The Factorization Method for a parabolic-elliptic problem

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Let $\overline{\Omega} \subset B \subset \mathbb{R}^n$ be bounded domains with smooth boundaries, and $Q := B \setminus \overline{\Omega}$ connected.

Heat equation:

$$\partial_t (\chi_{\Omega}(x) u(x, t)) - \nabla \cdot (\kappa(x) \nabla u(x, t)) = 0$$

with $\kappa = 1$ on $Q$ and $\kappa(x) - 1 \in L^\infty(\Omega)$.

Boundary Measurements: $\Lambda_1 : g \mapsto u_1|_{\partial B}$, where

$$\partial_t (\chi_{\Omega} u_1) - \nabla \cdot (\kappa \nabla u_1) = 0 \quad \text{in} \ B_T := B \times (0, T),$$

$$\partial_t u_1 = g \quad \text{on} \ \partial B_T := \partial B \times (0, T),$$

$$u_1(x, 0) = 0 \quad \text{on} \ \Omega.$$

**Goal:** Reconstruct $\Omega$ from given boundary measurements $\Lambda_1$. 
Forward problem: Uniqueness

Lemma 1

If \( u_1 \in H^{1,0}(B_T) := L^2(0, T, H^1(B)) \) solves
\[
\partial_t(\chi \Omega u_1) - \nabla \cdot (\kappa \nabla u_1) = 0 \quad \text{in } B_T
\]
in the sense of distributions, then
\[
\partial_\nu u_1 \in H^{-\frac{1}{2},0}(\partial B_T),
\]
\[
u \mid \Omega \in W(0, T, H^1(\Omega), H^1(\Omega)^\prime) \subset C^0(0, T, L^2(\Omega)) \]
(with respect to \( H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^1(\Omega)^\prime \)).

Furthermore \( u_1 \in H^{1,0}(B_T) \) solves (1) iff \( u_1 \in H^{1,0}((Q \cup \Omega)_T) \) solves
\[
\partial_t u_1 - \nabla \cdot (\kappa \nabla u_1) = 0 \quad \text{in } \Omega_T, \quad [u_1]_{\partial \Omega} = 0,
\]
\[
\Delta u_1 = 0 \quad \text{in } Q_T, \quad [\kappa \partial_\nu u_1]_{\partial \Omega} = 0,
\]
and the solution is uniquely determined by \( \partial_\nu u_1 \mid_{\partial B} \) and \( u_1(x, 0) \mid_{\Omega} \).
**Forward problem: Existence**

**Lemma 2**

Given \( g \in H^{-\frac{1}{2},0}_{\Omega}(\partial B_T) \) there exists a solution \( u_1 \in H^{1,0}(B_T) \) of

\[
\partial_t (\chi_{\Omega} u_1) - \nabla \cdot (\kappa \nabla u_1) = 0 \quad \text{in} \ B_T := B \times (0,T),
\]

\[
\partial_{\nu} u_1 = g \quad \text{on} \ \partial B_T := \partial B \times (0,T),
\]

\[
u_1(x,0) = 0 \quad \text{on} \ \Omega.
\]

**Proof.**

1. **Equivalent variational problem:** Find \( u_1 \in H^{1,0}(B_T) \) that solves

\[
\int_0^T \int_B \kappa \nabla u_1 \nabla v \, dx \, dt - \int_0^T \langle (v|\Omega)', u_1|\Omega \rangle \, dt = \int_0^T \langle g(t), v|_{\partial B} \rangle \, dt
\]

for all \( v \in H^{1,0}(B_T) \) with \( v|_{\Omega} \in W(0,T, H^1(\Omega), H^1(\Omega)') \) and \( v(x,T) = 0 \) on \( \Omega. \)

2. **Existence of a solution follows from Lion’s Projection Lemma.**
A reference problem

Reference Measurements (without inclusion $\Omega$):

$\Lambda_0 : g \mapsto u_0|_{\partial B},$

where

$\Delta u_0(x, t) = 0 \quad \text{in } B \times (0, T') \quad (2)$

$\partial_{\nu} u_0 = g \quad \text{on } \partial B \times (0, T) \quad (3)$

**Lemma 3** Given $g \in H_{\phi}^{-\frac{1}{2}, 0}(\partial B_T)$ there exists a solution $u_0 \in H^{1, 0}(B_T)$ to (2), (3). $u_0$ is unique up to addition of $u(x, t) = c(t)$ with $c \in L^2(0, T, \mathbb{R}).$

Factoring out $L^2(0, T, \mathbb{R})$ and using $L^2_\phi(\partial B_T) \hookrightarrow H_{\phi}^{-\frac{1}{2}, 0}(\partial B_T)$

**Updated Goal:** Reconstruct $\Omega$ from given

$\Lambda_0, \Lambda_1 : L^2_\phi(\partial B_T) \rightarrow L^2_\phi(\partial B_T)$
Virtual Measurements

\( \psi \): given boundary flux on \( \partial \Omega_T \)

\[
L : \ H^\frac{1}{2},^0(\partial \Omega_T) \rightarrow L^2(\partial B_T),
\]

\[
\psi \mapsto v|_{\partial B},
\]

where

\[
\Delta v(x, t) = 0 \quad \text{in } Q_T, \quad (4)
\]

\[
\partial_{\nu} v = 0 \quad \text{on } \partial B_T, \quad (5)
\]

\[
\partial_{\nu} v = \psi \quad \text{on } \partial \Omega_T. \quad (6)
\]

\( \mathcal{R}(L) \) determines \( \Omega \):

\[
|_{\partial \Omega} \in \mathcal{R}(L) \quad \text{if and only if } z \in \Omega,
\]

where \( v_z \) solves (4) in \( B_T \setminus \{z\} \), \( v_z \) solves (5), \( v_z \) suff. singular in \( z \in B \), e.g. the dipole functions from EIT (constant in time, cf. [Brühl, Hanke]).
Key identity of the Factorization Method (for other problems!):

\[ \mathcal{R}(L) = \mathcal{R}((\Lambda_0 - \Lambda_1)^{1/2}). \]

\( \sim \mathcal{R}(L) \) (and thus \( \Omega \)) can be computed from the measurements.

Such an identity

- was originally developed by Kirsch for Inverse Scattering
- is known (under suitable conditions on the inclusion) for
  - Electrostatics (Hähner)
  - EIT (Brühl, Hanke), also with different electrode models (Brühl, Hanke, Hyvönen) and in the half space (Schappel)
  - Diffusion tomography (Kirsch), also with Robin B.C. (Hyvönen)
  - general real elliptic problems (G.)

Does a similar identity hold in this parabolic-elliptic case?
Main Result

Theorem 4

With \( \tilde{\Lambda} := \Lambda_0 - \frac{1}{2}(\Lambda_1 + \Lambda_1^*) \) we have

\[
\mathcal{R}(\tilde{\Lambda}^{1/2}) \subseteq \mathcal{R}(L) = L\left(H_0^{-\frac{1}{2},0}(\partial\Omega_T)\right),
\]

\[
\mathcal{R}(\tilde{\Lambda}^{1/2}) \supseteq L\left(H^{\frac{1}{4}}(0,T,H_0^{-\frac{1}{2}}(\partial\Omega))\right),
\]

Corollary 5

With appropriate functions \( v_z \) that solve

\[
\begin{align*}
\Delta v_z(x,t) &= 0 \quad \text{in } B_T \setminus \{z\}, \\
\partial_{\nu}v_z|_{\partial B} &= 0,
\end{align*}
\]

this yields

\[
z \in \Omega \quad \text{if and only if} \quad v_z|_{\partial B} \in \mathcal{R}(\tilde{\Lambda}^{1/2}).
\]
Sketch of the Proof

Factorization \( \Lambda_0 - \Lambda_1 = (Lt)F(Lt)^*, \) with \( F \neq F^*. \)

\[ \tilde{\Lambda} = \Lambda_0 - \frac{1}{2}(\Lambda_1 + \Lambda_1^*) = (Lt)\tilde{F}(Lt)^*, \] with \( \tilde{F} = \frac{1}{2}(F + F^*) = \tilde{F}^*. \)

If \( \|Ax\| \leq \|Bx\| \) for all \( x \) then \( \mathcal{R}(A^*) \subseteq \mathcal{R}(B^*). \)

\[ \mathcal{R}(\tilde{\Lambda}^{1/2}) = \mathcal{R}(Lt\tilde{F}^{1/2}) \subseteq \mathcal{R}(L). \]

\[ (\tilde{F}\phi, \phi)_{H^{\frac{1}{2}, 0}_\Omega(\partial\Omega_T)} \geq c\|F_1\phi\|^2_{H^{-\frac{1}{2}, -\frac{1}{4}}(\partial\Omega_T)}. \]

\[ \mathcal{R}(\tilde{F}^{1/2}) \supseteq \mathcal{R}(F_1^*I^*), \text{ where } I : H^{-\frac{1}{2}, 0}_\Omega(\partial\Omega_T) \hookrightarrow H^{-\frac{1}{2}, -\frac{1}{4}}(\partial\Omega_T). \]

\[ \mathcal{R}(Lt\tilde{F}^{1/2}) \supseteq H^{\frac{1}{4}}(0, T, H^{-\frac{1}{2}}_\Omega(\partial\Omega)). \]
Final Remarks

A possible choice for $v_z$ are the dipole functions from EIT (constant in time), e.g. for the unit circle in $\mathbb{R}^2$ (cf. [Brühl]):

$$
\left. v_z \right|_{\partial B} = \frac{1}{\pi} \frac{(z - x) \cdot d}{|z - x|^2} \in \mathcal{R}(\Lambda^{1/2}) \quad \text{iff} \quad z \in \Omega.
$$

holds for all directions $d \in \mathbb{R}^2, |d| = 1$.

Corollary 2 contains the theoretical result:

$$
\Omega \text{ is uniquely determined by } \Lambda_1.
$$

$\Omega$ does not have to be connected.

Unlike EIT the case $\kappa < 1$ cannot be treated by simply interchanging $\Lambda_0$ and $\Lambda_1$. 