mapsto \|\nabla u_{h_{z,d}}^{(\mathrm{hom})}(x)\|_{L^2(D_1 \setminus D_0)}$ shows D_1.

Factorization Method

Plotting $z \mapsto \|\nabla u_{h_{z,d}}^{(\mathrm{hom})}(x)\|_{L^2(D_1 \setminus D_0)}$ shows D_1.

Up to multiplicative constants,

$$\|\nabla u_{h_{z,d}}^{(\mathrm{hom})}(x)\|^2_{L^2(D_1 \setminus D_0)}$$
$$\eqsim \int_{D_1 \setminus D_0} \sigma |\nabla u_{h_{z,d}}^{(\mathrm{hom})}(x)|^2 \, \mathrm{d}x \approx h_{z,d} \cdot (\mathbb{U}^{(0)} - \mathbb{U}^{(1)}) h_{z,d}$$
$$= \left| (\mathbb{U}^{(0)} - \mathbb{U}^{(1)})^{-1/2} \Phi_{z,d} \right|^2$$

Plotting $\left| (\mathbb{U}^{(0)} - \mathbb{U}^{(1)})^{-1/2} \Phi_{z,d} \right|^2$ shows D_1. (Factorization Method)

Up to multiplicative constants everything holds without linearisation!

Physical justification

FM indicator: $z \mapsto \left|(\mathbb{U}^{(0)} - \mathbb{U}^{(1)})^{-1/2}\Phi_{z,d}\right|^2$.

Physical justification of the FM *(H., Seo, Woo)*:

Plot of FM indicator distinguishes object from well-separated points.

Well-separated points are those in which the current can be made large without making it large in the object.

- Justifies FM for realistic, discrete settings
- Consistent with continuous setting, where current can be concentrated everywhere in object's connected complement *(H. '08)*.

Real data

Reconstructions for real data measured on human lung *(Gisa, H.)*: