Detecting objects by low-frequency electromagnetic imaging

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**Setting**

$\mathcal{M}$: measurement device

$\Omega$: magnetic object

Apply surface currents $J$ on $\mathcal{M}$ (time-harmonic with frequency $\omega$).

\[ \Rightarrow \text{electromagnetic field } (E^\omega, H^\omega) \]

(time-harmonic with frequency $\omega$)

Measure field on $\mathcal{M}$ (and try to locate $\Omega$ from it).

wavelength $\approx 15 \text{ km} \gg$ size of object $\approx 10 \text{ cm}$

($\Rightarrow$ frequency $\omega$ very small)

**What happens when $\omega \rightarrow 0$?**
Maxwell’s equations

Time-harmonic Maxwell’s equations

\[
\begin{align*}
\text{curl } H^\omega + i \omega \varepsilon E^\omega &= J \quad \text{in } \mathbb{R}^3, \\
- \text{curl } E^\omega + i \omega \mu H^\omega &= 0 \quad \text{in } \mathbb{R}^3, \\
\text{div}(\varepsilon E^\omega) &= 0 \quad \text{in } \mathbb{R}^3, \\
\text{div}(\mu H^\omega) &= 0 \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

Silver-Müller radiation condition (RC)

\[
\int_{\partial B_\rho} |\nu \wedge \sqrt{\mu} H^\omega + \sqrt{\varepsilon} E^\omega|^2 \, d\sigma = o(1), \quad \rho \to \infty.
\]

\begin{align*}
E^\omega &\quad \text{electric field} & \varepsilon &\quad \text{dielectricity (\textit{const.} around } \mathcal{M}) \\
H^\omega &\quad \text{magnetic field} & \mu &\quad \text{permeability (magnetic properties)} \\
\omega &\quad \text{frequency} & J &\quad \text{applied currents, } \text{div } J = 0, \text{ supp } J \subseteq \mathcal{M}
\end{align*}

relative parameter values: \( \varepsilon = 1, \mu = 1 \) outside some bounded domain
Formal asymptotic analysis

Solve Maxwell’s equations for $E^\omega$:

\[
\begin{align*}
\text{curl} \left( \frac{1}{\mu} \text{curl} E^\omega \right) - \omega^2 \epsilon E^\omega &= i \omega J \quad \text{in } \mathbb{R}^3, \\
\text{div}(\epsilon E^\omega) &= 0 \quad \text{in } \mathbb{R}^3 \quad \text{(redundant)},
\end{align*}
\]

Real frequency 20 kHz $\sim \rightarrow$ relative parameter $\omega \approx 4 \times 10^{-4} \text{ m}^{-1}$

Neglecting terms in $\omega^2$ suggests that $E^\omega \approx i \omega E$, with

\[
\begin{align*}
\text{curl} \left( \frac{1}{\mu} \text{curl} E \right) &= J \\
\text{div}(\epsilon E) &= 0 \quad \text{(not redundant anymore)}
\end{align*}
\]

Rigorous asymptotic analysis (for fixed incoming waves):


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$E^\omega \approx i \omega E$, where $\text{curl} \frac{1}{\mu} \text{curl} E = J, \quad \text{div}(\epsilon E) = 0$

$B := \text{curl} E$ solves

$\text{curl} \frac{1}{\mu} B = J, \quad \text{div} B = 0.$

$\implies B$ is the magnetostatic field generated by a steady current $J$ (Ampère’s Law).

$B = \text{curl} E \implies E$ is a vector potential of $B$

(unique up to addition of $A$ with $\text{curl} A = 0$, i.e. up to $A = \nabla \varphi$).

$\text{div}(\epsilon E) = 0$ determines $E$ uniquely (so-called Coulomb gage).

$\implies \text{curl} \frac{1}{\mu} \text{curl} E = J, \quad \text{div}(\epsilon E) = 0$ describe magnetostatics.

Figure based on [http://de.wikipedia.org/wiki/Bild:RechteHand.png](http://de.wikipedia.org/wiki/Bild:RechteHand.png), published under the GNU Free Documentation License (FDL) by "Frau Holle".
Rigorous mathematical results

Assume that

\[ J \in TL^2_\diamond (\mathcal{M}; \mathbb{C}^3), \text{ i.e. } J \in TL^2(\mathcal{M}; \mathbb{C}^3), \text{ div}_\mathcal{M} J = 0 \text{ and } \nu \cdot J|_{\partial \mathcal{M}} = 0. \]

\[ \epsilon, \mu \in L^\infty_+ (\mathbb{R}^3; \mathbb{R}) \text{ are identical to } 1 \text{ outside some bounded domain.} \]

\[ \epsilon \text{ is constant in some neighborhood of } \mathcal{M} \]

Theorem

- There exists a unique solution \( E \) of the magnetostatic equations.

- For every bounded domain \( D \), there exists \( C > 0, \omega_0 > 0 \), such that for every \( 0 < \omega < \omega_0 \) and every \( J \in TL^2_\diamond(\mathcal{M}) \) there is a unique solution \( E^\omega \) of Maxwell’s equations and

\[
\| E^\omega - i \omega E \|_{H(\text{curl}, D)} \leq C \omega^3 \| J \|_{TL^2_\diamond(\mathcal{M})}.
\]
Measurements

\[ \mathcal{M} \]

- Permeability \( \mu(x) = 1 + \mu_1 \chi_{\Omega}(x), \mu_1 > 0 \)
- Apply surface currents \( J \) on \( \mathcal{M} \)
- Electromagnetic field \( (E^\omega, H^\omega) \)
- Measure field on \( \mathcal{M} \)

"Full set of measurements" corresponds to measurement operator

\[ \Lambda^\omega : \left\{ \begin{array}{ccc} TL^2_\diamond(\mathcal{M}; \mathbb{C}^3) & \rightarrow & TL^2_\diamond(\mathcal{M}; \mathbb{C}^3), \\ J & \mapsto & E^\omega|_{\mathcal{M}}, \end{array} \right. \]

\( E^\omega \) solves Maxwell’s eq.

Magnetostatic measurements would be

\[ \Lambda : \left\{ \begin{array}{ccc} TL^2_\diamond(\mathcal{M}; \mathbb{C}^3) & \rightarrow & TL^2_\diamond(\mathcal{M}; \mathbb{C}^3), \\ J & \mapsto & E_\tau|_{\mathcal{M}}, \end{array} \right. \]

\( E \) solves magnetostatic eq.

\[ \Lambda = \frac{1}{i\omega} \Lambda^\omega + O(\omega^2) \quad \text{in} \quad \mathcal{L}(TL^2_\diamond(\mathcal{M}; \mathbb{C}^3), TL^2_\diamond(\mathcal{M}; \mathbb{C}^3)) \]
Inverse Problem

To reconstruct $\Omega$ we apply the so-called Factorization Method:

- Method was originally developed by Kirsch (1998) for far-field measurements in inverse scattering (Helmholtz equation).
- Method was generalized to EIT by Brühl and Hanke (1999).
- Method works for harmonic vector fields (Kress, 2002) and for far-field measurements for Maxwell’s equations (Kirsch, 2004).
- Method works for general real elliptic equations (G, 2005).

Here:

- Magnetostatic equations are real elliptic differential equation.

  - $\Omega$ can be reconstructed from magnetostatic measurements $\Lambda$

  \[ \Lambda = \frac{1}{i\omega} \Lambda^\omega + O(\omega^2) \]

  - $\Omega$ can be reconstructed from electromagnetic measurements $\Lambda^\omega$ (when the frequency $\omega$ is small).
Factorization Method compares $\Lambda$ with reference measurements $\Lambda_0$ (reference = without object $\Omega$)

- Range identity:
  \[ \mathcal{R}((\Lambda - \Lambda_0)^{1/2}) = \mathcal{R}(L), \]
  with some auxiliary operator $L$.

  $\mapsto$ $\mathcal{R}(L)$ is determined by the measurements $\Lambda, \Lambda_0$.

- Test functions: For points $z$ below $\mathcal{M}$
  \[ z \in \Omega \quad \text{if and only if} \quad (v_z)_\tau |_{\mathcal{M}} \in \mathcal{R}(L) \]
  with certain functions $v_z$ having a singularity in $z$.

  $\mapsto$ Object $\Omega$ can be located from $\mathcal{R}(L)$. 

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Numerical results

Detection algorithm: For every point $z$ on a sampling grid below $\mathcal{M}$:

- Test whether $(v_z)_\tau |_{\mathcal{M}} \in \mathcal{R}((\Lambda - \Lambda_0)^{1/2})$.
- If yes, mark point as "inside object $\Omega$".

Christoph Schneider tested this method with his code from the BMBF project "HuMin/MD – Metal detectors for humanitarian demining".

- Measurement device $\mathcal{M}$: 40 cm $\times$ 40 cm
- Scatterer ("the mine"): Ball, 8 cm diameter, 15 cm $-$ 20 cm below $\mathcal{M}$
- Permeability "$\mu = \infty$" in $\Omega$
- Frequency $19.2 \text{ kHz} \sim \omega \approx 4 \times 10^{-4} \text{ m}^{-1}$
- Currents imposed / electric fields measured on 100 "loops" on $\mathcal{M}$
- Simulated data (BEM) using code from K. Erhard, Göttingen
Numerical results - asymptotics

Numerical test for convergence

\[ \omega \mapsto \left\| \frac{1}{i \omega} \tilde{\Lambda}^\omega - \tilde{\Lambda} \right\| / \| \tilde{\Lambda} \|, \]

where

\[ \tilde{\Lambda}^\omega \approx \Lambda^\omega - \Lambda^0 \]
\[ \tilde{\Lambda} := \tilde{\Lambda}^{10^{-7}} \approx \Lambda - \Lambda_0 \]

are calculated with the forward solver from Göttingen.

\[ \sim \quad \Lambda = \frac{1}{i \omega} \tilde{\Lambda}^\omega + O(\omega^2) \]
Ball with radius $r = 4\text{cm}$ located $15\text{cm}$ below $M$. 

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Numerical results - reconstruction

Ball with radius $r = 4cm$ located $20cm$ below $\mathcal{M}$