Detecting inclusions of mixed type in optical tomography

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Optical tomography:

Use low-energy visible or near infrared light for medical imaging

General forward model:

Photon transport models (Boltzmann transport equation)

For highly scattering media:

Diffusion approximation for photon density $u$:

$$\nabla \cdot \sigma \nabla u - \mu u = 0$$

$\sigma > 0$ : diffusion coefficient

$\mu > 0$ : absorption coefficient

Inverse problem of (diffusive) optical tomography:

Reconstruct (properties of) $\sigma$ and $\mu$ from pairs of Neumann and Dirichlet boundary values of $u$. 

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Mathematical formulation

\[ \Omega \subset \mathbb{R}^n \] smoothly bounded domain, \( \sigma, \mu \in L^\infty(\Omega) \).

For every input flux \( f \in L^2(\partial\Omega) \) there exists a unique solution \( u \in H^1(\Omega) \) of

\[ \nabla \cdot \sigma \nabla u - \mu u = 0 \quad \text{in} \ \Omega, \quad \nu \cdot \sigma \nabla u \big|_{\partial\Omega} = f \quad \text{on} \ \partial\Omega, \]

The Neumann-to-Dirichlet map

\[ \Lambda : f \mapsto u|_{\partial\Omega}, \quad L^2(\partial\Omega) \rightarrow L^2(\partial\Omega), \]

is linear, compact and self-adjoint and can also be considered as an isomorphism from \( H^{-1/2}(\partial\Omega) \) to \( H^{1/2}(\partial\Omega) \).

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Detected inclusions

Consider the special case that

\[ \sigma = 1 + \kappa, \quad \mu = 1 + \eta \]

where \( \kappa, \eta \in L^\infty(\Omega) \) are compactly supported in \( \Omega \).

Goal:

Determine support of \( \kappa \) and \( \eta \) ("the inclusions") from comparing \( \Lambda \)
with reference operator \( \Lambda_0 : f \mapsto u_0|_{\partial\Omega} \), where

\[ \Delta u_0 - u_0 = 0 \quad \text{in } \Omega, \quad \partial_{\nu} u_0|_{\partial\Omega} = f \quad \text{on } \partial\Omega, \]

i.e., the Neumann-to-Dirichlet operator of a domain without
inclusions.

Fairly recent, fast methods for detecting inclusions:

Linear Sampling / Factorization Method

Virtual measurements

$D$: smoothly bounded inclusion, i.e. the support of $\kappa$ and $\eta$. $\Omega \setminus \overline{D}$ conn.

LSM/FM are based on a relation between the (real) measurements $\Lambda$ and the so-called virtual measurements

$$L : \psi \in H^{-1/2}(\partial D) \mapsto v|_{\partial \Omega} \in L^2(\partial \Omega),$$

where $v$ solves $\Delta v - v = 0$ outside $D$ with $\partial_\nu v|_{\partial \Omega} = 0$.

$\mathcal{R}(L)$ determines $D$:

(Traces) of singular solutions $\Phi_z$ of

$$\Delta \Phi_z - \Phi_z = \delta_z, \quad \partial_\nu \Phi_z|_{\partial \Omega} = 0$$

belong to $\mathcal{R}(L)$ if and only if $z \in D$.

*If $\mathcal{R}(L)$ is known, then the inclusions can be found by testing for each point $z \in \Omega$ whether $\Phi_z \in \mathcal{R}(L)$ or not.*
Relation between (real) measurements $\Lambda$ and virtual measurements $L$:

LSM: $\mathcal{R}(L) \supseteq \mathcal{R}(\Lambda_0 - \Lambda)$

$$\rightsquigarrow \{ z \mid \Phi_z \in R(\Lambda_0 - \Lambda) \} \subseteq D$$

(holds for all kinds of inclusions).

FM: $\mathcal{R}(L) = \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2})$

$$\rightsquigarrow \left\{ z \mid \Phi_z \in R(|\Lambda_0 - \Lambda|^{1/2}) \right\} = D$$

(needs additional assumptions on inclusions).

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Properties of LSM/FM:

- Range test
  \[ \Phi_z \in \mathcal{R}(\Lambda_0 - \Lambda), \text{ resp., } \Phi_z \in \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2}) \]
  is easy to implement and extremely fast (no iterations, no forward solutions!)

- FM also yields theoretical uniqueness result.

However,

- LSM is only guaranteed to find a subset of the inclusion.
- FM needs additional assumptions on inclusions.
- Implementation of the range test needs additional threshold parameter. Finding a convergent threshold choosing strategy is still an open problem!

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Known results

Assume that either $\kappa, \eta \geq 0$ or $\kappa, \eta \leq 0$.

**Known results for optical tomography:**

- FM works if conductivity jump $\kappa$ is strictly positive (or negative) and the absorption jump $\eta$ does not interfere with injectivity of $\Lambda_0 - \Lambda$.  \((\text{Hyv"onen 2004, Kirsch 2005, G. 2006})\)
- FM works if $\kappa = 0$ and absorption jump $\eta$ is strictly positive (or negative).  \((\text{Hyv"onen 2005})\)

**Here:**

- Treat combinations of absorption and conductivity perturbations.
- Treat smooth transitions between inclusions and background \((\text{analogous result for EIT: G. and Hyv"onen 2007})\)
- Treat inclusions with unconnected complements.
Concepts of support

\[ g_1, g_2 \in L^\infty(\Omega), \text{ supp } g_1, \text{ supp } g_2 \subset \Omega \]

**Definition** \((\partial \Omega\text{-support})\):

\( \text{supp}_{\partial \Omega} g_1 \) is the complement of the set of all \( x \in \Omega \) for which there exists a (relatively) open connected \( U \subset \overline{\Omega} \) with \( x \in U, \partial \Omega \cap U \neq \emptyset, g_1|_U = 0 \) a.e.

Combined \(\partial \Omega\text{-support}):

\[ \text{supp}_{\partial \Omega} (g_1, g_2) := \text{supp}_{\partial \Omega}(|g_1| + |g_2|). \]

*(goes back to analogous definition of the infinity-support by Kusiak and Sylvester 2003)*

"\( \text{supp}_{\partial \Omega} (g_1, g_2) \) is closed and contains the support of \( g_1 \) and \( g_2 \) together with all the holes that cannot be connected to the boundary without crossing the support."
Concepts of support

\( g_1, g_2 \in L^\infty(\Omega), \text{supp} \, g_1, \text{supp} \, g_2 \subset \Omega \)

**Definition** shaded set:

\( \text{sh}(g_1, g_2) \) is the set of all \( y \in \Omega \) for which there exists a smooth domain \( D \subset \Omega \), with \( \overline{D} \subset \Omega \) and \( \Omega \setminus \overline{D} \) connected, such that \( y \in D \) and for each \( z \in \partial D \) there exist constants \( \epsilon_z, r_z > 0 \) such that

\[
|g_j| > \epsilon_z I \quad \text{almost everywhere in } B_{r_z}(z),
\]

for \( j = 1 \) or \( j = 2 \).

"\( \text{sh}(g_1, g_2) \) is open and contains \( x \in \Omega \) if one cannot travel from \( x \) to the boundary \( \partial \Omega \) without going over a strictly positive hump in \( |g_1| \) or in \( |g_2| \)."
Examples

horizontal lines: $\kappa = 1$
vertical lines: $\eta = 1$

$\text{supp}_{\partial \Omega} (\kappa, \eta)$
$\text{sh}(\kappa, \eta)$

Here: $\overline{\text{sh}(\kappa, \eta)} = \text{supp}_{\partial \Omega} (\kappa, \eta)$. 

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Examples

horizontal lines: $\kappa = 1$
vertical lines: $\eta = 1$

$\text{supp}_{\partial \Omega} (\kappa, \eta)$  $\text{sh}(\kappa, \eta)$

In general: $\overline{\text{sh}(\kappa, \eta)} \subsetneq \text{supp}_{\partial \Omega} (\kappa, \eta)$.
Main result

Theorem

Assume that either $\kappa, \eta \geq 0$ or $\kappa, \eta \leq 0$.

\[
\Phi_y|_{\partial \Omega} \in \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2}) \implies y \in \text{supp}_{\partial \Omega}(\kappa, \eta)
\]
\[
\Phi_y|_{\partial \Omega} \notin \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2}) \implies y \notin \text{sh}(\kappa, \eta)
\]

In other words, the set of points detected by the FM

\[
\{ y \in \Omega \mid \Phi_y|_{\partial \Omega} \in \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2}) \}
\]

lies between $\text{sh}(\kappa, \eta)$ and $\text{supp}_{\partial \Omega}(\kappa, \eta)$. 
Proof

Main ideas of the proof:

- **Monotonicity properties of NtD-Mappings:** Let \( \sigma_1 \leq \sigma_2 \leq \sigma_3 \) and \( \mu_1 \leq \mu_2 \leq \mu_3 \). Then

\[
\mathcal{R} \left\{ (\Lambda_1 - \Lambda_2)^{1/2} \right\}, \quad \mathcal{R} \left\{ (\Lambda_2 - \Lambda_3)^{1/2} \right\} \subseteq \mathcal{R} \left\{ (\Lambda_1 - \Lambda_3)^{1/2} \right\}.
\]

\( \Rightarrow \) A point is detected by FM if it has a neighbourhood for which FM works. (FM can be applied locally.)

- Every point that is "surrounded by detected points" is itself detected by the FM.

\( \Rightarrow \) Holes in the inclusion are (falsely) detected by FM.
Remarks

Result treats "lower order perturbation" $\eta$ and "higher order perturbation" $\kappa$ in a symmetric way.

Result needs no regularity or jump conditions for $\eta$ or $\kappa$.

Result includes known results on FM for optical tomography except for points on the inclusions boundary.

Result is easily extended to anisotropic diffusivities, non-constant background diffusion or absorption, partial boundary data or other real elliptic equations (e.g., Lamé equations in linear elasticity).

However, perturbations $\eta$ or $\kappa$ have to be in the same direction. FM for indefinite problems is still an open question!
Simulated data for the forward problem

- $\Omega$: two-dimensional unit disk
- background diffusion $\sigma_0 = 0.05$, absorption $\mu_0 = 0.5$ corresponding to the optical parameters of a neonatal head (cf. Arridge 1999, Hyvönen 2007)
- In inclusions background parameters are doubled.
- Neumann-to-Dirichlet boundary maps $\Lambda_0 - \Lambda$ approximated in trigonometric basis functions using commercial finite element software Comsol.
Implementation of the FM

Let \((v_j, \lambda_j)\) be the spectral decomposition of \(\Lambda_0 - \Lambda\).

Picard criterion:

\[ \Phi_y|\partial\Omega \in \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2}) \]

if and only if

\[ f(y) := \frac{1}{\|\Phi_y|\partial\Omega\|_{L^2(\partial\Omega)}^2} \sum_{j=1}^{\infty} \frac{|\langle\Phi_y|\partial\Omega, v_j\rangle_{L^2(\partial\Omega)}|^2}{|\lambda_j|} < \infty. \]

Using a SVD of the finite-dimensional approximation to \(\Lambda_0 - \Lambda\) one defines a finite series \(\tilde{f}(y) \approx f(y)\).

\[ \tilde{f}(y) \text{ large, when } \Phi_y|\partial\Omega \not\in \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2}). \]

\[ \tilde{f}(y) \text{ small, otherwise.} \]

\[ \tilde{f}(y) \text{ large, when } \Phi_y|\partial\Omega \not\in \mathcal{R}(|\Lambda_0 - \Lambda|^{1/2}). \]

\[ \tilde{f}(y) \text{ small, otherwise.} \]

A plot of (a monotone function of) \(\tilde{f}\) should reveal the inclusion.

Convergent treshold choosing strategy for \(\tilde{f}\) is still an open problem!
Numerical results

horizontal lines: $\kappa = \sigma_0$
vertical lines: $\eta = \mu_0$

$\tilde{f}(y)$
contour lines of $\tilde{f}$
Numerical results

horizontal lines: $\kappa = \sigma_0$

vertical lines: $\eta = \mu_0$

$\tilde{f}(y)$

contour lines of $\tilde{f}$

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Numerical results

Numerical results with 0.1% noise
Example with smooth transition

Example from EIT with smooth transition (from G. and Hyvönen 2007)

real conductivity

\[ \tilde{f}(y) \]

contour lines of \( \tilde{f} \)
Conclusions

- FM simultaneously detects diffusive and absorbing inclusions in optical tomography.
- Holes inside the inclusion are falsely detected by the FM.
- FM detects also smooth deviations, not only jumps.
- Open problems (not only for optical tomography):
  - FM for indefinite problems
  - Convergent treshold choosing strategy for the range test.