Detecting Interfaces in a Parabolic-Elliptic Problem from Surface Measurements

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A parabolic-elliptic problem

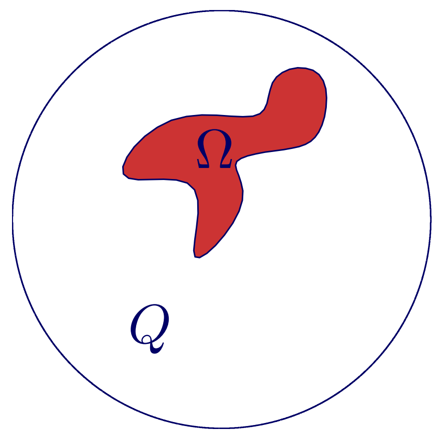
Heat equation: $$\partial_t(c(x)u(x,t)) - \nabla\cdot(\kappa(x)\nabla u(x,t)) = 0$$ in $B = Q \cup \overline{\Omega}$

$u(x,t)$: temperature $c(x)$: heat capacity $\kappa(x)$: heat conductivity

Special case:

$$c(x) = \begin{cases} 1 & \text{in } \Omega \\ 0 & \text{in } Q \end{cases}$$

$$\kappa(x) = \begin{cases} \kappa_1 > 1 & \text{in } \Omega \\ 1 & \text{in } Q \end{cases}$$



parabolic equation in Ω
("$\partial_t u - \Delta u = 0$")

elliptic equation in Q
("$\Delta u = 0$")

Motivation:

- Domain with inclusions of much higher heat capacity
- Electrically conducting objects in a non-conducting background illuminated by low-freqency electromagnetic waves

Direct problem / Inverse Problem

Direct Problem: For every heat flux g there is a unique solution u_1 of

$$\begin{aligned} \partial_t(\chi_\Omega u_1) - \nabla \cdot (\kappa \nabla u_1) &= 0, && (1)\\ \partial_\nu u_1|_{\partial B} &= g, && (2)\\ u_1(x,0)|_\Omega &= 0. && (3) \end{aligned}$$

(can be proven in appropriate Sobolev spaces using Lions Projection Lemma.)

Inverse Problem: Given a complete set of measurements

$$\Lambda_1 : \; g \mapsto u_1|_{\partial B}, \quad u_1 \text{ solves (1)–(3)},$$

reconstruct the interface $\partial\Omega$ resp. the inclusion Ω.

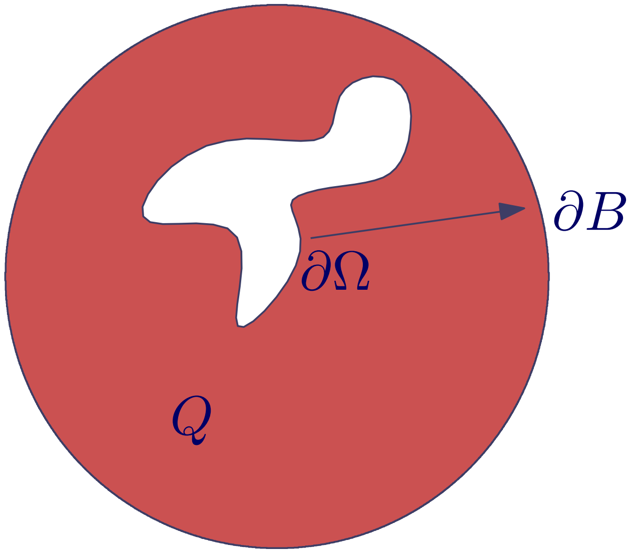
To solve the inverse problem we compare Λ_1 to reference measurements

$$\Lambda_0 : \; g \mapsto u_0|_{\partial B}, \quad u_0 \text{ solves } \Delta u_0 = 0, \quad \partial_\nu u_0|_{\partial B} = g,$$

i. e. measurements without an inclusion Ω.

Goal: Reconstruct Ω from given Λ_0 and Λ_1.

Virtual Measurements



ψ: given boundary flux on $\partial\Omega$

$$L: \psi \mapsto v|_{\partial B}, \quad \text{where}$$

$$\begin{aligned} \Delta v(x,t) &= 0 && \text{in } Q\times]0,T[, && (4)\\ \partial_\nu v|_{\partial B} &= 0 && \text{on } \partial B, && (5)\\ \partial_\nu v|_{\partial\Omega} &= \psi && \text{on } \partial\Omega. && (6) \end{aligned}$$

$\mathcal{R}(L)$ determines Ω:

$$v_z|_{\partial B} \in \mathcal{R}(L) \quad \text{if and only if} \quad z \in \Omega$$

where v_z solves (4) in $B \setminus \{z\}$, v_z solves (5), v_z suff. singular in $z \in B$,

(e. g. a partial derivative of the Green's function for the Laplacian)

Factorization Method

Key identity of the so-called Factorization Method (for other problems!):

$$\mathcal{R}(L) = \mathcal{R}((\Lambda_0 - \Lambda_1)^{1/2}).$$

$\rightsquigarrow \mathcal{R}(L)$ (and thus Ω) can be computed from the measurements.

Such a range identity

- was originally developed by Kirsch for Inverse Scattering
- is known (under suitable conditions on the inclusion) for
  - Electrostatics (Hähner)
  - EIT (Brühl, Hanke), also with different electrode models (Brühl, Hanke, Hyvönen) and in the half space (Schappel)
  - Diffusion tomography (Kirsch), also with Robin B.C. (Hyvönen)
  - general real elliptic problems (G.)

Does a similar identity hold in this parabolic-elliptic case?

Main Result

- Range inclusions:

$$\mathcal{R}(\tilde{\Lambda}^{1/2}) \subseteq \mathcal{R}(L),$$
$$\mathcal{R}(\tilde{\Lambda}^{1/2}) \supseteq \mathcal{R}(L|_V),$$

$\tilde{\Lambda}$: symmetric part of $\Lambda_1 - \Lambda_0$,

V: space of boundary fluxes with certain temporal smoothness

- Existence of singular functions v_z with

$$v_z|_{\partial B} \in \mathcal{R}(L) \quad \text{if and only if} \quad z \in \Omega,$$

and $\partial_\nu v_z|_{\partial\Omega} \in V$.

$$\rightsquigarrow \quad z \in \Omega \quad \text{if and only if} \quad v_z|_{\partial B} \in \mathcal{R}(\tilde{\Lambda}^{1/2}).$$

Sketch of the proof

$$\mathcal{R}(\tilde{\Lambda}^{1/2}) \subseteq \mathcal{R}(L),$$

$$\mathcal{R}(\tilde{\Lambda}^{1/2}) \supseteq \mathcal{R}(L|_V),$$

- Factorization:

$$\tilde{\Lambda} = LFL^*$$

- If $\|Ax\| \leq \|Bx\|$ for all x then $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$.

$\rightsquigarrow$ $\mathcal{R}(\tilde{\Lambda}^{1/2}) = \mathcal{R}(LF^{1/2}) \subseteq \mathcal{R}(L)$.

- Coercivity condition for F

$\rightsquigarrow$ $\mathcal{R}(F^{1/2}) \supseteq H^{\frac{1}{4}}(0,T,H_\diamond^{-\frac{1}{2}}(\partial\Omega)) =: V$.

Consequences / Remarks

- Theoretical result:

$$\partial\Omega \text{ is uniquely determined by } \Lambda_1,$$

i. e. the interface is uniquely determined by measuring all pairs of heat flux and temperature on ∂B.

- Range test $v_z|_{\partial B} \in \mathcal{R}(\tilde{\Lambda}^{1/2})$ can be implemented numerically

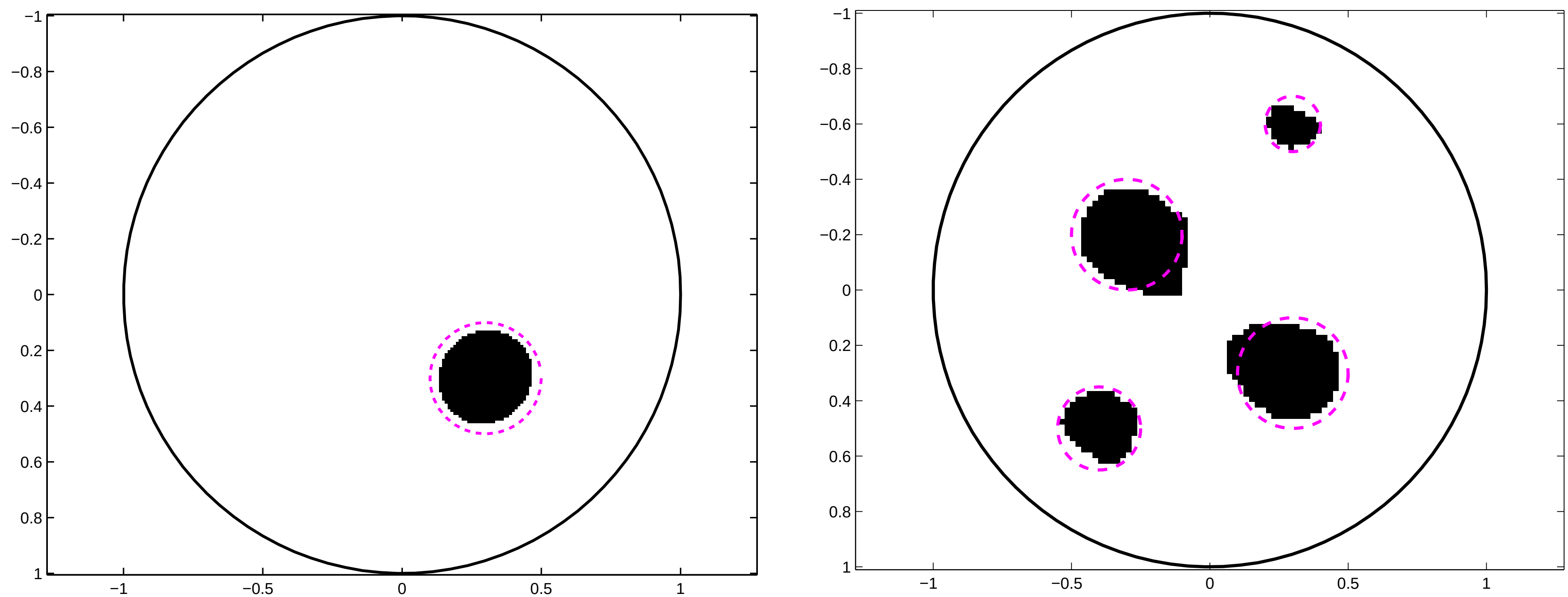
$\rightsquigarrow$ practical reconstruction algorithm.

- Ω does not have to be connected. No apriori information about the number of connected components is needed.

- Usual numerical implementation of range test leads to the problem of determining the convergence of a series from finitely many summands.

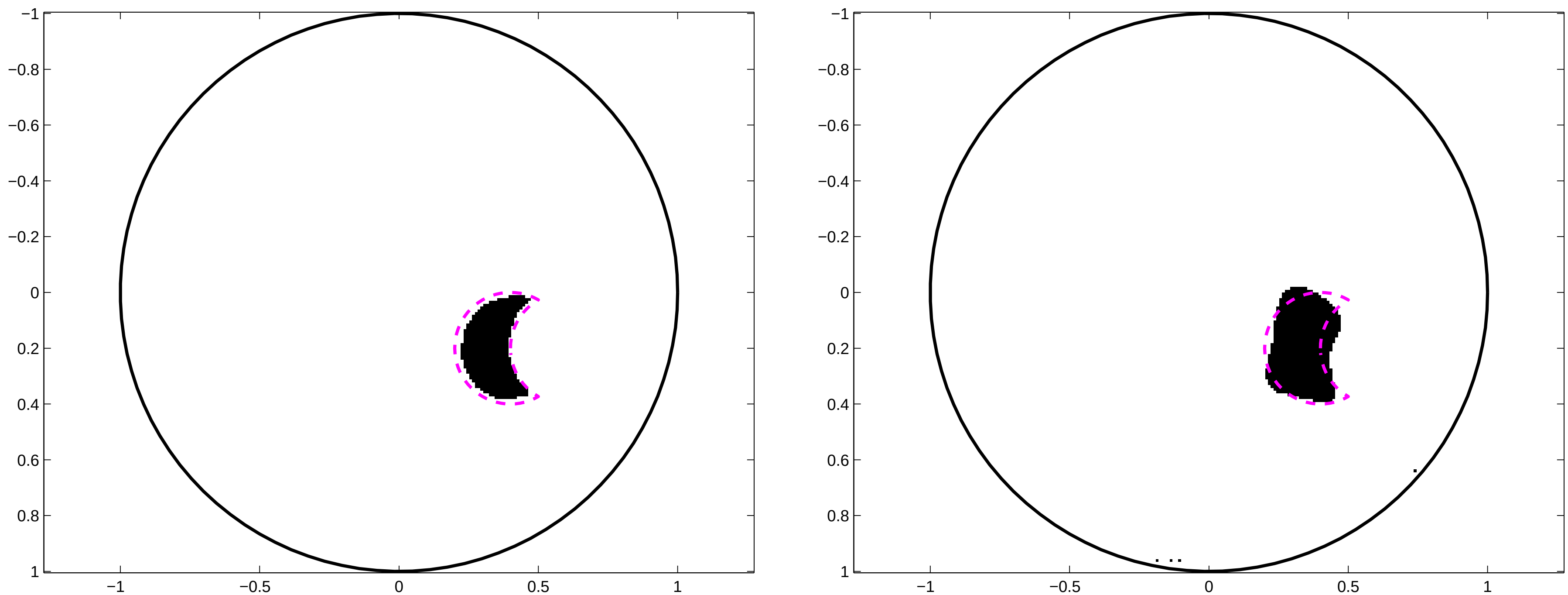
$\rightsquigarrow$ treshold problem.

Numerical results



Reconstruction of single and multiple inclusions.

Numerical results



Reconstruction of a nonconvex inclusion
(left: no noise, right: 0.1% noise)