The Factorization Method for a parabolic-elliptic problem

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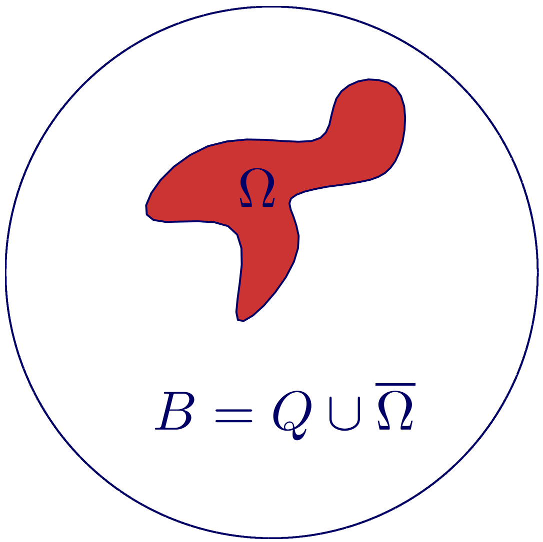
A parabolic-elliptic problem



Let

$$\overline{\Omega} \subset B \subset \mathbb{R}^n$$

be bounded domains with smooth boundaries, and $Q := B \setminus \overline{\Omega}$ connected.

Heat equation:

$$\partial_t(\chi_\Omega(x)u(x,t)) - \nabla \cdot (\kappa(x)\nabla u(x,t)) = 0$$

with $\kappa = 1$ on Q and $\kappa(x) - 1 \in L^\infty_+(\Omega)$.

Boundary Measurements: $\Lambda_1 : g \mapsto u_1|_{\partial B}$, where

$$\begin{aligned} \partial_t(\chi_\Omega u_1) - \nabla \cdot (\kappa \nabla u_1) &= 0 && \text{in } B_T := B \times (0,T), \\ \partial_\nu u_1 &= g && \text{on } \partial B_T := \partial B \times (0,T), \\ u_1(x,0) &= 0 && \text{on } \Omega. \end{aligned}$$

Goal: Reconstruct Ω from given boundary measurements Λ_1.

Forward problem: Uniqueness

Lemma 1

If $u_1 \in H^{1,0}(B_T) := L^2(0,T,H^1(B))$ *solves*

$$\partial_t(\chi_\Omega u_1) - \nabla \cdot (\kappa \nabla u_1) = 0 \quad \text{in } B_T \tag{1}$$

in the sense of distributions, then

$$\begin{aligned} \partial_\nu u_1 &\in H^{-\frac{1}{2},0}(\partial B_T), \\ u_1|_\Omega &\in W(0,T,H^1(\Omega),H^1(\Omega)') \subset C^0(0,T,L^2(\Omega)) \\ &\quad \text{(with respect to } H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^1(\Omega)'). \end{aligned}$$

Furthermore $u_1 \in H^{1,0}(B_T)$ *solves (1) iff* $u_1 \in H^{1,0}((Q \cup \Omega)_T)$ *solves*

$$\begin{aligned} \partial_t u_1 - \nabla \cdot (\kappa \nabla u_1) &= 0 \quad \text{in } \Omega_T, & [u_1]_{\partial\Omega} &= 0, \\ \Delta u_1 &= 0 \quad \text{in } Q_T, & [\kappa \partial_\nu u_1]_{\partial\Omega} &= 0, \end{aligned}$$

and the solution ist uniquely determined by $\partial_\nu u_1|_{\partial B}$ *and* $u_1(x,0)|_\Omega$.

Forward problem: Existence

Lemma 2

Given $g \in H_\diamond^{-\frac{1}{2},0}(\partial B_T)$ *there exists a solution* $u_1 \in H^{1,0}(B_T)$ *of*

$$\begin{aligned} \partial_t(\chi_\Omega u_1) - \nabla\cdot(\kappa\nabla u_1) &= 0 && \textit{in } B_T := B\times(0,T),\\ \partial_\nu u_1 &= g && \textit{on } \partial B_T := \partial B\times(0,T),\\ u_1(x,0) &= 0 && \textit{on } \Omega. \end{aligned}$$

Proof.

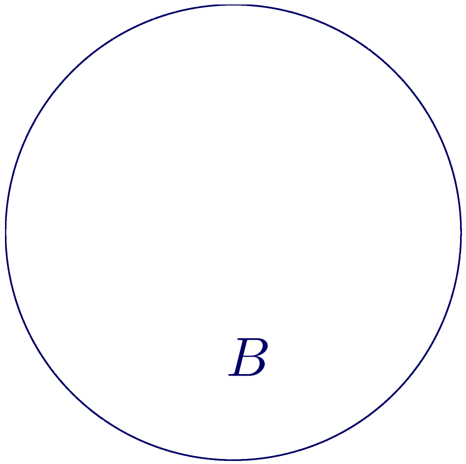
1. Equivalent variational problem: Find $u_1 \in H^{1,0}(B_T)$ that solves

$$\int_0^T\int_B \kappa\nabla u_1\nabla v\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle (v|_\Omega)', u_1|_\Omega\rangle\,\mathrm{d}t = \int_0^T \langle g(t), v|_{\partial B}\rangle\,\mathrm{d}t$$

for all $v \in H^{1,0}(B_T)$ with $v|_\Omega \in W(0,T,H^1(\Omega),H^1(\Omega)')$ and $v(x,T) = 0$ on Ω.

2. Existence of a solution follows from Lion's Projection Lemma.

A reference problem



Reference Measurements (without inclusion Ω):

$$\Lambda_0 : \; g \mapsto u_0|_{\partial B},$$

where

$$\begin{aligned} \Delta u_0(x,t) &= 0 \quad \text{in } B \times (0,T) & (2) \\ \partial_\nu u_0 &= g \quad \text{on } \partial B \times (0,T) & (3) \end{aligned}$$

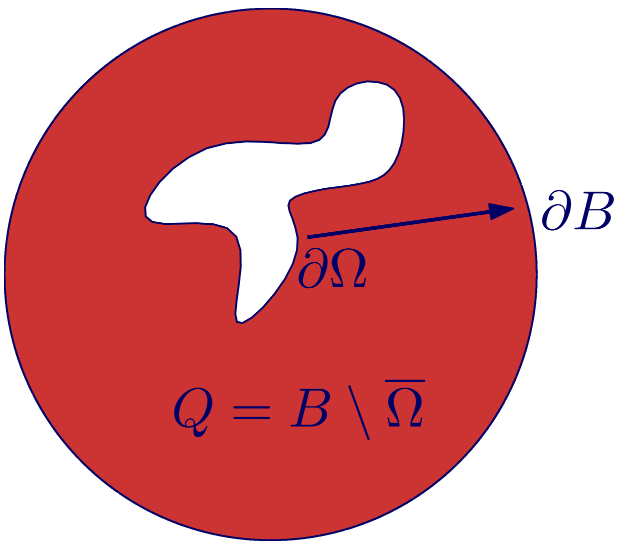
Lemma 3 *Given $g \in H_\diamond^{-\frac{1}{2},0}(\partial B_T)$ there exists a solution $u_0 \in H^{1,0}(B_T)$ to (2), (3).*
u_0 is unique up to addition of $u(x,t) = c(t)$ with $c \in L^2(0,T,\mathbb{R})$.

Factoring out $L^2(0,T,\mathbb{R})$ and using $L^2_\diamond(\partial B_T) \hookrightarrow H_\diamond^{-\frac{1}{2},0}(\partial B_T)$

$\rightsquigarrow$ Updated Goal: Reconstruct Ω from given

$$\Lambda_0, \Lambda_1 : \; L^2_\diamond(\partial B_T) \to L^2_\diamond(\partial B_T)$$

Virtual Measurements



ψ: given boundary flux on $\partial\Omega_T$

$$\begin{aligned} L: \quad H_\diamond^{-\frac{1}{2},0}(\partial\Omega_T) &\to L^2_\diamond(\partial B_T), \\ \psi &\mapsto v|_{\partial B}, \end{aligned}$$

where

$$\Delta v(x,t) = 0 \quad \text{in } Q_T, \tag{4}$$
$$\partial_\nu v = 0 \quad \text{on } \partial B_T, \tag{5}$$
$$\partial_\nu v = \psi \quad \text{on } \partial\Omega_T. \tag{6}$$

$\mathcal{R}(L)$ determines Ω:

$$v_z|_{\partial\Omega} \in \mathcal{R}(L) \quad \text{if and only if} \quad z \in \Omega$$

where v_z solves (4) in $B_T \setminus \{z\}$, v_z solves (5), v_z suff. singular in $z \in B$, e. g. the dipole functions from EIT (constant in time, cf. [Brühl, Hanke]).

Factorization Method

Key identity of the Factorization Method (for other problems!):

$$\mathcal{R}(L) = \mathcal{R}((\Lambda_0 - \Lambda_1)^{1/2}).$$

$\rightsquigarrow \mathcal{R}(L)$ (and thus Ω) can be computed from the measurements.

Such an identity

- was originally developed by Kirsch for Inverse Scattering
- is known (under suitable conditions on the inclusion) for
  - Electrostatics (Hähner)
  - EIT (Brühl, Hanke), also with different electrode models (Brühl, Hanke, Hyvönen) and in the half space (Schappel)
  - Diffusion tomography (Kirsch), also with Robin B.C. (Hyvönen)
  - general real elliptic problems (G.)

Does a similar identity hold in this parabolic-elliptic case?

Main Result

Theorem 4 *With* $\tilde{\Lambda} := \Lambda_0 - \frac{1}{2}(\Lambda_1 + \Lambda_1^*)$ *we have*

$$\mathcal{R}(\tilde{\Lambda}^{1/2}) \subseteq \mathcal{R}(L) = L\left(H_\diamond^{-\frac{1}{2},0}(\partial\Omega_T)\right),$$

$$\mathcal{R}(\tilde{\Lambda}^{1/2}) \supseteq L\left(H^{\frac{1}{4}}(0,T,H_\diamond^{-\frac{1}{2}}(\partial\Omega))\right),$$

Corollary 5 *With appropriate functions* v_z *that solve*

$$\begin{aligned} \Delta v_z(x,t) &= 0 \quad \text{in } B_T \setminus \{z\}, && v_z \text{ suff. singular in } z \in B, \\ \partial_\nu v_z|_{\partial B} &= 0, && \partial_\nu v_z|_{\partial\Omega} \in H^{\frac{1}{4}}(0,T,H_\diamond^{-\frac{1}{2}}(\partial\Omega)), \end{aligned}$$

this yields

$$z \in \Omega \quad \textit{if and only if} \quad v_z|_{\partial B} \in \mathcal{R}(\tilde{\Lambda}^{1/2}).$$

Sketch of the Proof

- Factorization $\Lambda_0 - \Lambda_1 = (L\iota)F(L\iota)^*$, with $F \neq F^*$.

  $\rightsquigarrow$ $\tilde{\Lambda} = \Lambda_0 - \frac{1}{2}(\Lambda_1 + \Lambda_1^*) = (L\iota)\tilde{F}(L\iota)^*$, with $\tilde{F} = \frac{1}{2}(F + F^*) = \tilde{F}^*$.

- If $\|Ax\| \leq \|Bx\|$ for all x then $\mathcal{R}(A^*) \subseteq \mathcal{R}(B^*)$.

  $\rightsquigarrow$ $\mathcal{R}(\tilde{\Lambda}^{1/2}) = \mathcal{R}(L\iota\tilde{F}^{1/2}) \subseteq \mathcal{R}(L)$.

- $(\tilde{F}\phi, \phi)_{H_\diamond^{\frac{1}{2},0}(\partial\Omega_T)} \geq c\|F_1\phi\|^2_{H^{-\frac{1}{2},-\frac{1}{4}}(\partial\Omega_T)}$.

  $\rightsquigarrow$ $\mathcal{R}(\tilde{F}^{1/2}) \supseteq \mathcal{R}(F_1^* I^*)$, where $I:\ H_\diamond^{-\frac{1}{2},0}(\partial\Omega_T) \hookrightarrow H^{-\frac{1}{2},-\frac{1}{4}}(\partial\Omega_T)$.

  $\rightsquigarrow$ $\mathcal{R}(\iota\tilde{F}^{1/2}) \supseteq H^{\frac{1}{4}}(0, T, H_\diamond^{-\frac{1}{2}}(\partial\Omega))$.

Final Remarks

- A possible choice for v_z are the dipole functions from EIT (constant in time), e. g. for the unit circle in $\mathbb{R}^2$ (cf. [Brühl]):

$$v_z|_{\partial B} = \frac{1}{\pi} \frac{(z-x)\cdot d}{|z-x|^2} \in \mathcal{R}(\tilde{\Lambda}^{1/2}) \quad \text{iff} \quad z \in \Omega.$$

  holds for all directions $d \in \mathbb{R}^2$, $|d| = 1$.

- Corollary 2 contains the theoretical result:

  Ω is uniquely determined by Λ_1.

- Ω does not have to be connected.
- Unlike EIT the case $\kappa < 1$ cannot be treated by simply interchanging Λ_0 and Λ_1.