Detecting objects by low-frequency electromagnetic imaging

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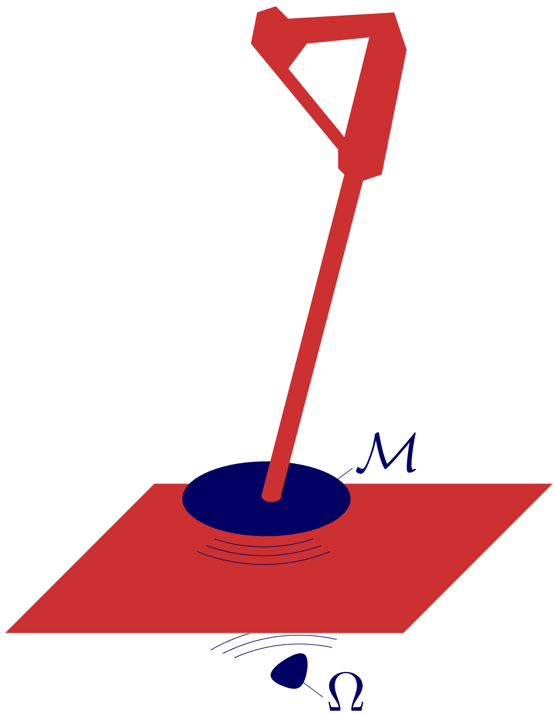
Setting



$\mathcal{M}$: measurement device

Ω: magnetic object

- Apply surface currents J on $\mathcal{M}$ (time-harmonic with frequency ω).

$\rightsquigarrow$ electromagnetic field (E^ω, H^ω) (time-harmonic with frequency ω)

- Measure field on $\mathcal{M}$ (and try to locate Ω from it).

wavelength $\approx 15\,\mathrm{km}$ $\gg$ size of object $\approx 10\,\mathrm{cm}$

($\rightsquigarrow$ frequency ω very small)

What happens when $\omega \to 0$?

Maxwell's equations

Time-harmonic Maxwell's equations

$$\begin{aligned} \operatorname{curl} H^\omega + \mathrm{i}\,\omega\epsilon E^\omega &= J && \text{in } \mathbb{R}^3,\\ -\operatorname{curl} E^\omega + \mathrm{i}\,\omega\mu H^\omega &= 0 && \text{in } \mathbb{R}^3,\\ \operatorname{div}(\epsilon E^\omega) &= 0 && \text{in } \mathbb{R}^3,\\ \operatorname{div}(\mu H^\omega) &= 0 && \text{in } \mathbb{R}^3, \end{aligned}$$

Silver-Müller radiation condition (RC)

$$\int_{\partial B_\rho} \left|\nu \wedge \sqrt{\mu} H^\omega + \sqrt{\epsilon} E^\omega\right|^2 \mathrm{d}\sigma = o(1), \quad \rho \to \infty.$$

E^ω:	electric field	ϵ:	dielectricity ($= const.$ around $\mathcal{M}$)
H^ω:	magnetic field	μ:	permeability (magnetic properties)
ω:	frequency	J:	applied currents, $\operatorname{div} J = 0$, $\operatorname{supp} J \subseteq \mathcal{M}$

relative parameter values: $\epsilon = 1$, $\mu = 1$ outside some bounded domain

Formal asymptotic analysis

Solve Maxwell's equations for E^ω:

$$\begin{aligned} \operatorname{curl}\frac{1}{\mu}\operatorname{curl}E^\omega - \omega^2\epsilon E^\omega &= \mathrm{i}\,\omega J && \text{in } \mathbb{R}^3, \\ \operatorname{div}(\epsilon E^\omega) &= 0 && \text{in } \mathbb{R}^3 \quad \text{(redundant)}, \end{aligned}$$

Real frequency $20\,\mathrm{kHz}$ $\rightsquigarrow$ relative parameter $\omega \approx 4 \times 10^{-4}\,\mathrm{m}^{-1}$

Neglecting terms in ω^2 suggests that $E^\omega \approx \mathrm{i}\,\omega E$, with

$$\begin{aligned} \operatorname{curl}\frac{1}{\mu}\operatorname{curl}E &= J \\ \operatorname{div}(\epsilon E) &= 0 \quad \text{(not redundant anymore)} \end{aligned}$$

Rigorous asymptotic analysis (for fixed incoming waves):

Ammari, Nedelec: *Low Frequency electromagnetic scattering*, SIAM J. Math. Anal., 2000.

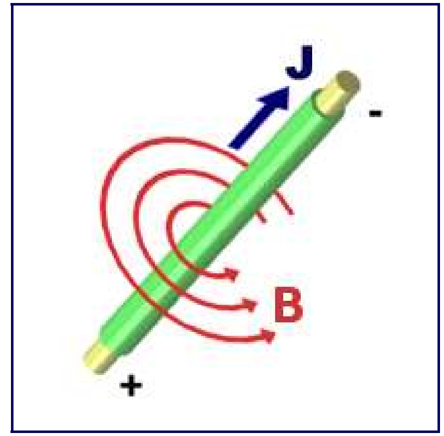
Interpretation

$E^\omega \approx \mathrm{i}\,\omega E$, where $\operatorname{curl} \frac{1}{\mu} \operatorname{curl} E = J, \quad \operatorname{div}(\epsilon E) = 0$

- $B := \operatorname{curl} E$ solves

$$\operatorname{curl} \tfrac{1}{\mu} B = J, \quad \operatorname{div} B = 0.$$

$\rightsquigarrow$ B is the magnetostatic field generated by a steady current J (*Ampère's Law).*



- $B = \operatorname{curl} E$ $\rightsquigarrow$ E is a vector potential of B (unique up to addition of A with $\operatorname{curl} A = 0$, i. e. up to $A = \nabla\varphi$).
- $\operatorname{div}(\epsilon E) = 0$ determines E uniquely (so-called *Coulomb gage*).

$\rightsquigarrow$ $\operatorname{curl} \frac{1}{\mu} \operatorname{curl} E = J, \quad \operatorname{div}(\epsilon E) = 0$ describe magnetostatics.

Rigorous mathematical results

Assume that

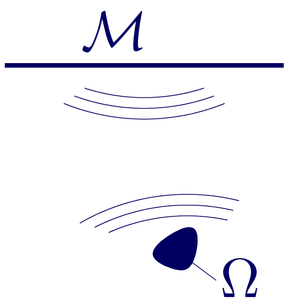
- $J \in TL^2_\diamond(\mathcal{M};\mathbb{C}^3)$, i. e. $J \in TL^2(\mathcal{M};\mathbb{C}^3)$, $\operatorname{div}_\mathcal{M} J = 0$ and $\nu \cdot J|_{\partial M} = 0$.
- $\epsilon, \mu \in L^\infty_+(\mathbb{R}^3;\mathbb{R})$ are identical to 1 outside some bounded domain.
- ϵ is constant in some neighborhood of $\mathcal{M}$

Theorem

- There exists a unique solution E of the magnetostatic equations.
- For every bounded domain D, there exists $C > 0$, $\omega_0 > 0$, such that for every $0 < \omega < \omega_0$ and every $J \in TL^2_\diamond(\mathcal{M})$ there is a unique solution E^ω of Maxwell's equations and

$$\|E^\omega - \mathrm{i}\,\omega E\|_{H(\mathrm{curl},D)} \leq C\omega^3 \|J\|_{TL^2_\diamond(\mathcal{M})}.$$

Measurements



- Permeability $\mu(x) = 1 + \mu_1\,\chi_\Omega(x)$, $\mu_1 > 0$
- Apply surface currents J on $\mathcal{M}$
- $\rightsquigarrow$ electromagnetic field (E^ω, H^ω)
- Measure field on $\mathcal{M}$

"Full set of measurements" corresponds to measurement operator

$$\Lambda^\omega : \begin{cases} TL^2_\diamond(\mathcal{M};\mathbb{C}^3) & \to & TL^2_\diamond(\mathcal{M};\mathbb{C}^3), \\ J & \mapsto & E^\omega_\tau|_\mathcal{M}, \end{cases}$$

E^ω solves Maxwell's eq.

Magnetostatic measurements would be

$$\Lambda : \begin{cases} TL^2_\diamond(\mathcal{M};\mathbb{C}^3) & \to & TL^2_\diamond(\mathcal{M};\mathbb{C}^3), \\ J & \mapsto & E_\tau|_\mathcal{M}, \end{cases}$$

E solves magnetostatic eq.

$$\Lambda = \frac{1}{i\omega}\Lambda^\omega + O(\omega^2) \quad \text{in } \mathcal{L}(TL^2_\diamond(\mathcal{M};\mathbb{C}^3), TL^2_\diamond(\mathcal{M};\mathbb{C}^3))$$

Inverse Problem

To reconstruct Ω we apply the so-called Factorization Method:

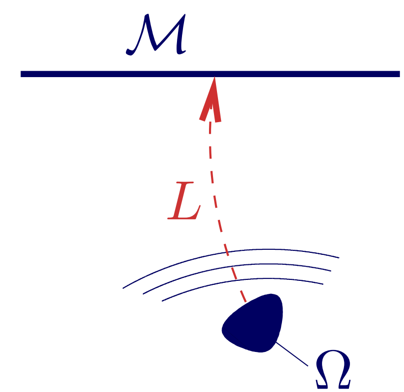
- Method was originally developed by Kirsch (1998) for far-field measurements in inverse scattering (Helmholtz equation)
- Method was generalized to EIT by Brühl and Hanke (1999).
- Method works for harmonic vector fields (Kress, 2002) and for far-field measurements for Maxwell's equations (Kirsch, 2004)
- Method works for general real elliptic equations (G, 2005)

Here:

- Magnetostatic equations are real elliptic differential equation.
  - $\rightsquigarrow$ Ω can be reconstructed from magnetostatic measurements Λ
- $\Lambda = \frac{1}{i\omega}\Lambda^\omega + O(\omega^2)$
  - $\rightsquigarrow$ Ω can be reconstructed from electromagnetic measurements Λ^ω (when the frequency ω is small).

Factorization Method

Factorization Method compares Λ with reference measurements Λ_0
(reference = without object Ω)



- Range identity:

$$\mathcal{R}((\Lambda - \Lambda_0)^{1/2}) = \mathcal{R}(L),$$

with some auxiliary operator L.

$\rightsquigarrow$ $\mathcal{R}(L)$ is determined by the measurements Λ, Λ_0.

- Test functions: For points z below $\mathcal{M}$

$$z \in \Omega \quad \text{if and only if} \quad (v_z)_\tau|_{\mathcal{M}} \in \mathcal{R}(L)$$

with certain functions v_z having a singularity in z.

$\rightsquigarrow$ Object Ω can be located from $\mathcal{R}(L)$.

Numerical results

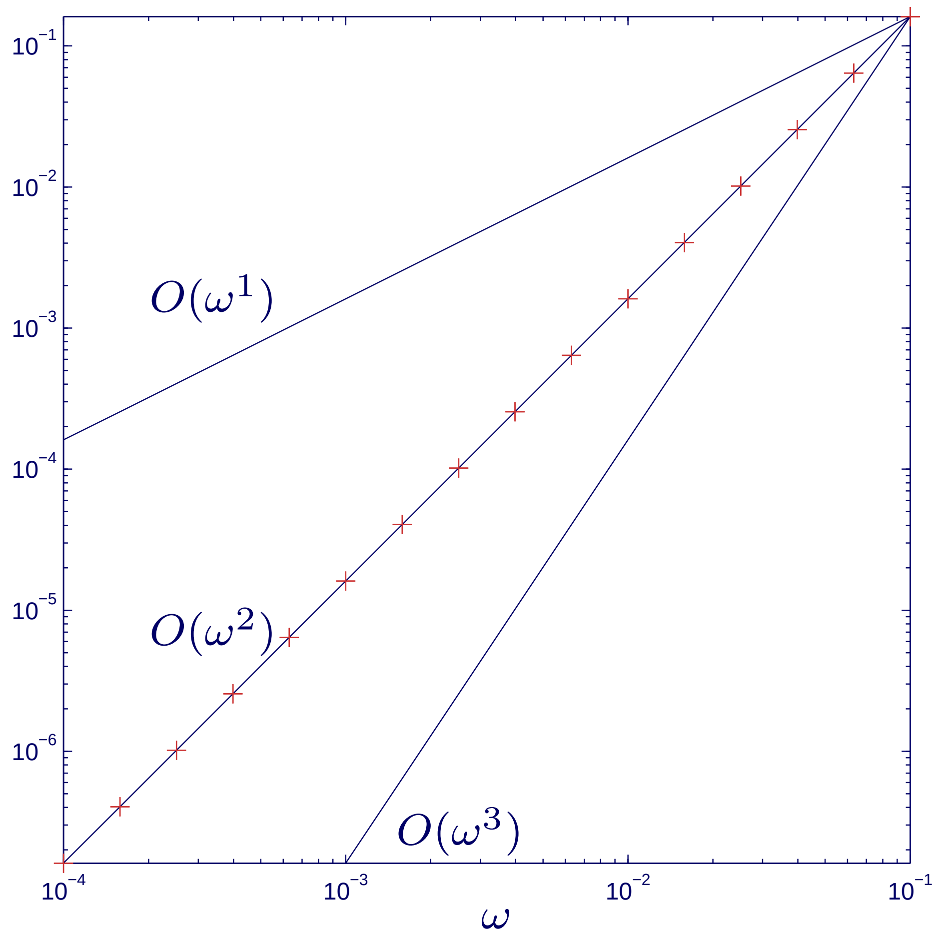
Detection algorithm: For every point z on a sampling grid below $\mathcal{M}$:

- Test whether $(v_z)_\tau|_\mathcal{M} \in \mathcal{R}((\Lambda - \Lambda_0)^{1/2})$.
- If yes, mark point as ”inside object Ω”.

Christoph Schneider tested this method with his code from the BMBF project "HuMin/MD – Metal detectors for humanitarian demining".

- Measurement device $\mathcal{M}$: $40\,\mathrm{cm} \times 40\,\mathrm{cm}$
- Scatterer ("the mine"): Ball, $8\,\mathrm{cm}$ diameter, $15\,\mathrm{cm} - 20\,\mathrm{cm}$ below $\mathcal{M}$
- Permeability ”$\mu = \infty$” in Ω
- Frequency $19{,}2\,\mathrm{kHz} \rightsquigarrow \omega \approx 4 \times 10^{-4}\,\mathrm{m}^{-1}$
- Currents imposed / electric fields measured on 100 ”loops” on $\mathcal{M}$
- Simulated data (BEM) using code from K. Erhard, Göttingen

Numerical results - asymptotics



Numerical test for convergence

$$\omega \mapsto \left\| \frac{1}{\mathrm{i}\,\omega} \tilde{\Lambda}^{\omega} - \tilde{\Lambda} \right\| \Big/ \|\tilde{\Lambda}\|,$$

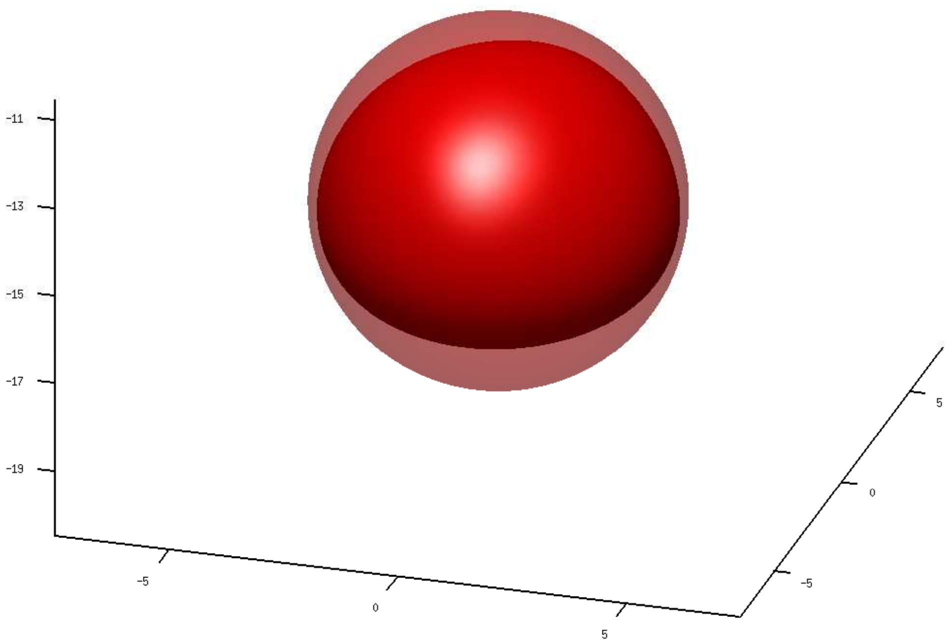
where

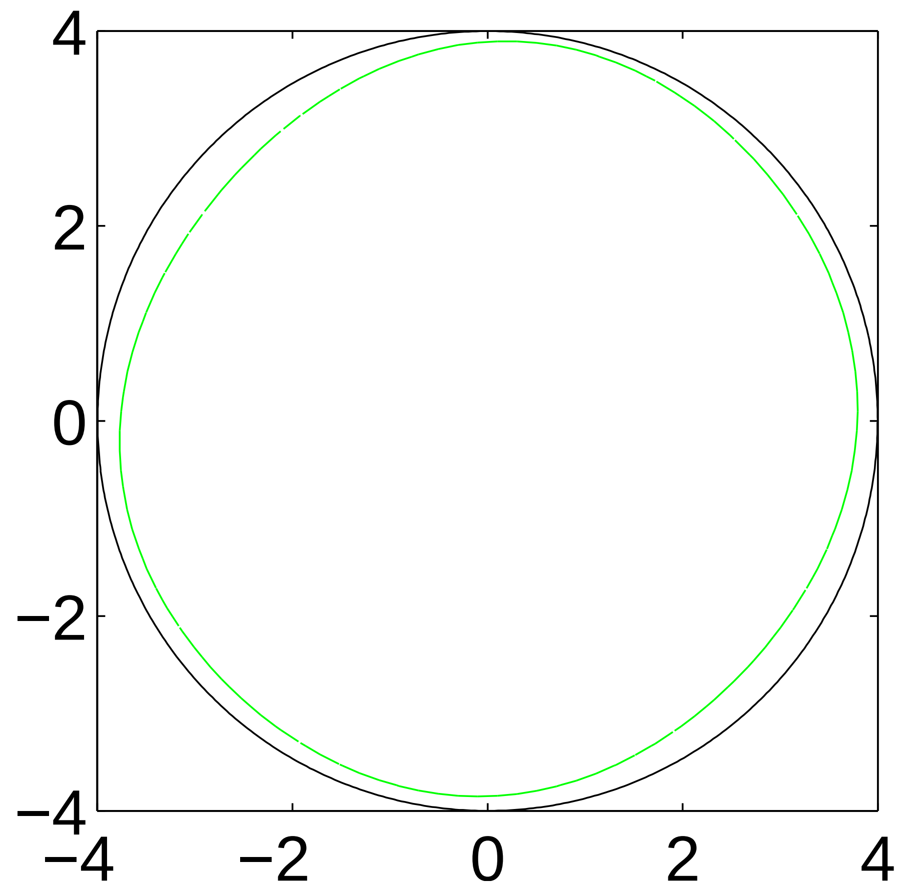
$$\begin{aligned} \tilde{\Lambda}^{\omega} &\approx \Lambda^{\omega} - \Lambda_0^{\omega} \\ \tilde{\Lambda} := \tilde{\Lambda}^{10^{-7}} &\approx \Lambda - \Lambda_0 \end{aligned}$$

are calculated with the forward solver from Göttingen.

$$\rightsquigarrow \quad \Lambda = \frac{1}{\mathrm{i}\,\omega} \tilde{\Lambda}^{\omega} + O(\omega^2)$$

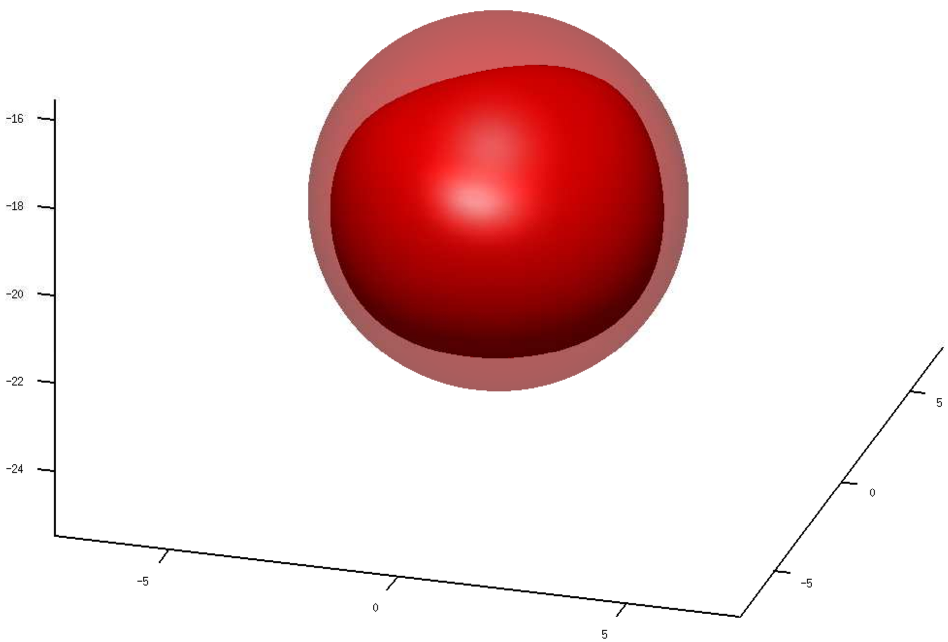
Numerical results - reconstruction

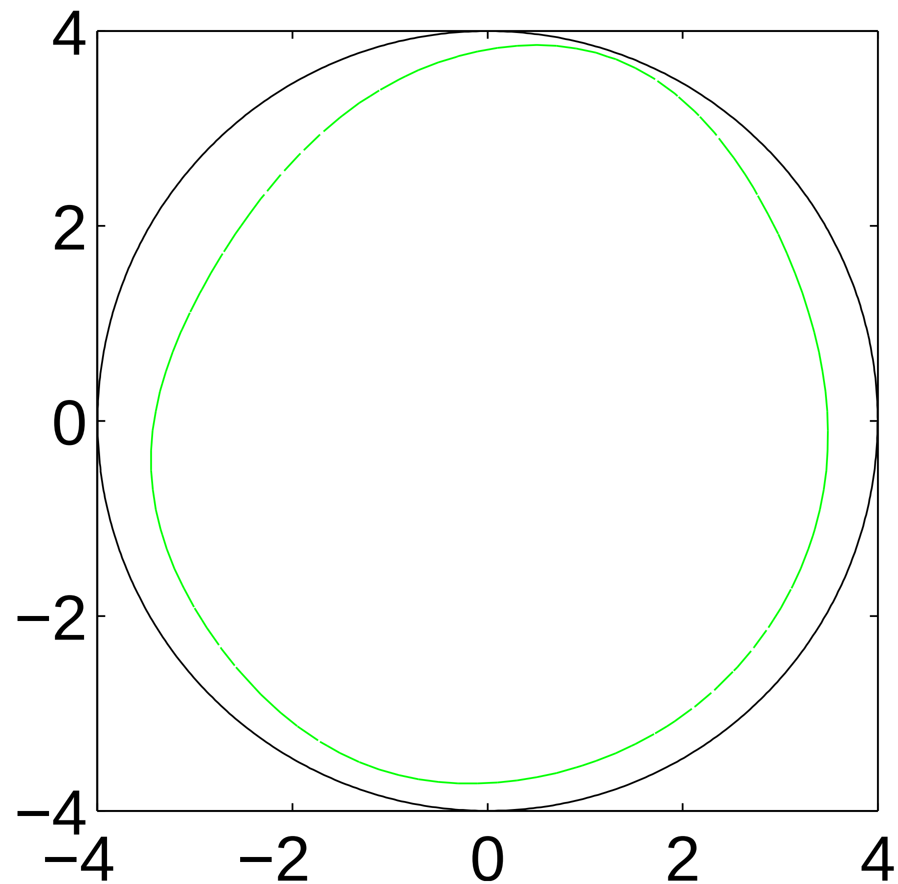




Ball with radius $r = 4cm$ located $15cm$ below $\mathcal{M}$

Numerical results - reconstruction





Ball with radius $r = 4cm$ located $20cm$ below $\mathcal{M}$