Sampling methods for low-frequency electromagnetic imaging

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Low-frequency electromagnetics

Several physically relevant settings for LF-EM imaging.

- Charges/currents applied away from conducting medium
  - Inverse scattering problems
    - Electric charges \(\rightarrow\) Electrostatics, Laplace-equation
    - Current loops \(\rightarrow\) Magnetostatics, curl-curl-equation
    - Eddy current in conducting objects, parabolic-elliptic equations

- Currents applied directly to conducting medium
  - Electrical Impedance Tomography
    - freq. \(< 1\text{kHz}\) \(\rightarrow\) static currents, real conductivity
    - \(1\text{kHz} < \text{freq.} < 100\text{kHz}\) \(\rightarrow\) phase shifts, complex conductivity

In this talk: EIT with complex conductivity.
Electrical impedance tomography

Apply one or several input currents to a body and measure the resulting voltages

Goal: Obtain an image of the interior conductivity distribution.

Possible advantages:
- EIT may be less harmful than other tomography techniques,
- Conductivity contrast is high in many medical applications
Electrical impedance tomography

Idealized model for EIT:

- Apply all possible currents \( g \) on boundary of the body \( B \)
- Electric potential \( u \) that solves
  \[
  \nabla \cdot \gamma^\omega \nabla u = 0, \quad \gamma^\omega \partial_\nu u|_{\partial B} = g
  \]
  \((\gamma^\omega = \sigma + i\omega\epsilon, \sigma \text{ real conductivity, } \epsilon \text{ dielectricity})\)
- Measure corresponding potential \( u \) everywhere on \( \partial B \).
- Current-to-voltage map \( \Lambda : g \mapsto u|_{\partial B} \).

Direct Problem: (Standard theory for linear, elliptic PDEs):

- For \( \sigma \in L^\infty_+ (B) \), \( \epsilon \in L^\infty (B) \), \( g \in L^2_\circ (\partial B) := \{ \tilde{g} \in L^2 : \int \tilde{g} = 0 \} \)
  there exists a unique solution \( u \in H^1_\circ (B) := H^1 (B)/\mathbb{C} \).
- Current-to-voltage map \( \Lambda : L^2_\circ (\partial B) \to L^2_\circ (\partial B) \) is a compact, linear operator.
Detecting inclusions in EIT

Special case of EIT: locate inclusions in known background medium.

Current-to-voltage map with inclusion:

\[ \Lambda_1 : g \mapsto u_1|_{\partial B}, \]

where \( u_1 \) solves

\[ \nabla \cdot \gamma^\omega \nabla u_1 = 0 \quad \gamma^\omega \partial_n u_1|_{\partial B} = g \]

with \( \gamma^\omega = \gamma_0^\omega + \gamma_\Omega^\omega \chi_\Omega, \)

(\( \gamma_0^\omega \in \mathbb{C}: \) background conductivity).

Current-to-voltage map without inclusion:

\[ \Lambda_0 : g \mapsto u_0|_{\partial B}, \]

where \( u_0 \) solves analogous equation with \( \gamma^\omega = \gamma_0^\omega. \)

"Reference measurements"

Goal: Locate \( \Omega \) from comparing \( \Lambda_1 \) with \( \Lambda_0. \)
Factorization Method:

\[ \Phi_z|_{\partial B} \in \mathcal{R}(|\Lambda_0 - \Lambda_1|^{1/2}) \quad \text{if and only if} \quad z \in \Omega \]

where

\[ \nabla_x \cdot \gamma_0^\omega \nabla_x \Phi_z(x) = d \cdot \nabla_x \delta_z(x), \quad \gamma_0^\omega \partial_\nu(x) \Phi_z|_{\partial B} = 0 \]

(electric potential of dipole in point \( z \) with arbitrary direction \( d \)).

\( \Rightarrow \) Measurements \( \Lambda_1, \Lambda_0 \) determine \( \Omega \).

Numerical implementation:

- Calculate regularized approximation of preimage \( |\Lambda_0 - \Lambda_1|^{-1/2}\Phi_z|_{\partial B} \)
- Plot norm of this approximation as function of \( z \). ("Indicator function")
  - Larger norm \( \Rightarrow \) preimage does not exist \( \Rightarrow z \notin \Omega \).
  - Smaller norm \( \Rightarrow \) preimage exists \( \Rightarrow z \in \Omega \).
History and known results

FM relies on characterization of $\Omega$ via $R(\Lambda_0 - \Lambda_1^{1/2})$.

- originally developed by Kirsch, 1998 for inverse scattering problems,
- generalized to real conductivity EIT with inclusions "with sharp jumps" and connected complement (Brühl and Hanke, 1999),
- extended to electrode models (Hyvönen, Hakula, Pursiainen, Lechleiter),
- detects inclusions with complex conductivity in real conductivity background (Kirsch, 2005),
- detects inclusions "without sharp jumps" (G. and Hyvönen 2007), "fills up holes" in case of disconn. complements (G. and Hyvönen 2008).

Here: Variant of FM that detects inclusion in complex conductivity case and works without reference measurements.
Reference measurements

- Factorization method uses difference $\Lambda_1 - \Lambda_0$ between
  - actual measurements $\Lambda_1$
  - reference measurements $\Lambda_0$ at an inclusion-free body

**Advantage:** If reference measurements are available then systematic errors cancel out, e.g., forward modeling errors about the body geometry.

**Disadvantage:** If reference measurements have to be simulated (or calculated analytically) then forward modeling errors have a large impact on the reconstructions.

In medical application, reference measurements at an inclusion-free body are usually not available.
Example

\[ \Lambda_1 : \text{circle} \]
\[ \Lambda_0 : \text{circle} \]
FM for circle

\[ \Lambda_1 : \text{ellipse} \]
\[ \Lambda_0 : \text{circle} \]
FM for circle

\[ \Lambda_1 : \text{ellipse} \]
\[ \Lambda_0 : \text{ellipse} \]
FM for circle

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Possible solution

Replace reference measurements by measurements at another frequency:

1. Given two frequencies $\omega, \tau > 0$, conductivities $\gamma^\omega, \gamma^\tau$ and NtDs $\Lambda^\omega, \Lambda^\tau$
2. Assume that for all $x$ outside the inclusion $\Omega$
   \[ \gamma^\omega(x) = \gamma^\omega_0 \in \mathbb{C} \quad \text{and} \quad \gamma^\tau(x) = \gamma^\tau_0 \in \mathbb{C} \]
3. Using $\gamma^\omega_0 \Lambda^\omega$ and $\gamma^\tau_0 \Lambda^\tau$ scales down conductivity outside $\Omega$ to 1.
4. Difference $\gamma^\omega_0 \Lambda^\omega - \gamma^\tau_0 \Lambda^\tau$ should have similar properties to $\Lambda_1 - \Lambda_0$.

**FM should also work with** $\gamma^\omega_0 \Lambda^\omega - \gamma^\tau_0 \Lambda^\tau$ **instead of** $\Lambda_1 - \Lambda_0$.

5. For non-zero frequencies, $\gamma^\omega_0 \Lambda^\omega$ is not self-adjoint, so we will have to use its real or imaginary part
   \[ \Im(A) := \frac{1}{2i}(A - A^*), \quad \Re(A) := \frac{1}{2}(A + A^*) \]
   for an operator $A : L^2_\diamond(S) \to L^2_\diamond(S)$. 

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Theorem (G, Seo 2008)

Let $\Omega$ have a connected complement,

$$\gamma^\omega(x) = \gamma_0 + \gamma^\omega_\Omega(x) \chi_\Omega(x), \quad \text{and} \quad \gamma^\tau(x) = \gamma_0 + \gamma^\tau_\Omega(x) \chi_\Omega(x).$$

If $\Im \left( \frac{\gamma^\omega_\Omega}{\gamma_0} \right) \in L^\infty_+(\Omega)$ or $-\Im \left( \frac{\gamma^\omega_\Omega}{\gamma_0} \right) \in L^\infty_+(\Omega)$, then

$$z \in \Omega \quad \text{if and only if} \quad \Phi_z|_{\partial B} \in \mathcal{R} \left( |\Im \left( \sigma^\omega_0 \Lambda_\omega \right)|^{1/2} \right),$$

If $\Re \left( \frac{\sigma^\tau_\Omega}{\sigma_0^\tau} \right) - \Re \left( \frac{\sigma^\tau_0}{\sigma_0^\tau} \right) - \frac{\Im \left( \frac{\sigma^\tau_\Omega}{\sigma_0^\tau} \right)^2}{\Re \left( \frac{\sigma^\tau_\Omega}{\sigma_0^\tau} \right)} \in L^\infty_+(\Omega)$, then

$$z \in \Omega \quad \text{if and only if} \quad \Phi_z|_{\partial B} \in \mathcal{R} \left( |\Re \left( \sigma^\omega_0 \Lambda_\omega - \sigma_0^\tau \Lambda_\tau \right)|^{1/2} \right).$$

($\tau = 0$ possible and same assertion also holds with interchanged $\omega$ and $\tau$).

*FM can be used on single non-zero frequency data or on frequency-difference data.*

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Interpretation

\[ z \in \Omega \text{ if and only if } \Phi_z|_{\partial B} \in \mathcal{R} \left( |\Im(\sigma_0^\omega \Lambda_\omega)|^{1/2} \right), \]

Using \( \Im(\ldots) \) compares \( \Lambda_\omega \) to its own adjoint ("phase information").

\[ \Rightarrow \text{ Inclusion can be found from measurements at a single non-zero frequency without any reference measurements.} \]

\[ z \in \Omega \text{ if and only if } \Phi_z|_{\partial B} \in \mathcal{R} \left( |\Re(\sigma_0^\omega \Lambda_\omega - \sigma_0^\tau \Lambda_\tau)|^{1/2} \right). \]

\( \Re(\ldots) \) compares two different frequencies.

\[ \Rightarrow \text{ Reference measurements can be replaced by measurements at a different frequency, e.g. by comparing static with non-zero frequency measurements.} \]

\[ \omega = 0 \text{ ("static measurements")}: \quad \text{freq.} < 1\text{khz} \]

\[ \omega > 0, \text{ but still low frequency}: \quad 1\text{khz} < \text{freq.} < 500\text{khz} \]
Numerical example

(Conductivities: \( \sigma = 0.3 - 0.2\chi_\Omega(x) \), \( \sigma_\omega(x) = 0.3 + 0.1i - 0.2\chi_\Omega(x) \).)
Numerical example

Reconstructions of an ellipse-shaped body that is wrongly assumed to be a circle.
Conclusions

Simulating reference data makes Factorization Method vulnerable to forward modeling errors.

Using frequency-difference measurements instead strongly improves FMs robustness. Results are comparable to those with correct reference data.

Open problems:

Scaling the conductivity by simple multiplication only works for constant background conductivity.

Unsolved problems in the theory of FM: convergent threshold choice, definiteness properties.