Low-frequency electromagnetic imaging with the factorization method

Bastian Gebauer

gebauer@math.uni-mainz.de

Institut für Mathematik, Joh. Gutenberg-Universität Mainz, Germany

Joint work with Martin Hanke & Christoph Schneider, University of Mainz

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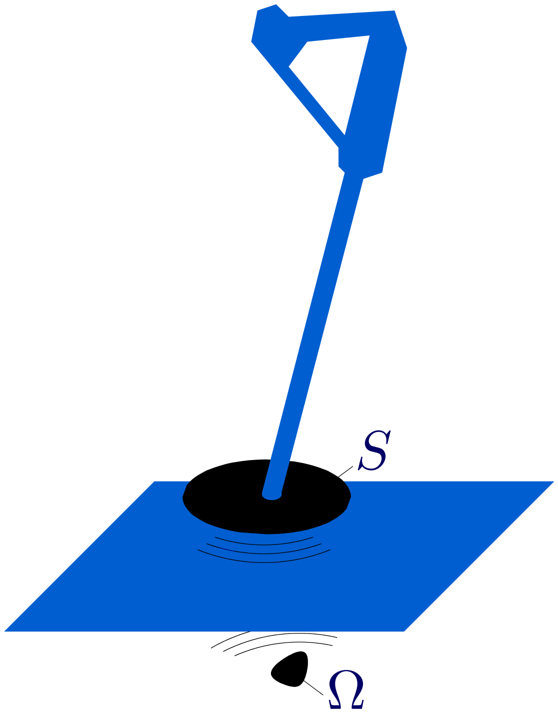
Setting I: Scattering



S: Measurement device

Ω: Magnetic / dielectric object

- Apply surface currents J on S (time-harmonic with frequency ω).

$\rightsquigarrow$ Electromagnetic field (E^ω, H^ω) (time-harmonic with frequency ω)

- Measure field on S (and try to locate Ω from it).

Idealistic assumption:

- Measure (tangential component of) $E^\omega|_S$ for all possible J

$\rightsquigarrow$ Measurement operator: $M^\omega : J \mapsto \gamma_\tau E^\omega|_S$

Goal: Locate Ω from the measurements M^ω.

Maxwell's equations

Time-harmonic Maxwell's equations

$$\begin{aligned} \operatorname{curl} H^\omega + \mathrm{i}\,\omega\epsilon E^\omega &= J \quad \text{in } \mathbb{R}^3, \\ -\operatorname{curl} E^\omega + \mathrm{i}\,\omega\mu H^\omega &= 0 \quad \text{in } \mathbb{R}^3. \end{aligned}$$

Silver-Müller radiation condition (RC)

$$\int_{\partial B_\rho} \left| \nu \wedge \sqrt{\mu} H^\omega + \sqrt{\epsilon} E^\omega \right|^2 \mathrm{d}\sigma = o(1), \quad \rho \to \infty.$$

E^ω:	electric field	ϵ:	dielectricity
H^ω:	magnetic field	μ:	permeability
ω:	frequency	J:	applied currents, $\operatorname{supp} J \subseteq S$

More idealistic assumptions: $\epsilon = 1$, $\mu = 1$ outside the object Ω

Typical metal detectors work at very low frequencies:

frequency ≈ 20kHz, wavelength ≈ 15km, $\omega \approx 4 \times 10^{-4}\text{m}^{-1}$

Forward Problem

Eliminate H^ω from Maxwell's equations:

$$\operatorname{curl}\frac{1}{\mu}\operatorname{curl}E^\omega - \omega^2\epsilon E^\omega = \mathrm{i}\omega J \quad \text{in } \mathbb{R}^3, \tag{1}$$

$$+ \text{ radiation condition.} \tag{RC}$$

Function space: $E^\omega \in H_{\mathrm{loc}}(\operatorname{curl},\mathbb{R}^3;\mathbb{C}^3)$

$\rightsquigarrow$ $\begin{cases} \text{Left side of (1) makes sense (in } \mathcal{D}'(\mathbb{R}^3;\mathbb{C}^3)\text{)}, \\ E^\omega \text{ has tangential trace on } S\text{: } \gamma_t E^\omega|_S \in TH^{-1/2}(\operatorname{curl},S). \end{cases}$

Under certain conditions (1)+(RC) have a unique solution for all

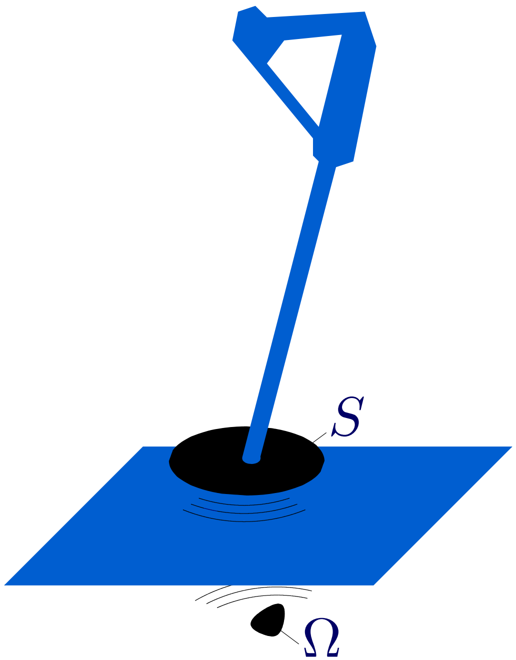
$$J \in TH^{-1/2}(\operatorname{div},S) = TH^{-1/2}(\operatorname{curl},S)'$$

and the solution depends continuously on J.

$\rightsquigarrow$
$$M^\omega : TH^{-1/2}(\operatorname{div},S) \to TH^{-1/2}(\operatorname{curl},S), \quad J \mapsto \gamma_\tau E^\omega|_S$$

is a continuous, linear operator.

Scattered Field

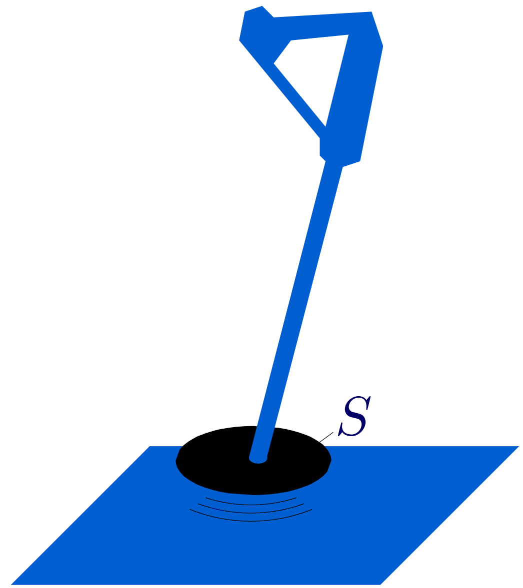


$$M_t^\omega : \; J \mapsto \gamma_\tau E_t^\omega,$$

E_t^ω solution for

$$\epsilon = 1 + \epsilon_1 \chi_\Omega(x)$$
$$\mu = 1 + \mu_1 \chi_\Omega(x)$$

"total field"

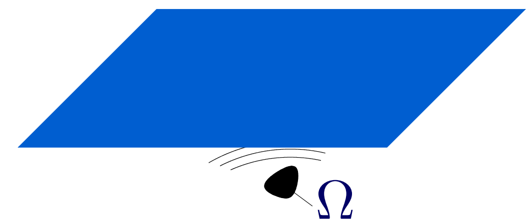


$$M_i^\omega : \; J \mapsto \gamma_\tau E_i^\omega,$$

E_i^ω solution for

$$\epsilon = 1, \quad \mu = 1$$

"incoming field"



$$M_s^\omega := M_t^\omega - M_i^\omega$$

"scattered field"

Detecting the scatterer

Goal: Locate Ω from the measurements M_s^ω.

Promising approach: linear sampling / factorization methods

- Non-iterative (no forward solutions of 3D Maxwell's equations)
- Yield pointwise, binary criterion whether $z \in \Omega$ or not
  $\rightsquigarrow$ Detect Ω by checking this criterion for every z below S ("sampling/probing")
- Independent of number and type of scatterers
- FM also implies theoretical uniqueness results.
- Based on functions $E_{z,d}^\omega$ with singularity in sampling point z:

$$\operatorname{curl}\operatorname{curl} E_{z,d}^\omega - \omega^2 E_{z,d}^\omega = \mathrm{i}\omega\delta_z d \quad \text{in } \mathbb{R}^3, \qquad + \text{(RC)}$$

(electric field of a point current in point z with direction d).

LSM / FM

$E^{\omega}_{z,d}$: electric field of a point current in point z with direction d.

- Linear Sampling Method (Colton, Kirsch 1996):

$$\gamma_\tau E^{\omega}_{z,d} \in \mathcal{R}(M^{\omega}_s) \quad \Longrightarrow \quad z \in \Omega$$

(holds for every z below S and every direction d).
$\rightsquigarrow$ (LSM) finds a subset of Ω.

- Factorization Method (Kirsch 1998):

$$\gamma_\tau E^{\omega}_{z,d} \in \mathcal{R}(|M^{\omega}_s|^{1/2}) \quad \Longleftrightarrow \quad z \in \Omega \qquad (*)$$

(holds for similar problems).
$\rightsquigarrow$ (FM) finds Ω (If (*) holds!).

(*) only known to hold for far-field measurements (Kirsch, 2004).

In this talk: FM works in the low-frequency limit
(actually: in various low-frequency limits).

Low-frequency asymptotics

Maxwell's equation $\qquad \operatorname{curl}\frac{1}{\mu}\operatorname{curl}E^\omega - \omega^2\epsilon E^\omega = \mathrm{i}\omega J \quad$ in $\mathbb{R}^3$

also implies $\qquad \operatorname{div}(\epsilon E^\omega) = \frac{1}{\mathrm{i}\omega}\operatorname{div}J \quad$ in $\mathbb{R}^3$.

(Time-harmonic formulation of conservation of surface charges ρ

$\operatorname{div}J = -\partial_t\rho, \qquad \operatorname{div}(\epsilon E^\omega) = \rho$.)

Formal asymptotic analysis for $\operatorname{div}J \neq 0$:

$$E^\omega = \frac{1}{\mathrm{i}\omega}\nabla\varphi + O(\omega), \quad \text{where } \operatorname{div}(\epsilon\nabla\varphi) = \operatorname{div}J.$$

Rigorous analysis (for fixed incoming waves): Ammari, Nédélec, 2000

(Low frequency electromagnetic scattering, SIAM J. Math. Anal.)

Interpretation:

$\frac{1}{\mathrm{i}\omega}\varphi$: electrostatic potential created by surface charges $\rho = \frac{1}{\mathrm{i}\omega}\operatorname{div}J$.

Electrostatic measurements

Consequence for the measurements $M_s^\omega: \ J \mapsto \gamma_\tau E_s^\omega$

$$M_s^\omega \approx -\frac{1}{\mathrm{i}\omega} \nabla_S \Lambda_s \nabla_S^*, \qquad J \overset{\nabla_S^*}{\longmapsto} \operatorname{div} J = \rho \overset{\Lambda_s}{\longmapsto} \varphi|_S \overset{\nabla_S}{\longmapsto} \gamma_\tau \nabla \varphi,$$

with the electrostatic measurement operator $\Lambda_s = \Lambda_t - \Lambda_i$,

$$\Lambda_t : \begin{cases} H^{-1/2}(S) \to H^{1/2}(S), \\ \rho \mapsto \varphi_t|_S, \end{cases} \qquad \Lambda_i : \begin{cases} H^{-1/2}(S) \to H^{1/2}(S), \\ \rho \mapsto \varphi_i|_S, \end{cases}$$

$$\operatorname{div}(\epsilon_t \nabla \varphi_t) = \rho \qquad \operatorname{div}(\epsilon_i \nabla \varphi_i) = 0$$

$$\epsilon_t = 1 + \epsilon_1 \chi_\Omega \qquad \epsilon_i = 1$$

"electrostatic measurements with object" — *"electrostatic measurements "without object"*

LF measurements are essentially electrostatic measurements.

Current loops

- In practice: currents will be applied along closed loops.

$\rightsquigarrow$ $\operatorname{div} J = 0$

- Also the electric field can only be measured along closed loops.

$\rightsquigarrow$ More realistic model for the measurements:

$$j^* M^\omega j : \; TL^2_\diamond(S) \to TL^2_\diamond(S)$$

where $j : \; TL^2_\diamond(S) = \{v \in TL^2(S), \operatorname{div} v = 0\} \hookrightarrow TH^{-1/2}(\operatorname{div}, S)$.

- j^* "factors out gradient fields", in particular

$$j^*(\nabla_S \Lambda_s \nabla_S^*) j = 0.$$

Electrostatic effects do not appear in practice.

Asymptotics again

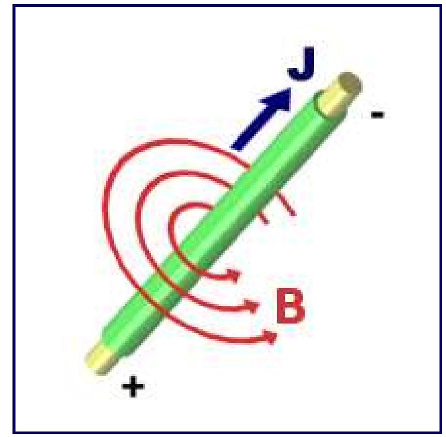
$$\operatorname{div} J = 0 \quad \rightsquigarrow \quad E^\omega = \mathrm{i}\omega E + O(\omega^3), \text{ with } \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E = J, \quad \operatorname{div}(\epsilon E) = 0.$$

(Rigorous asymptotic analysis and existence theory: G., 2006)

- $B := \operatorname{curl} E$ solves

$$\operatorname{curl} \tfrac{1}{\mu} B = J, \quad \operatorname{div} B = 0.$$

$\rightsquigarrow$ B is the magnetostatic field generated by a steady current J (*Ampère's Law*).



- $B = \frac{1}{\mathrm{i}} \operatorname{curl} E \quad \rightsquigarrow \quad E$ is a vector potential of B (unique up to addition of A with $\operatorname{curl} A = 0$, i. e. up to $A = \nabla\varphi$).
- $\operatorname{div}(\epsilon E) = 0$ determines E uniquely (so-called *Coulomb gage).*

$\rightsquigarrow$ *E is (a potential of) the magnetostatic field induced by J.*

(Figure based on `http://de.wikipedia.org/wiki/Bild:RechteHand.png`*)*

Magnetostatic measurements

Consequence for the measurements $j^* M_s^\omega j: \; J \mapsto \gamma_\tau E_s^\omega$

$$j^* M_s^\omega j \approx -\mathrm{i}\omega M_s,$$

with the magnetostatic measurement operator $M_s = M_t - M_i$,

$$M_t: \begin{cases} TL^2_\diamond(S) \to TL^2_\diamond(S)', \\ J \mapsto \gamma_\tau E_t|_S, \end{cases} \qquad M_i: \begin{cases} TL^2_\diamond(S) \to TL^2_\diamond(S)', \\ J \mapsto \gamma_\tau E_i|_S, \end{cases}$$

$$\operatorname{curl} \frac{1}{\mu_t} \operatorname{curl} E_t = J \qquad \operatorname{curl} \frac{1}{\mu_i} \operatorname{curl} E_i = J$$

$$\operatorname{div} E_t = 0 \qquad \operatorname{div} E_i = 0$$

$$\mu_t = 1 + \mu_1 \chi_\Omega \qquad \mu_i = 1$$

"magnetostatic measurements with object" — *"magnetostatic measurements "without object"*

(Note that replacing $\operatorname{div} \epsilon E = 0$ with $\operatorname{div} E = 0$ changes E only by a gradient field.)

LF measurements are essentially magnetostatic measurements.

Eddy currents

What happens if the object has a finite conductivity $\sigma\chi_\Omega > 0$?

$$\operatorname{curl}\frac{1}{\mu}\operatorname{curl} E^\omega - \omega^2\epsilon E^\omega = \mathrm{i}\omega(J+\sigma E^\omega)$$

Low frequency asymptotics in the time domain lead to

$$\partial_t(\sigma E) - \operatorname{curl}\frac{1}{\mu}\operatorname{curl} E = -\partial_t J,$$

which is parabolic in the object ($\sigma > 0$) and elliptic outside ($\sigma = 0$).

(Ammari, Buffa, Nédélec, 2000, SIAM J. Math. Anal.)

Scalar model problem (heat equation)

$$\partial_t(\chi_\Omega u) - \operatorname{grad}\kappa\operatorname{div} u = 0$$

describes domain of low heat capacity with inclusion of high heat capacity.

LF asymptotics

Low-frequency asymptotics for the scattering measurements

$$M_s^\omega : \; J \mapsto \gamma_\tau E_s^\omega.$$

- If $\operatorname{div} J \neq 0$ (presence of surface charges)

$$M_s^\omega \approx -\frac{1}{\mathrm{i}\omega} \nabla_S \Lambda_s \nabla_S^*$$

essentially consists of electrostatic measurements Λ_s.

- More realistic: $\operatorname{div} J = 0$ (currents applied along closed loops)

$$j^* M_s^\omega j \approx -\mathrm{i}\omega M_s$$

are essentially magnetostatic measurements Λ_s.

- Conducting objects lead to parabolic-elliptic, eddy-current problems.

Factorization Method

Factorization Method for the three cases:

- FM works for electrostatic limit: *(Haehner 1999, G. 2006)*

$$z \in \Omega \quad \Longleftrightarrow \quad \gamma_\tau E_{z,d} \in \mathcal{R}(|\nabla_S \Lambda_s \nabla_S^*|^{1/2}) \approx \mathcal{R}(|M_s^\omega|^{1/2})$$

($E_{z,d}$: electrostatic field of a dipole in z with direction d).

- FM works for magnetostatic limit: *(G, Hanke and Schneider 2008)*

$$z \in \Omega \quad \Longleftrightarrow \quad \gamma_\tau G_{z,d} \in \mathcal{R}(|M_s|^{1/2}) \approx \mathcal{R}(|j^* M_s^\omega j|^{1/2}) \qquad (*)$$

($G_{z,d}$: vector potential of the magnetostatic field of a magnetic dipole)

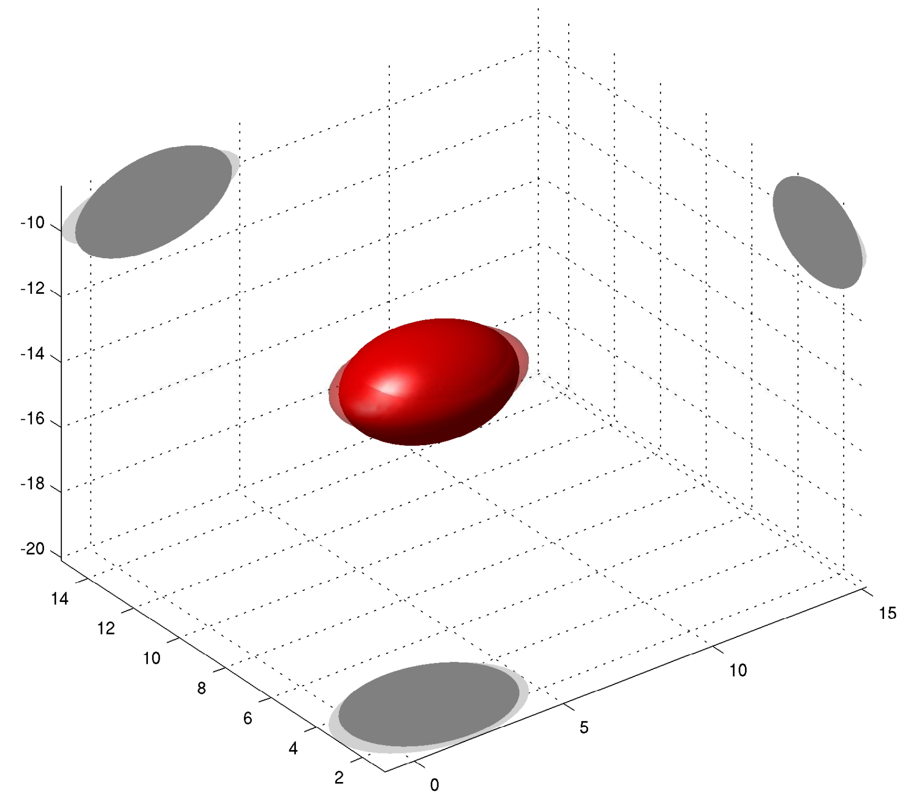
- FM works for parabolic-elliptic scalar model problem *(Frühauf, G and Scherzer 2007)*

$\rightsquigarrow$ We expect that (*) also holds for conducting, diamagnetic objects.

Numerical result

Conducting object in dielectric halfspace:

- Measurement device S:
  square at height $z = 5\text{cm}$,
  size $32\text{cm} \times 32\text{cm}$
- $\omega = 4 \times 10^{-4}$, i. e. freq. $\approx 20\text{kHz}$
- Ω: copper ellipsoid,
  15cm below S,
  in dielectric halfspace
  ($z > 0$: air, $z < 0$: humid earth)
- Currents imposed / electric fields measured on a 6×6 grid on S



Forward solver: BEM from Schulz, Erhard, Potthast, Göttingen

Implementation of FM: C. Schneider, Mainz

Outlook: EIT

Even more low-frequency asymptotics:

- In this talk: measurement device S separated from object by non-conducting medium
  $\rightsquigarrow$ *wave scattering*
- Currents imposed directly to a conducting medium
  $\rightsquigarrow$ *electrical impedance tomography*

  Again, modelling equations are LF-asymptotics of Maxwell's equ.:

$$\nabla \cdot \gamma^\omega \nabla u^\omega = 0, \qquad \gamma^\omega \partial_\nu u^\omega|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases}$$

  - $\omega = 0$: static currents, real conductivity $\gamma^\omega = \sigma$.
  - $\omega^2 = 0$: phase shifts due to complex conductivity $\gamma^\omega = \sigma + \mathrm{i}\epsilon\omega$.

Factorization Method

Factorization Method also woks for EIT:

- FM works for real conductivity (frequency $< 1\mathrm{kHz}$).
  (Brühl and Hanke 1999)

$$z \in \Omega \quad \Longleftrightarrow \quad \Phi_z|_S \in \mathcal{R}(|\Lambda|^{1/2})$$

Ω: inclusion where conductivity differs from known background,
Φ_z: electric potential of a dipole in z,
Λ: difference of current-voltage measurements and reference measurements at inclusion-free body

- FM works for complex conductivity ($1\mathrm{kHz} < \mathrm{frequ.} < 500\mathrm{kHz}$).
  (Kirsch 2005, G. and Seo 2008)

$$z \in \Omega \quad \Longleftrightarrow \quad \Phi_z|_S \in \mathcal{R}(|\Lambda|^{1/2})$$

Λ: weigthed frequency-difference current-voltage measurements (no reference measurements needed)