Electric potentials with localized divergence properties

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Overview

Motivation and existence of localized potentials
Constructions and numerical examples
Relation and consequences for the Factorization Method
Motivation and existence
Electrical impedance tomography

**B**: bounded domain

**S**: relatively open subset

**σ** ∈ **L**^∞**(B)**: electrical conductivity in **B**

**g** ∈ **L**^2**(S)**: applied current on **S**

Electric potential **u** ∈ **H**^1**(B)** that solves

\[ \nabla \cdot \sigma \nabla u = 0, \quad \sigma \partial_{\nu} u_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases} \]

**EIT**: Measure **u|S** for one or several input currents **g** and reconstruct (properties of) **σ** from it.

Additional assumption: **σ** satisfies (UCP), i.e., if **u|T** = 0, **σ∂ₜ u|T** = 0 for some open part **T** of a surface in **B** then **u** = 0.
Localized potentials

Can we localize (the energy of) the potentials in given subsets?

Restrictions:

- High energy parts have to be connected to the boundary.
- Because of (UCP) zero energy parts are not possible.

Goal: sequences \((g_n)\) such that energy of \((u_n)\) diverges on some subset while tending to zero on another.

Energy of a potential \(u: \int \sigma |\nabla u|^2 \, dx \approx \int |\nabla u|^2 \, dx \approx \|u\|_{H^1_0(\Omega)}^2\)
Theoretical motivation

Calderon problem with local data:
Is $\sigma$ uniquely determined by the full (local) current-to-voltage map

$$\Lambda_\sigma : L^2_\phi(S) \to L^2_\phi(S), \quad g \mapsto u|_S \ ?$$

Monotonicity property:

$$\int_B (\sigma_1 - \sigma_2) |\nabla u_2|^2 \, dx \geq \langle (\Lambda_{\sigma_2} - \Lambda_{\sigma_1})g, g \rangle \geq \int_B (\sigma_1 - \sigma_2) |\nabla u_1|^2 \, dx$$

$\implies$ If $\sigma_1 - \sigma_2 > 0$ in some region where we can localize the electric energy $|\nabla u_1|^2$ then $\Lambda_{\sigma_1} \neq \Lambda_{\sigma_2}$.

"A higher conductivity in such a region can not be balanced out."

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Virtual Measurements

\[ f \in (H^1_\dagger(\Omega))' : \text{applied source on } \Omega \]

\[ L_\Omega : (H^1_\dagger(\Omega))' \to L^2_\dagger(S), \quad f \mapsto u|_S, \]

where \( u \in H^1_\dagger(B) \) solves

\[ \int_B \sigma \nabla u \cdot \nabla v \, dx = \langle f, v|_\Omega \rangle \quad \text{for all } v \in H^1_\dagger(B). \]

If \( \overline{\Omega} \subset B: \quad \nabla \cdot \sigma \nabla u = f, \quad \sigma \partial_\nu u|_{\partial B} = 0. \]

(UCP) yields: If \( \overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset \), \( B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2) \) is connected and its boundary contains \( S \) then \( \mathcal{R}(L_{\Omega_1}) \cap \mathcal{R}(L_{\Omega_2}) = 0. \)

Dual operator \( L'_\Omega : L^2_\dagger(S) \to H^1_\dagger(\Omega), \quad g \mapsto u|_\Omega \), where \( u \) solves

\[ \nabla \cdot \sigma \nabla u = 0, \quad \sigma \partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{else.} \end{cases} \]
Some functional analysis

Lemma
Let $X, Y$ be two reflexive Banach spaces, $A \in \mathcal{L}(X, Y)$, $y \in Y$. Then

$$y \in \mathcal{R}(A) \iff |\langle y', y \rangle| \leq C \|A'y'\| \quad \forall y' \in Y'.$$

Corollary
If $\|L'_{\Omega_1}g\| \leq C \|L'_{\Omega_2}g\|$ for all applied currents $g$, i.e., $\|u|_{\Omega_1}\| \leq C \|u|_{\Omega_2}\|$ for the corresponding potentials $u$, then $\mathcal{R}(L_{\Omega_1}) \subseteq \mathcal{R}(L_{\Omega_2})$.

Contraposition
If $\mathcal{R}(L_{\Omega_1}) \nsubseteq \mathcal{R}(L_{\Omega_2})$ then there exist currents $(g_n)$ such that the corresponding potentials $(u_n)$ satisfy

$$\|u_n|_{\Omega_1}\|_{H^1_0(\Omega_1)} \to \infty \quad \text{and} \quad \|u_n|_{\Omega_2}\|_{H^1_0(\Omega_2)} \to 0.$$
Existence of localized potentials

**Theorem**

If $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, $B \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$ is connected and its boundary contains $S$, then there exists currents $(g_n)$ such that (the energy of) the corresponding potentials $(u_n)$ diverges on $\Omega_1$ while tending to zero on $\Omega_2$, i.e.,

$$\int_{\Omega_1} \sigma |\nabla u_n|^2 \, dx \to \infty, \quad \text{and} \quad \int_{\Omega_2} \sigma |\nabla u_n|^2 \, dx \to 0.$$

**Result uses only ellipticity properties, thus also holds e.g. for linear elasticity, electro- and magnetostatics.**
Theoretical consequence

Corollary
Let $\sigma_1, \sigma_2 \in L^\infty_+(B)$ satisfy (UCP) and $\Lambda_{\sigma_1}, \Lambda_{\sigma_2}$ be the corresponding current-to-volage-maps.
If $\sigma_2 \geq \sigma_1$ in some neighbourhood $V$ of $S$ and $\sigma_2 - \sigma_1 \in L^\infty_+(U)$ for some open $U \subseteq V$ then there exists $(g_n)$ such that
$$\langle (\Lambda_{\sigma_2} - \Lambda_{\sigma_1})g_n, g_n \rangle \to \infty,$$
so in particular $\Lambda_{\sigma_2} \neq \Lambda_{\sigma_1}$.

Consequence:
Piecewise constant conductivities $\sigma$ are uniquely determined by $\Lambda_\sigma$ (cf. e.g. Druskin, 1998, for local boundary data for of a halfspace)
Construction
More constructive version of the functional analysis:
Let \( h \in \mathcal{R}(L_{\Omega_1}) \), \( h \notin \mathcal{R}(L_{\Omega_2}) \). Define \( \gamma_\alpha \in L^2_\infty(S) \) by
\[
\gamma_\alpha = (L_{\Omega_2}L^*_{\Omega_2} + \alpha I)^{-1}h, \quad \alpha > 0.
\]
Then
\[
\|L'_{\Omega_2} \gamma_\alpha\|^2 \leq C \|L'_{\Omega_1} \gamma_\alpha\| \quad \text{and} \quad \|L'_{\Omega_2} \gamma_\alpha\| \to \infty \quad \text{for} \ \alpha \to 0.
\]
So for the currents \( g_\alpha := \frac{1}{\|L'_{\Omega_2} \gamma_\alpha\|^{3/2}} \gamma_\alpha \), the corresponding potentials \( u_\alpha \) satisfy
\[
\|L'_{\Omega_1} g_\alpha\|^2 \approx \int_{\Omega_1} \sigma |\nabla u_\alpha|^2 \, dx \to \infty,
\]
\[
\|L'_{\Omega_2} g_\alpha\|^2 \approx \int_{\Omega_2} \sigma |\nabla u_\alpha|^2 \, dx \to 0.
\]
Construction

Even more specific for $\sigma = 1$:

- $h_z$: boundary data of an electric dipole in $z \in B$, i.e., $h_z = u_z|_S$, where

$$\Delta u_z = d \cdot \nabla \delta_z, \quad \partial_{\nu} u_z|_{\partial B} = 0$$

($d \in \mathbb{R}^n$, $|d| = 1$ fixed arbitrary direction).

- If $B \setminus \overline{\Omega}$ is connected and its boundary contains $S$, then for $z \notin \partial \Omega$:

$$h_z \in \mathcal{R}(L_{\Omega}) \quad \text{iff} \quad z \in \Omega.$$

- Define currents $g_{\alpha,z}$, potentials $u_{\alpha,z}$ as on the last slide, then

$$\int_{\Omega_1} \sigma |\nabla u_{\alpha,z}|^2 \, dx \to \infty, \quad \int_{\Omega_2} \sigma |\nabla u_{\alpha,z}|^2 \, dx \to 0$$

for every neighbourhood $\Omega_1$ of $z$. 
Numerical example

Electric potentials can be localized in very general domains but the problem is very ill-posed.

$z = (0, 0)$, color axis scaled logarithmically and cropped.

$S = \partial B$

$\rightsquigarrow$ Electric potentials can be localized in very general domains but the problem is very ill-posed.
Factorization Method
Factorization Method

Special case of EIT: locate inclusions in known background medium.

Current-to-voltage map with inclusion:
\[
\Lambda_1 : g \mapsto u_1|_{\partial B},
\]
where \( u_1 \) solves \( \nabla \cdot \sigma \nabla u_1 = 0, \partial_\nu u_1|_{\partial B} = g \)
with \( \sigma = 1 + \sigma_1 \chi_\Omega, \sigma_1 > 0 \).

Current-to-voltage map without inclusion:
\[
\Lambda_0 : g \mapsto u_0|_{\partial B},
\]
where \( u_0 \) solves \( \Delta u_0 = 0, \partial_\nu u_0|_{\partial B} = g \)
with \( \sigma = 1 \).

Factorization method reconstructs \( \Omega \) from \( \Lambda := \Lambda_0 - \Lambda_1 \)
by using that for \( z \notin \partial \Omega \):
\[
h_z \in \mathcal{R}(\Lambda^{1/2}) \quad \text{iff} \quad z \in \Omega.
\]
Factorization Method

Factorization Method: for \( z \notin \partial \Omega \)
\[
z \in \Omega \quad \text{iff} \quad h_z \in \mathcal{R}(\Lambda^{1/2}).
\]

Numerical implementations usually calculate regularized preimage
\[
\psi_{z,\alpha} \approx \Lambda^{-1/2} h_z
\]
and use that (for \( z \notin \partial \Omega \))
\[
z \in \Omega \quad \text{iff} \quad \|\psi_{z,\alpha}\| \text{ bounded}.
\]

\( \psi_{z,\alpha} \approx \Lambda^{-1/2} h_z \) has no physical interpretation.

Similar criterion (for \( z \notin \partial \Omega \)) uses \( g_{z,\alpha} := \Lambda^* (\Lambda \Lambda^* + \alpha I)^{-1} h_z \approx \Lambda^{-1} h_z \)
\[
z \in \Omega \quad \text{iff} \quad \|L'_{\Omega} g_{z,\alpha}\| \text{ bounded}.
\]

Physical interpretation: \( L'_{\Omega} g_{z,\alpha} = u_{z,\alpha} |_{\Omega} \) with electric potential \( u_{z,\alpha} \)
generated by \( g_{z,\alpha} \) (in homogeneous body).
Lemma
There exist $C, C' > 0$:

$$|\nabla u_{z,\alpha}(z)| \leq C \|L_{\Omega}g_{z,\alpha}\| \quad \text{for } z \in \Omega,$$

$$|\nabla u_{z,\alpha}(z)| \geq C' \|L_{\Omega}g_{z,\alpha}\|^2 \quad \text{for } z \notin \Omega.$$

Variant of the Factorization Method (for $z \notin \partial \Omega$):

$$z \in \Omega \iff |\nabla u_{z,\alpha}(z)| \text{ bounded}.$$  

(EIT-analogue of Arens’ variant of the FM / LSM for inverse scattering).

Interpretation in context of localized potentials:
For fixed $z \notin \Omega$ (and after appropriate scaling) $u_{z,\alpha}(\cdot)$ are potentials with

- energy tending to zero on $\Omega$,
- energy tending to infinity on every neighbourhood of $z$. 

Bastian Gebauer: "Electric potentials with localized divergence properties"
In theory, electric potentials can be localized on quite arbitrary regions as long as they are connected to the boundary.

In practice, such potentials can be calculated by solving ill-posed equations.

The approximated preimages of the Factorization Method (after appropriate scaling) can be interpreted as potentials that localize outside the unknown inclusions.