Detecting Interfaces in a Parabolic-Elliptic Problem from Surface Measurements

Bastian Gebauer
gebauer@math.uni-mainz.de
Johannes Gutenberg-Universität Mainz, Germany

Joint work with Florian Fruehauf & Otmar Scherzer, University of Innsbruck

SIAM Conference on Imaging Science,
Minneapolis, Minnesota, May 15–17, 2006
A parabolic-elliptic problem

Heat equation: \[ \partial_t (c(x)u(x, t)) - \nabla \cdot (\kappa(x) \nabla u(x, t)) = 0 \]

in \( B = Q \cup \overline{\Omega} \)

\( u(x, t) \): temperature \hspace{1cm} \( c(x) \): heat capacity \hspace{1cm} \( \kappa(x) \): heat conductivity

Special case:

\[ c(x) = \begin{cases} 1 & \text{in } \Omega \\ 0 & \text{in } Q \end{cases} \]

parabolic equation in \( \Omega \)

\[ \text{"} \partial_t u - \Delta u = 0 \text{"} \]

\[ \kappa(x) = \begin{cases} \kappa_1 > 1 & \text{in } \Omega \\ 1 & \text{in } Q \end{cases} \]

elliptic equation in \( Q \)

\[ \text{"} \Delta u = 0 \text{"} \]

Motivation:

- Domain with inclusions of much higher heat capacity
- Electrically conducting objects in a non-conducting background illuminated by low-frequency electromagnetic waves

B. Gebauer: "Detecting Interfaces in a Parabolic-Elliptic Problem from Surface Measurements"
Direct problem / Inverse Problem

Direct Problem: For every heat flux $g$ there is a unique solution $u_1$ of

$$\partial_t (\chi_\Omega u_1) - \nabla \cdot (\kappa \nabla u_1) = 0,$$  \hspace{1cm} (1)
$$\partial_\nu u_1|_{\partial B} = g,$$  \hspace{1cm} (2)
$$u_1(x, 0)|_\Omega = 0.$$  \hspace{1cm} (3)

(can be proven in appropriate Sobolev spaces using Lions Projection Lemma.)

Inverse Problem: Given a complete set of measurements

$$\Lambda_1 : g \mapsto u_1|_{\partial B}, \quad u_1 \text{ solves (1)–(3)},$$
reconstruct the interface $\partial \Omega$ resp. the inclusion $\Omega$.

To solve the inverse problem we compare $\Lambda_1$ to reference measurements

$$\Lambda_0 : g \mapsto u_0|_{\partial B}, \quad u_0 \text{ solves } \Delta u_0 = 0, \quad \partial_\nu u_0|_{\partial B} = g,$$
i. e. measurements without an inclusion $\Omega$.

Goal: Reconstruct $\Omega$ from given $\Lambda_0$ and $\Lambda_1$. 
Virtual Measurements

$\psi$: given boundary flux on $\partial \Omega$

$L : \psi \mapsto v|_{\partial B}$, where

$$\Delta v(x,t) = 0 \quad \text{in} \quad Q \times ]0, T[, \quad (4)$$

$$\partial_{\nu} v|_{\partial B} = 0 \quad \text{on} \quad \partial B, \quad (5)$$

$$\partial_{\nu} v|_{\partial \Omega} = \psi \quad \text{on} \quad \partial \Omega. \quad (6)$$

$\mathcal{R}(L)$ determines $\Omega$:

$$v_z|_{\partial B} \in \mathcal{R}(L) \quad \text{if and only if} \quad z \in \Omega$$

where $v_z$ solves (4) in $B \setminus \{z\}$, $v_z$ solves (5), $v_z$ suff. singular in $z \in B$,

(e.g. a partial derivative of the Green's function for the Laplacian)
**Factorization Method**

Key identity of the so-called Factorization Method (for other problems!):

\[ \mathcal{R}(L) = \mathcal{R}((\Lambda_0 - \Lambda_1)^{1/2}). \]

\( \mathcal{R}(L) \) (and thus \( \Omega \)) can be computed from the measurements.

Such a range identity

- was originally developed by Kirsch for Inverse Scattering
- is known (under suitable conditions on the inclusion) for
  - Electrostatics (Hähner)
  - EIT (Brühl, Hanke), also with different electrode models (Brühl, Hanke, Hyvönen) and in the half space (Schappel)
  - Diffusion tomography (Kirsch), also with Robin B.C. (Hyvönen)
  - general real elliptic problems (G.)

*Does a similar identity hold in this parabolic-elliptic case?*
Main Result

Range inclusions:

\[
\mathcal{R}(\tilde{\Lambda}^{1/2}) \subseteq \mathcal{R}(L), \\
\mathcal{R}(\tilde{\Lambda}^{1/2}) \supseteq \mathcal{R}(L|_V),
\]

\(\tilde{\Lambda}\): symmetric part of \(\Lambda_1 - \Lambda_0\),

\(V\): space of boundary fluxes with certain temporal smoothness

Existence of singular functions \(v_z\) with

\[
v_z|_{\partial B} \in \mathcal{R}(L) \quad \text{if and only if} \quad z \in \Omega,
\]

and \(\partial_{\nu}v_z|_{\partial \Omega} \in V\).

\[\leadsto\]

\(z \in \Omega\) if and only if \(v_z|_{\partial B} \in \mathcal{R}(\tilde{\Lambda}^{1/2})\).
Sketch of the proof

Factorization:

\[ \tilde{\Lambda} = LFL^* \]

If \( \|Ax\| \leq \|Bx\| \) for all \( x \) then \( \mathcal{R}(A^*) \subseteq \mathcal{R}(B^*) \).

\[ \implies \mathcal{R}(\tilde{\Lambda}^{1/2}) = \mathcal{R}(LF^{1/2}) \subseteq \mathcal{R}(L) \]

Coercivity condition for \( F \)

\[ \implies \mathcal{R}(F^{1/2}) \supseteq H^{\frac{1}{4}}(0, T, H^{-\frac{1}{2}}(\partial \Omega)) =: V \]
Consequences / Remarks

Theoretical result:

$\partial \Omega$ is uniquely determined by $\Lambda_1$, i.e. the interface is uniquely determined by measuring all pairs of heat flux and temperature on $\partial B$.

Range test $v_z|_{\partial B} \in \mathcal{R}(\tilde{\Lambda}^{1/2})$ can be implemented numerically $\Rightarrow$ practical reconstruction algorithm.

$\Omega$ does not have to be connected. No apriori information about the number of connected components is needed.

Usual numerical implementation of range test leads to the problem of determining the convergence of a series from finitely many summands.

$\Rightarrow$ threshold problem.
Numerical results

Reconstruction of single and multiple inclusions.

B. Gebauer: "Detecting Interfaces in a Parabolic-Elliptic Problem from Surface Measurements"
Numerical results

Reconstruction of a nonconvex inclusion
(left: no noise, right: 0.1% noise)