# Localized potentials and reconstruction methods for EIT

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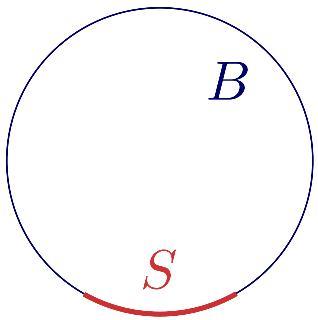
# Overview

- Motivation
- Existence of localized potentials
- Construction and numerical examples
- Detecting inclusions in EIT

# Motivation

# Electrical impedance tomography




| | |
|---|---|
| $B$: | bounded domain |
| $S \subseteq \partial B$: | relatively open subset |
| $\sigma \in L^\infty_+(B)$: | electrical conductivity in $B$ |
| $g \in L^2_\diamond(S)$: | applied current on $S$ |

$\rightsquigarrow$ Electric potential $u \in H^1_\diamond(B)$ that solves

$$\nabla \cdot \sigma \nabla u = 0, \qquad \sigma \partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}$$

EIT: Measure $u|_S$ for one or several input currents $g$ and reconstruct (properties of) $\sigma$ from it.

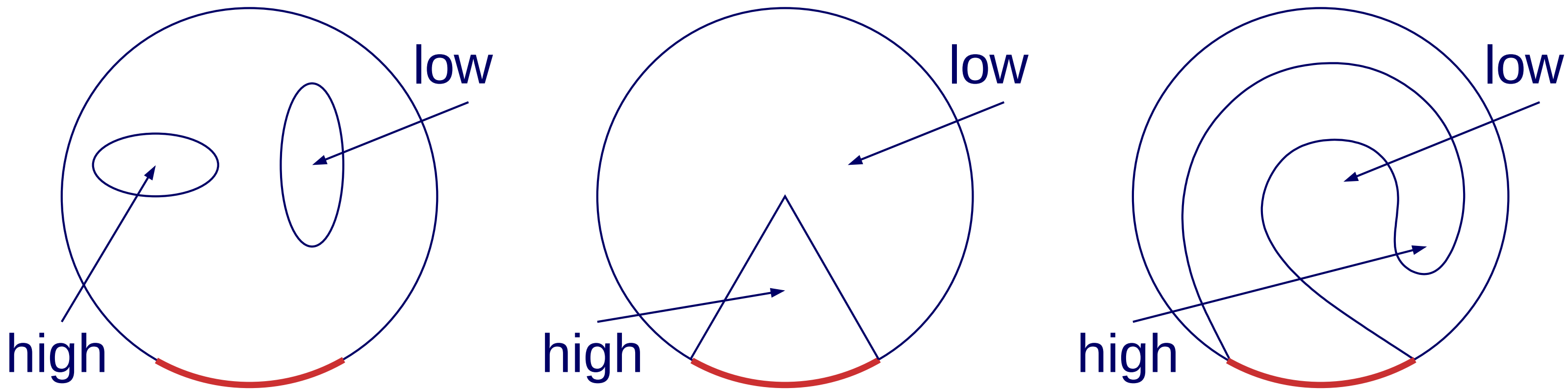
Regularity assumption:

$\sigma$ satisfies (UCP) in connected neighbourhoods $U$ of $S$, i.e.,

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } U, \quad \begin{cases} u|_S = 0,\ \sigma \partial_\nu u|_S = 0 & \Longrightarrow \quad u = 0. \\ u|_V = \text{const.}, V \subset U \text{ open} & \Longrightarrow \quad u = \text{const.} \end{cases}$$

# Localized potentials

*Can we localize (the energy of) the potentials in given subsets?*




Restrictions:

- High energy parts have to be connected to the boundary.
- Because of (UCP) zero energy parts are not possible.

⇝ Goal: sequences $(g_n)$ such that energy of $(u_n)$ diverges on some subset while tending to zero on another.

Energy of a potential $u$: $\int_\Omega \sigma |\nabla u|^2 \, \mathrm{d}x \approx \int_\Omega |\nabla u|^2 \, \mathrm{d}x \approx \|u\|^2_{H^1_\diamond(\Omega)}$

# Theoretical motivation

Calderon problem with partial data:

Is $\sigma$ uniquely determined by the (local) current-to-voltage map

$$\Lambda_\sigma : \; L^2_\diamond(S) \to L^2_\diamond(S), \quad g \mapsto u|_S \text{ ?}$$

For measurements on whole boundary $S = \partial B$:

- Identifiabilty question posed by Calderon 1980.
- For smooth $\sigma$ answered positively by Sylvester and Uhlmann 1987 for $n \geq 3$ and by Nachmann 1996 for $n = 2$.
- For $n = 2$ and general $\sigma \in L^\infty_+$ answered positively by Astala and Päivärinta 2006.
- Still an open question for general $\sigma \in L^\infty_+$ (with or without (UCP)) for $n \geq 3$.

# Theoretical motivation

Connection between Calderon problem (with $S \subseteq \partial B$) and localized potentials:

- Monotonicity property:

  Let $u_1$, $u_2$ be electric potentials for conductivities $\sigma_1$, $\sigma_2$ created by the same boundary current $g \in L^2_\diamond(S)$. Then

$$\int_B (\sigma_1 - \sigma_2)|\nabla u_2|^2 \, \mathrm{d}x \geq ((\Lambda_{\sigma_2} - \Lambda_{\sigma_1})g, g) \geq \int_B (\sigma_1 - \sigma_2)|\nabla u_1|^2 \, \mathrm{d}x.$$

$\rightsquigarrow$ If $\sigma_1 - \sigma_2 > 0$ in some region where we can localize the electric energy $|\nabla u_1|^2$ then $\Lambda_{\sigma_1} \neq \Lambda_{\sigma_2}$.

*"A higher conductivity in such a region can not be balanced out."*

# Known results on loc. potentials

- Potential can be concentrated around $z \in S$ if $\sigma \in C^2$ around $z$.

  Using Runge's approximation property the high energy part can be "shifted" into the interior of $B$.
  *(Kohn, Vogelius 1984+85)*

$\rightsquigarrow$ Piecewise analytic $\sigma$ is determined by local voltage-to-current map.

- For dimension $n \geq 3$ and additional condition on boundary parts, $C^2$-conductivities $\sigma$ are determined by local voltage-to-current map.
  *(Kenig, Sjöstrand, Uhlmann 2007)*

- If boundary part is part of a plane or sphere,

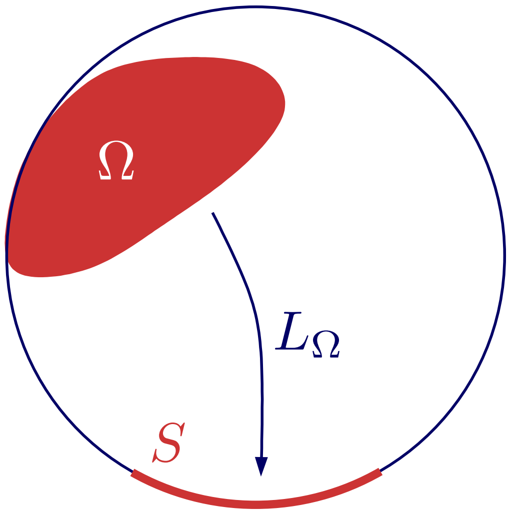
  $C^2$-conductivities $\sigma$ are determined by the current-to-voltage map.
  *(Isakov 2007)*

In this talk:

- Localized potentials exist for arbitrary $\sigma \in L^\infty_+$ that fulfill (UCP).

# Existence of localized potentials

# Virtual Measurements




$f \in (H^1_\diamond(\Omega))'$: applied source on $\Omega$

$$L_\Omega : \ (H^1_\diamond(\Omega))' \to L^2_\diamond(S), \quad f \mapsto u|_S,$$

where $u \in H^1_\diamond(B)$ solves

$$\int_B \sigma \nabla u \cdot \nabla v \, \mathrm{d}x = \langle f, v|_\Omega \rangle \quad \text{for all } v \in H^1_\diamond(B).$$

If $\overline{\Omega} \subset B$: $\quad \nabla \cdot \sigma \nabla u = f, \quad \sigma \partial_\nu u|_{\partial B} = 0.$

(UCP) yields: If $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ is connected and its boundary contains $S$ then $\mathcal{R}(L_{\Omega_1}) \cap \mathcal{R}(L_{\Omega_2}) = 0$.

Dual operator $L'_\Omega : \ L^2_\diamond(S) \to H^1_\diamond(\Omega), \quad g \mapsto u|_\Omega$, where $u$ solves

$$\nabla \cdot \sigma \nabla u = 0, \qquad \sigma \partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}$$

# Some functional analysis

**Lemma**

Let $X, Y$ be two reflexive Banach spaces, $A \in \mathcal{L}(X,Y)$, $y \in Y$. Then

$$y \in \mathcal{R}(A) \quad \text{iff} \quad |\langle y', y\rangle| \leq C\,\|A'y'\| \quad \forall y' \in Y'.$$

**Corollary**

If $\|L'_{\Omega_1} g\| \leq C\,\|L'_{\Omega_2} g\|$ for all applied currents $g$, i.e., $\|u|_{\Omega_1}\| \leq C\,\|u|_{\Omega_2}\|$ for the corresponding potentials $u$, then $\mathcal{R}(L_{\Omega_1}) \subseteq \mathcal{R}(L_{\Omega_2})$.

**Contraposition**

If $\mathcal{R}(L_{\Omega_1}) \not\subseteq \mathcal{R}(L_{\Omega_2})$ then there exist currents $(g_n)$ such that the corresponding potentials $(u_n)$ satisfy

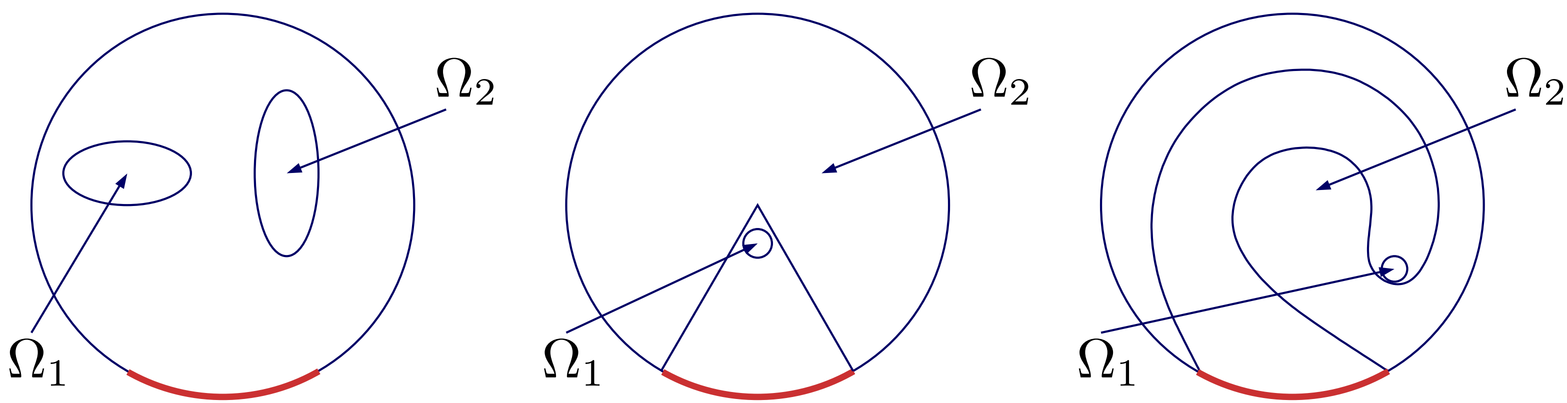
$$\|u_n|_{\Omega_1}\|_{H^1_\diamond(\Omega_1)} \to \infty \quad \text{and} \quad \|u_n|_{\Omega_2}\|_{H^1_\diamond(\Omega_2)} \to 0.$$

# Existence of localized potentials

Theorem

If $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ is connected and its boundary contains $S$, then there exists currents $(g_n)$ such that (the energy of) the corresponding potentials $(u_n)$ diverges on $\Omega_1$ while tending to zero on $\Omega_2$, i.e.,

$$\int_{\Omega_1} \sigma |\nabla u_n|^2 \, \mathrm{d}x \to \infty, \quad \text{and} \quad \int_{\Omega_2} \sigma |\nabla u_n|^2 \, \mathrm{d}x \to 0.$$



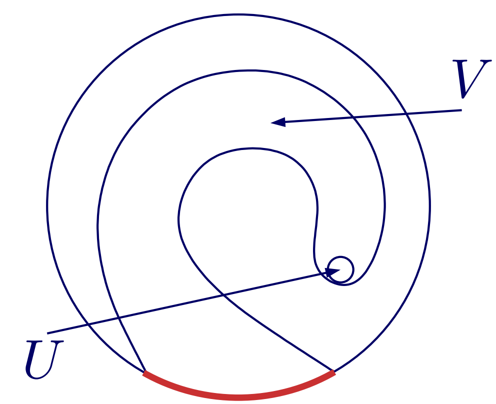

*Result uses only ellipticity properties, thus also holds e.g. for linear elasticity, electro- and magnetostatics.*

# Theoretical consequence

Corollary

Let $\sigma_1, \sigma_2 \in L^\infty_+(B)$ satisfy (UCP) and $\Lambda_{\sigma_1}$, $\Lambda_{\sigma_2}$ be the corresponding current-to-volage-maps.

If $\sigma_2 \geq \sigma_1$ in some neighbourhood $V$ of $S$ and $\sigma_2 - \sigma_1 \in L^\infty_+(U)$ for some open $U \subseteq V$ then there exists $(g_n)$ such that




$$\langle (\Lambda_{\sigma_2} - \Lambda_{\sigma_1}) g_n, g_n \rangle \to \infty,$$

so in particular $\Lambda_{\sigma_2} \neq \Lambda_{\sigma_1}$.

Consequences (already known from the Kohn-Vogelius result):

- $\sigma|_S$ and its derivatives on $S$ are uniquely determined by $\Lambda_\sigma$.
- Piecewise analytic conductivities $\sigma$ are uniquely determined by $\Lambda_\sigma$.

# More practical consequence

Lechleiter, Rieder (2008):

- studied EIT with more realistic electrode model (CEM) in discrete fem-setting
- used localized potentials to show injectivity of Frechét derivative of the conductivity-to-data-map
- deduced so-called ”tangential cone condition”

$\rightsquigarrow$ convergence of regularized Newton-like reconstruction algorithms

# Construction

# Construction

More constructive version of the functional analysis:

Let $h \in \mathcal{R}(L_{\Omega_1})$, $h \notin \mathcal{R}(L_{\Omega_2})$. Define $\gamma_\alpha \in L^2_\diamond(S)$ by

$$\gamma_\alpha = (L_{\Omega_2} L_{\Omega_2}^* + \alpha I)^{-1} h, \qquad \alpha > 0.$$

Then

$$\|L'_{\Omega_2}\gamma_\alpha\|^2 \leq C\,\|L'_{\Omega_1}\gamma_\alpha\| \quad \text{and} \quad \|L'_{\Omega_2}\gamma_\alpha\| \to \infty \quad \text{for } \alpha \to 0.$$

So for the currents $g_\alpha := \frac{1}{\|L'_{\Omega_2}\gamma_\alpha\|^{3/2}}\gamma_\alpha$, the corresponding potentials $u_\alpha$ satisfy

$$\|L'_{\Omega_1} g_\alpha\|^2 \approx \int_{\Omega_1} \sigma |\nabla u_\alpha|^2 \,\mathrm{d}x \to \infty,$$

$$\|L'_{\Omega_2} g_\alpha\|^2 \approx \int_{\Omega_2} \sigma |\nabla u_\alpha|^2 \,\mathrm{d}x \to 0.$$

# Construction

Even more specific for $\sigma = 1$:

- $h_z$: boundary data of a electric dipole in $z \in B$, i.e., $h_z = u_z|_S$, where

$$\Delta u_z = d \cdot \nabla \delta_z, \qquad \partial_\nu u_z|_{\partial B} = 0$$

($d \in \mathbb{R}^n$, $|d| = 1$ fixed arbitrary direction).

- If $B \setminus \overline{\Omega}$ is connected and its boundary contains $S$, then for $z \notin \partial\Omega$:

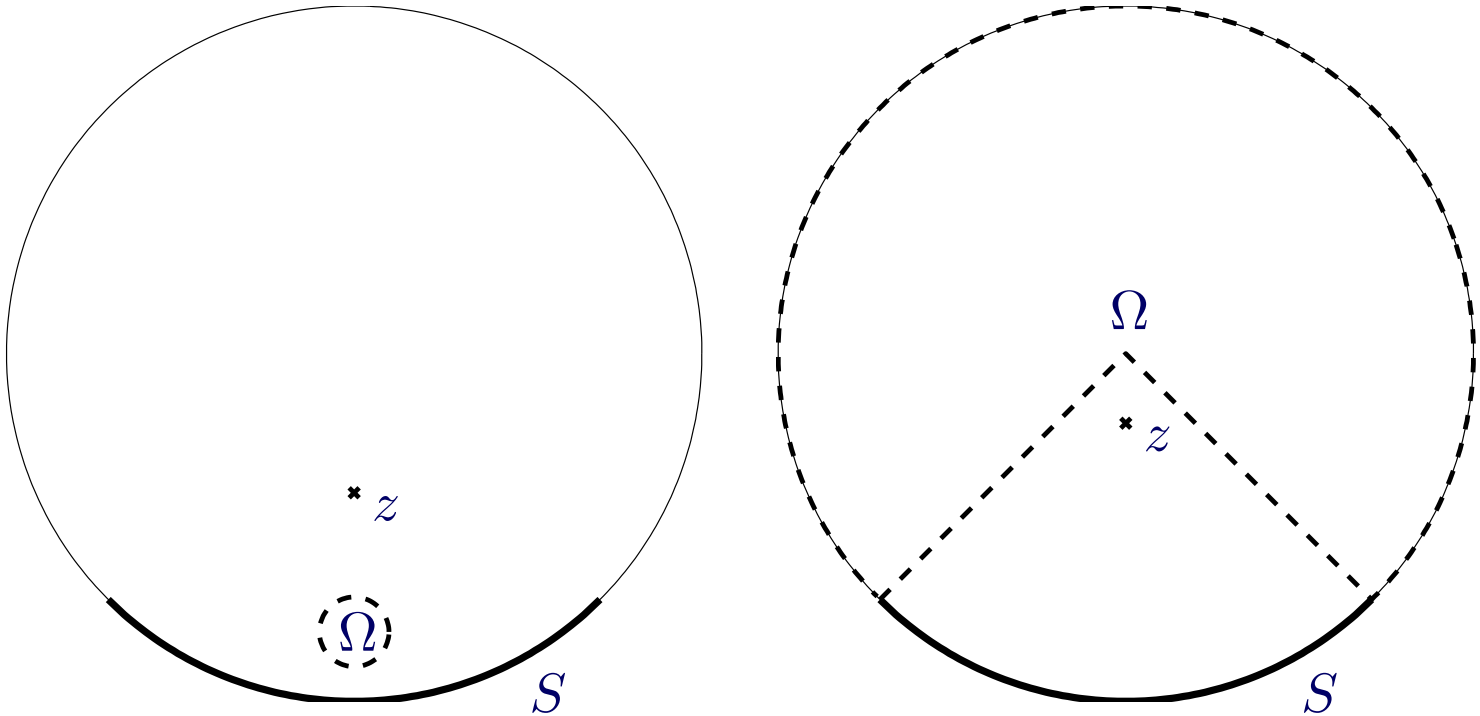
$$h_z \in \mathcal{R}(L_\Omega) \quad \text{iff} \quad z \in \Omega.$$

- Define currents $g_{\alpha,z}$, potentials $u_{\alpha,z}$ as on the last slide, then

$$\int_{\Omega_1} \sigma |\nabla u_{\alpha,z}|^2 \, \mathrm{d}x \to \infty, \qquad \int_{\Omega_2} \sigma |\nabla u_{\alpha,z}|^2 \, \mathrm{d}x \to 0$$

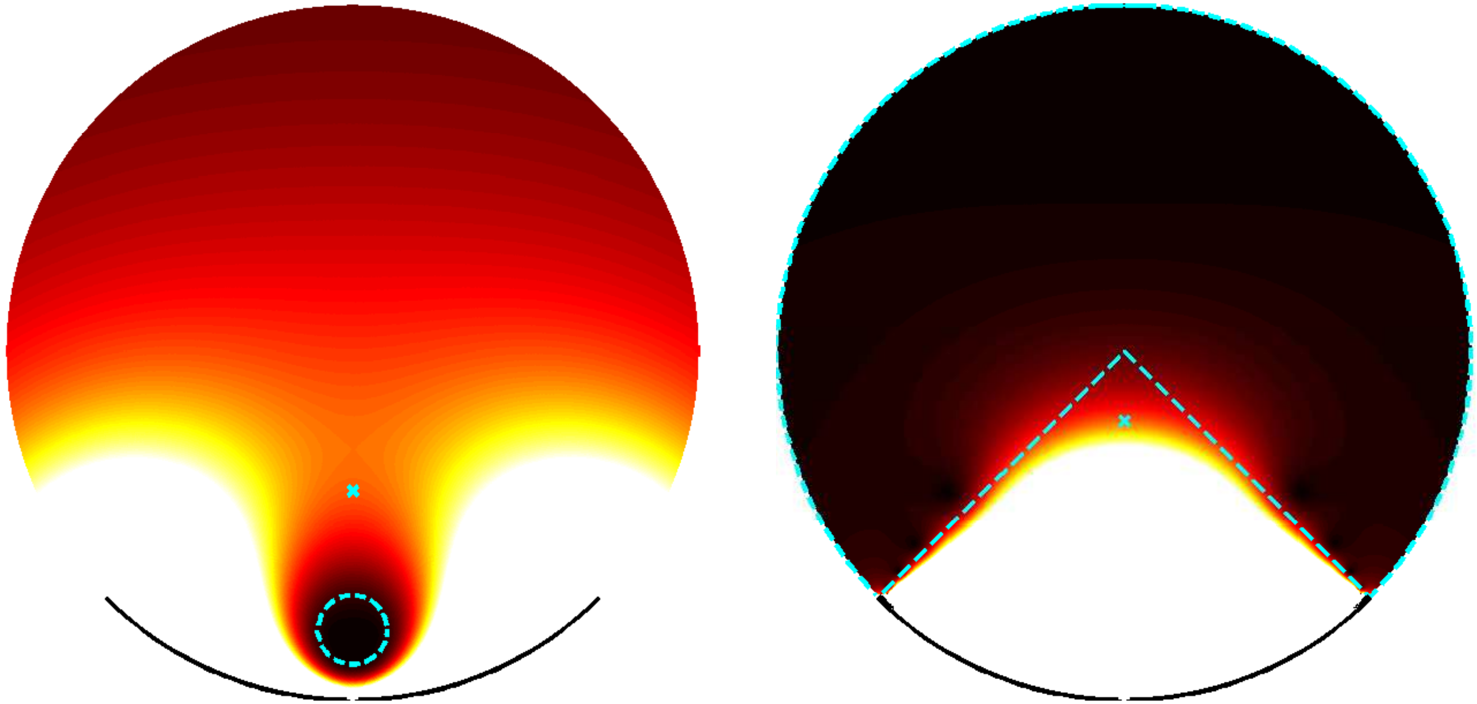
for every neighbourhood $\Omega_1$ of $z$.

# Numerical examples




Find a potential with high energy around $z$ and low energy in $\Omega$!

# Numerical examples



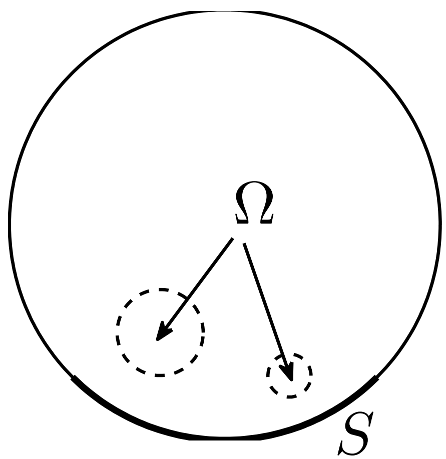
Plots of $|\nabla u_{\alpha,z}|$, $\alpha$ choosen "by hand", color axis cropped above $2|\nabla u_{\alpha,z}(z)|$.

$\rightsquigarrow$ *Electric potentials can be localized in very general domains but the problem is very ill-posed.*

# Detecting inclusions in EIT

# Detecting inclusions in EIT

Special case of EIT: locate inclusions in known background medium.



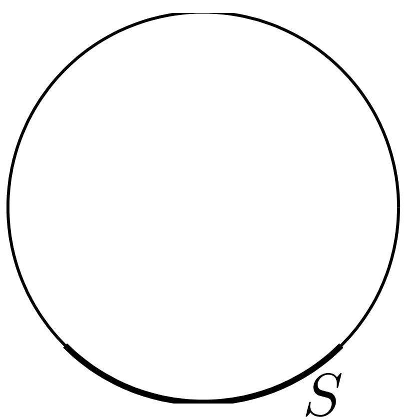

Current-to-voltage map with inclusion:

$$\Lambda_1 : \; g \mapsto u_1|_{\partial B},$$

where $u_1$ solves

$$\nabla \cdot \sigma \nabla u_1 = 0 \quad \partial_\nu u_1|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else,} \end{cases}$$

with $\sigma = 1 + \sigma_1 \chi_\Omega$, $\sigma_1 > 0$.




Current-to-voltage map without inclusion:

$$\Lambda_0 : \; g \mapsto u_0|_{\partial B},$$

where $u_0$ solves the analogous equation with $\sigma = 1$.

Goal: Reconstruct $\Omega$ from comparing $\Lambda_1$ with $\Lambda_0$.

# Virtual measurements again

Connection between $\Lambda_0 - \Lambda_1$ and virtual measurements $L_\Omega$:

Lemma

There exist $c, C > 0$ such that

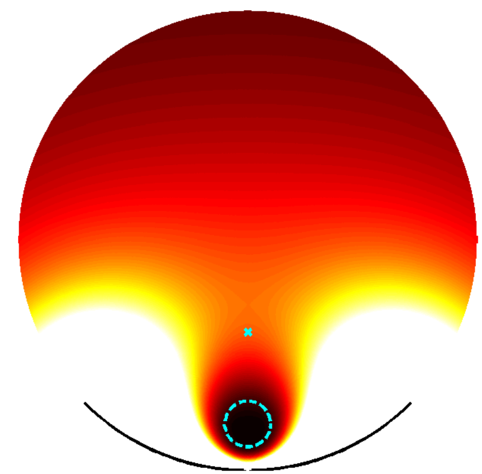
$$c \left\| L_\Omega^* g \right\|^2 \leq ((\Lambda_0 - \Lambda_1) g, g) \leq C \left\| L_\Omega^* g \right\|^2 \quad \text{for all } g \in L^2_\diamond(S),$$

so, roughly speaking, $L_\Omega L_\Omega^* \approx \Lambda_0 - \Lambda_1$.

Only $L_\Omega L_\Omega^*$ is needed to construct a localized potential that is large in some $z \notin \Omega$ and small in $\Omega$.

$\rightsquigarrow$ Simple reconstruction algorithm:

*Given a $z \notin \overline{\Omega}$ (must be known!), use $\Lambda := \Lambda_0 - \Lambda_1$ to create such a potential and locate $\Omega$ from it.*



# Reconstruction algorithm

Theorem

$z \notin \overline{\Omega}$, $h_z$: electric dipole in $z$, $g_z^\alpha := \Lambda^*(\Lambda\Lambda^* + \alpha I)^{-1} h_z \approx \Lambda^{-1} h_z$,
$u_z^\alpha$ hom. potential for currents $g_z^\alpha / (\Lambda g_z^\alpha, g_z^\alpha)^{3/2}$. Then

$$|\nabla u_z^\alpha(z)| \to \infty \quad \text{and} \quad |\nabla u_z^\alpha(x)| \to 0 \quad \text{for } x \in \Omega.$$

Connection to the Factorization Method *(slightly simplified)*:

- Factorization Method (Kirsch, Hanke, Brühl,... ):

  $$z \notin \Omega \text{ if and only if } \|\Lambda^{-1/2} h_z\| \to \infty.$$

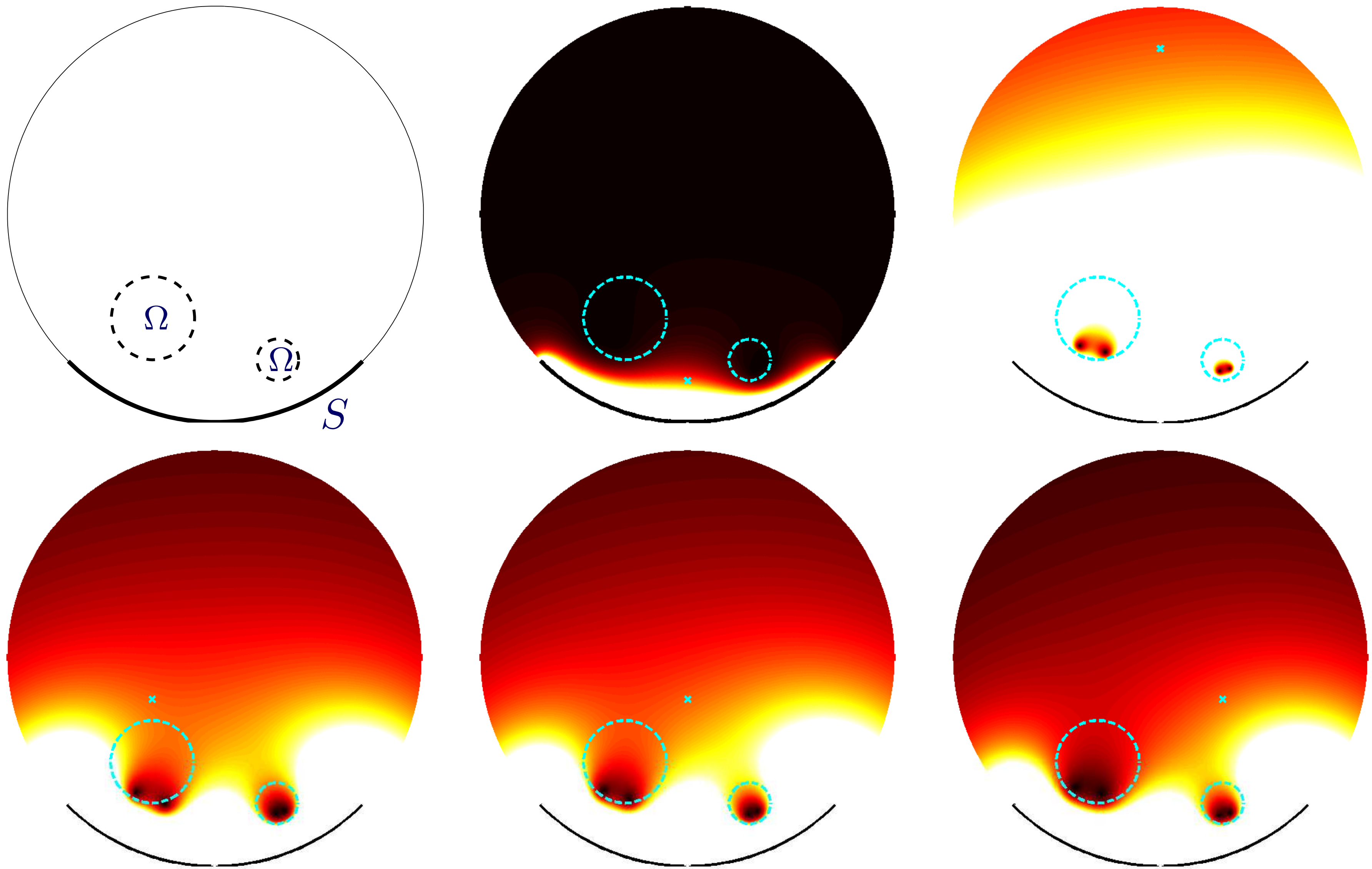
- EIT-analogue of Arens' variant of this criterion:

  $$z \notin \Omega \text{ if and only if } |\nabla v_z^\alpha(z)| \to \infty,$$

  where $v_z^\alpha$ hom. potential for currents $g_z^\alpha \approx \Lambda^{-1} h_z$.

- Here: $z \notin \overline{\Omega}$ fixed. $|\nabla u_z^\alpha(x)| \to 0$ if $x \in \Omega$

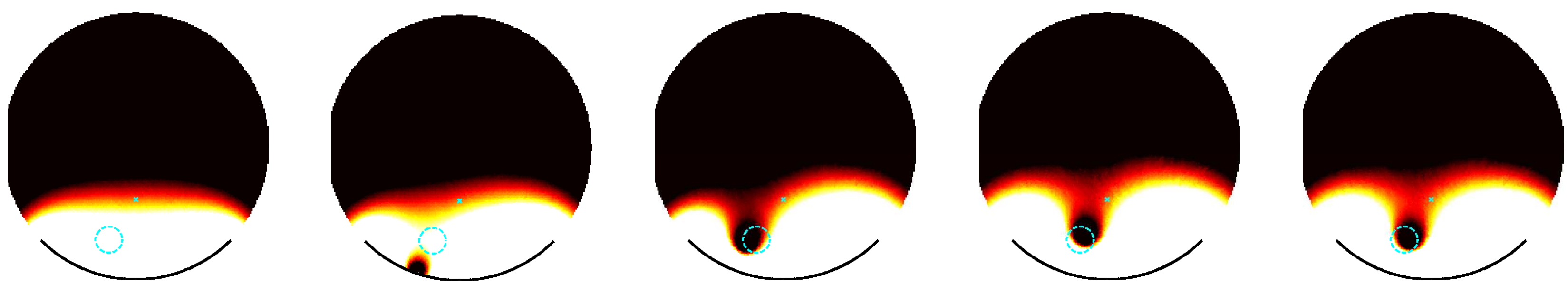
# Numerical example



# Outlook / Bold ideas

- Reconstruction algorithm needs to invert measurement matrix $\Lambda$ for only one right hand side.
- Can be done iteratively, e.g. CG, so that only a few applications of $\Lambda$ are needed.
- Application of $\Lambda$ $\Leftrightarrow$ one measurement

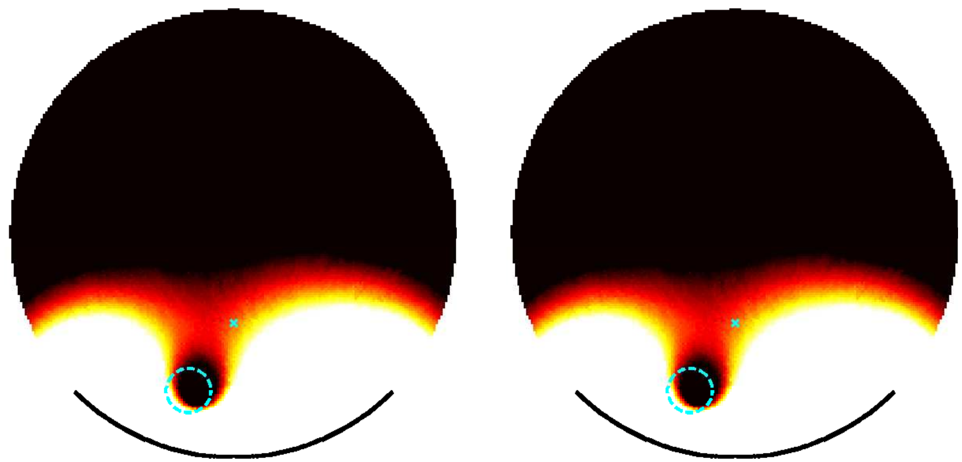
⇝ Method might work with only a few measurements.



⇝ Possible alternative to one-measurement techniques (cf., Kwon, Seo, Yoon 2002, Hanke 2008).

# Very bold idea

- Maybe this even works when inclusion moves between measurements...



- CG with 5 measurements taken for each plot,
- Object continuously moves between measurements

# Summary

- In theory, localized electric potentials exist for almost arbitrary conductivities and on almost arbitrary regions as long as they are connected to the boundary.
- Consequence for the Calderon problem for partial data:

  Two $L^\infty$-conductivities (with (UCP)) can be distinguished if one is larger in some part that can be connected to the boundary without crossing a sign change.
- In practice, localized potentials can be calculated by solving ill-posed equations.
- For detecting inclusions in EIT, quick rough reconstructions can be obtained from few measurements by calculating localized potentials.