Detecting magnetic objects using low frequency electromagnetic scattering

Bastian Gebauer

Institut für Mathematik
Johannes Gutenberg-Universität Mainz

gebauer@math.uni-mainz.de

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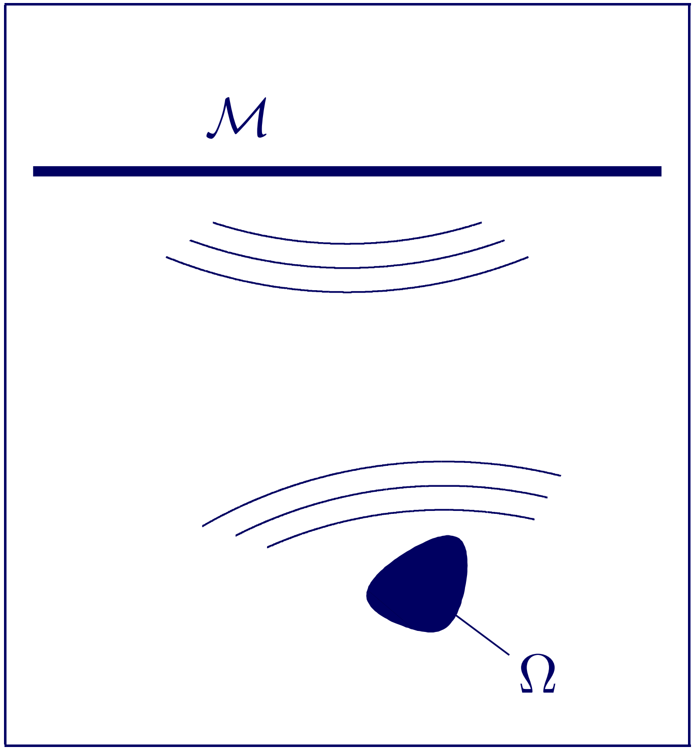
Setting



$\mathcal{M}$: measurement device
Ω: magnetic object

- Apply surface currents J on $\mathcal{M}$ (time-harmonic with frequency ω).

$\rightsquigarrow$ electromagnetic field (E^ω, H^ω) (time-harmonic with frequency ω)

- Measure field on $\mathcal{M}$ (and try to locate Ω from it).

wavelength $\approx 300\,\mathrm{km}$ $\gg$ size of object $\approx 10\,\mathrm{cm}$
($\rightsquigarrow$ frequency ω very small)

What happens when $\omega \to 0$?

Maxwell's equations

time-harmonic Maxwell's equations

$$\begin{aligned} \operatorname{curl} H^\omega + \mathrm{i}\,\omega\epsilon E^\omega &= J && \text{in } \mathbb{R}^3, \\ -\operatorname{curl} E^\omega + \mathrm{i}\,\omega\mu H^\omega &= 0 && \text{in } \mathbb{R}^3, \\ \operatorname{div}(\epsilon E^\omega) &= 0 && \text{in } \mathbb{R}^3, \\ \operatorname{div}(\mu H^\omega) &= 0 && \text{in } \mathbb{R}^3, \end{aligned}$$

Silver-Müller radiation condition (RC)

$$\int_{\partial B_\rho} \left| \nu \wedge \sqrt{\mu} H^\omega + \sqrt{\epsilon} E^\omega \right|^2 \mathrm{d}\sigma = o(1), \quad \rho \to \infty.$$

E^ω:	electric field	ϵ:	dielectricity ($= const.$ around $\mathcal{M}$)
H^ω:	magnetic field	μ:	permeability (magnetic properties)
ω:	frequency	J:	applied currents, $\operatorname{div} J = 0$, $\operatorname{supp} J \subseteq \mathcal{M}$

relative parameter values: $\epsilon = 1$, $\mu = 1$ outside some bounded domain

Formal asymptotic analysis

Solve Maxwell's equations for E^ω:

$$\begin{aligned} \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E^\omega - \omega^2 \epsilon E^\omega &= \mathrm{i}\,\omega J && \text{in } \mathbb{R}^3, \\ \operatorname{div}(\epsilon E^\omega) &= 0 && \text{in } \mathbb{R}^3 \quad \text{(redundant)}, \\ \int_{\partial B_\rho} |\nu \wedge \operatorname{curl} E^\omega + \mathrm{i}\,\omega E^\omega|^2 \,\mathrm{d}\sigma &= o(1), && \rho \to \infty. \end{aligned}$$

real frequency $1\,\mathrm{kHz}$ $\rightsquigarrow$ relative parameter $\omega \approx 2 \times 10^{-5}\,\mathrm{m}^{-1}$

Asymptotic analysis (formal):

Ansatz: $E^\omega = E_0 + \omega E_1 + \omega^2 E_2 + \ldots$

Rigorous analysis (for fixed incoming waves): Ammari, Nedelec, 2000 (*Low Frequency electromagnetic scattering, SIAM J. Math. Anal.)*

Formal asymptotic analysis

Asymptotic analysis: $E^\omega = E_0 + \omega E_1 + \omega^2 E_2 + \ldots$, where E_0, E_1, E_2 solve

$$\left.\begin{aligned} \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E_0 &= 0, \\ \operatorname{div}(\epsilon E_0) &= 0, \end{aligned}\right\} \rightsquigarrow E_0 = 0$$

$$\begin{aligned} \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E_1 &= \mathrm{i}\, J, \\ \operatorname{div}(\epsilon E_1) &= 0, \end{aligned}$$

$$\left.\begin{aligned} \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E_2 - \epsilon E_0 &= 0, \\ \operatorname{div}(\epsilon E_2) &= 0, \end{aligned}\right\} \rightsquigarrow E_2 = 0$$

(ignoring additional conditions at $x = \infty$)

$\overset{E:=E_1}{\rightsquigarrow}$ $E^\omega = \omega E + O(\omega^3)$, where $\quad \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E = \mathrm{i}\, J, \quad \operatorname{div}(\epsilon E) = 0.$

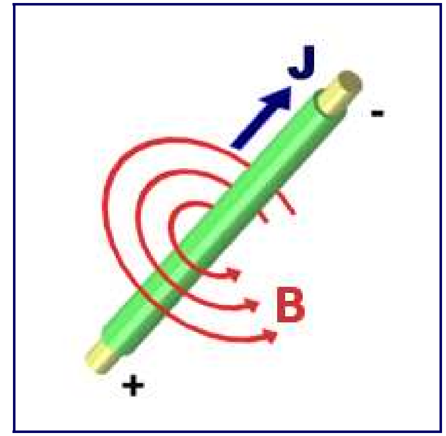
Interpretation

$$E^{\omega} = \omega E + O(\omega^3), \text{ where} \quad \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E = \mathrm{i}\, J, \quad \operatorname{div}(\epsilon E) = 0$$

- $B := \frac{1}{\mathrm{i}} \operatorname{curl} E$ solves

$$\operatorname{curl} \tfrac{1}{\mu} B = J, \quad \operatorname{div} B = 0.$$

$\rightsquigarrow$ B is the magnetostatic field generated by a steady current J (*Ampère's Law).*



- $B = \frac{1}{\mathrm{i}} \operatorname{curl} E \quad \rightsquigarrow \quad E$ is a vector potential of B (unique up to addition of A with $\operatorname{curl} A = 0$, i. e. up to $A = \nabla \varphi$).
- $\operatorname{div}(\epsilon E) = 0$ determines E uniquely (so-called *Coulomb gage).*

Figure based on `http://de.wikipedia.org/wiki/Bild:RechteHand.png`, published under the GNU Free Documentation License (FDL) by "Frau Holle".

Mathematical Formulation

Assume that

- $J \in TL^2_\diamond(\mathcal{M};\mathbb{C}^3)$, i. e. $J \in TL^2(\mathcal{M};\mathbb{C}^3)$, $\operatorname{div}_{\mathcal{M}} J = 0$ and $\nu \cdot J|_{\partial M} = 0$.
- $\epsilon, \mu \in L^\infty_+(\mathbb{R}^3;\mathbb{R})$ are identical to 1 outside some bounded domain.
- $\epsilon = 1$ in some neighborhood of $\mathcal{M}$

We seek a solution $E^\omega \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3;\mathbb{C}^3)$ of

$$\begin{aligned} \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E^\omega - \omega^2 \epsilon E^\omega &= \mathrm{i}\,\omega J \quad \text{in } \mathbb{R}^3, && (1)\\ \int_{\partial B_\rho} |\nu \wedge \operatorname{curl} E^\omega + \mathrm{i}\,\omega E^\omega|^2 \,\mathrm{d}\sigma &= o(1), \quad \rho \to \infty. && (2) \end{aligned}$$

(1) makes sense in $\mathcal{D}'(\mathbb{R}^3;\mathbb{C}^3)$.

Solutions of (1) are smooth where $\epsilon = 1$, $\mu = 1$, $J = 0$
$\rightsquigarrow$ (2) makes sense for solutions of (1).

Existence Theory

We seek a solution $E^\omega \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3; \mathbb{C}^3)$ of

$$\begin{aligned} \mathrm{curl}\, \frac{1}{\mu}\, \mathrm{curl}\, E^\omega - \omega^2 \epsilon E^\omega &= \mathrm{i}\,\omega J \quad \text{in } \mathbb{R}^3, && (1)\\ \int_{\partial B_\rho} |\nu \wedge \mathrm{curl}\, E^\omega + \mathrm{i}\,\omega E^\omega|^2 \,\mathrm{d}\sigma &= o(1), \quad \rho \to \infty. && (2) \end{aligned}$$

- Typically (1), (2) lead to a Fredholm equation

  $\rightsquigarrow$ Existence of a solution follows from uniqueness.

- Smooth coefficients ϵ, μ $\rightsquigarrow$ uniqueness, and thus existence

  (cf. e. g. Monk: *Finite Element Methods for Maxwell's Equations*.)

- Non-smooth coefficients $\rightsquigarrow$ existence/uniqueness is not guaranteed.

  Resonances may occur.

Magnetostatic equations

We seek a solution $E \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3; \mathbb{C}^3)$ of

$$\mathrm{curl}\,\frac{1}{\mu}\,\mathrm{curl}\,E = \mathrm{i}\,J \quad \text{in } \mathbb{R}^3, \qquad (3)$$

$$\mathrm{div}(\epsilon E) = 0 \quad \text{in } \mathbb{R}^3. \qquad (4)$$

For $\epsilon = 1$, there exists a unique solution in

$$W^1(\mathbb{R}^3; \mathbb{C}^3) := \left\{ u \,:\, (1 + |x|^2)^{-1/2} u \in L^2(\mathbb{R}^3; \mathbb{C}^3),\ \nabla u \in L^2(\mathbb{R}^3; \mathbb{C}^{3,3}) \right\},$$

(cf. e. g. Dautray, Lions: *Math. Analysis and Numerical Methods for Science and Technology*.)

This motivates (with suff. large ball B_r)

$$E|_{\mathbb{R}^3 \setminus \overline{B_r}} \in W^1(\mathbb{R}^3 \setminus \overline{B_r}; \mathbb{C}^3) \qquad (5)$$

Lemma

There exists a unique solution $E \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3)$ of (3), (4), (5).

Proof: Add appropriate $\nabla \varphi$.

Reduction to a bounded domain

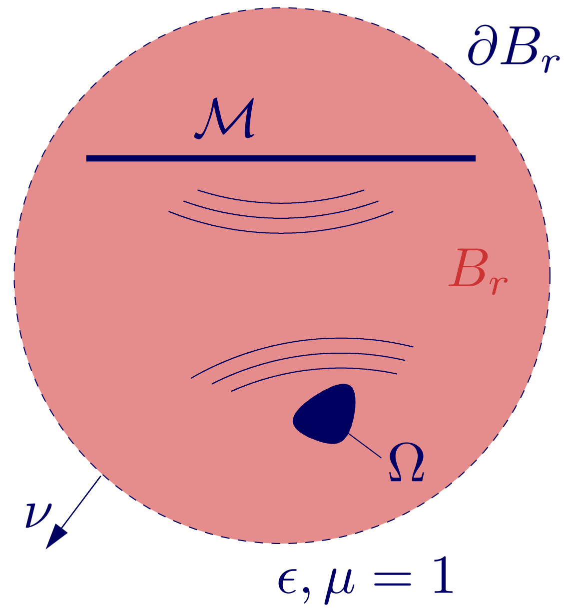
$\epsilon, \mu = 1$ outside a large ball B_r $\rightsquigarrow$ Introduce artificial boundary ∂B_r.

Replace Maxwell's equation by

$$\begin{aligned} \operatorname{curl} \tfrac{1}{\mu} \operatorname{curl} E^\omega - \omega^2 \epsilon E^\omega &= \mathrm{i}\,\omega J && \text{in } B_r, \\ T^\omega(E^\omega_\tau|_{\partial B_r}) &= \nu \wedge \operatorname{curl} E^\omega|_{\partial B_r}, \\ N^\omega(E^\omega_\tau|_{\partial B_r}) &= \nu \cdot E^\omega|_{\partial B_r}, \end{aligned}$$

and the magnetostatic equations by

$$\begin{aligned} \operatorname{curl} \tfrac{1}{\mu} \operatorname{curl} E &= \mathrm{i}\, J && \text{in } B_r, \\ \operatorname{div}(\epsilon E) &= 0 && \text{in } B_r, \\ T(E_\tau|_{\partial B_r}) &= \nu \wedge \operatorname{curl} E|_{\partial B_r}, \\ N(E_\tau|_{\partial B_r}) &= \nu \cdot E|_{\partial B_r}. \end{aligned}$$



T, T^ω, N, N^ω artificial (non-local) boundary conditions on ∂B_r, such that these equations become equivalent to those on $\mathbb{R}^3$.

Exterior calderon operators

- Exterior calderon operators for Maxwell's equation:

$$T^\omega:\ g\mapsto \nu\wedge \operatorname{curl} E^\omega|_{\partial B_r},\qquad N^\omega:\ g\mapsto \nu\cdot E^\omega|_{\partial B_r},$$

where $E^\omega \in H_{\mathrm{loc}}(\operatorname{curl},\mathbb{R}^3\setminus B_r;\mathbb{C}^3)$ solves

$$\left.\begin{array}{rcll} \operatorname{curl}\operatorname{curl} E^\omega - \omega^2 E^\omega &=& 0 & \text{in } \mathbb{R}^3\setminus \overline{B_r} \\ E^\omega_\tau|_{\partial B_r} &=& g & \text{on } \partial B_r \end{array}\right. + \text{(R.C.)}$$

- Exterior calderon operators for the magnetostatic equations:

$$T:\ g\mapsto \nu\wedge \operatorname{curl} E|_{\partial B_r},\qquad N:\ g\mapsto \nu\cdot E|_{\partial B_r},$$

where $E\in W^1(\mathbb{R}^3\setminus\overline{B_r};\mathbb{C}^3)$ solves

$$\begin{array}{rcll} \operatorname{curl}\operatorname{curl} E &=& 0 & \text{in } \mathbb{R}^3\setminus \overline{B_r} \\ \operatorname{div} E &=& 0 & \text{in } \mathbb{R}^3\setminus \overline{B_r} \\ E_\tau|_{\partial B_r} &=& g & \text{on } \partial B_r, \end{array}$$

$\rightsquigarrow$ Then the reduced problems on B_r are equivalent to those on $\mathbb{R}^3$.

Low frequency analysis

Lemma

For $\omega \to 0$ we have

$$T^\omega - T = O(\omega^2) \quad \text{in } \mathcal{L}(TH^{-1/2}(\mathrm{curl}, \partial B_r; \mathbb{C}^3), TH^{-1/2}(\mathrm{div}, \partial B_r; \mathbb{C}^3))$$

$$N^\omega - N = O(\omega^2) \quad \text{in } \mathcal{L}(TH^{-1/2}(\mathrm{curl}, \partial B_r; \mathbb{C}^3), H^{1/2}(\mathrm{div}, \partial B_r))$$

Proof:

Use explicit representations for T, T^ω, N, N^ω in terms of spherical harmonics and vector spherical harmonics.

Low frequency analysis

Theorem

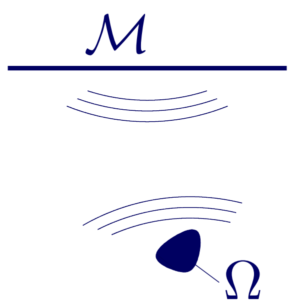
There exist $C > 0$, $\omega_0 > 0$, such that for every $0 < \omega < \omega_0$ and every $J \in TL^2_\diamond(\mathcal{M})$ there is a unique solution E^ω of Maxwell's equations and

$$\|E^\omega - \omega E\|_{H(\mathrm{curl},B_r)} \leq C\omega^3 \|J\|_{TL^2_\diamond(\mathcal{M})}. \qquad (6)$$

Proof

- Reduce both problems to bounded domain B_r.
- Standard variational formulation (equivalent to Maxwell's equations)
  - ⇝ Fredholm equation
  - ⇝ Existence of solution is equivalent to uniqueness.
- New variational formulation (not equivalent!)
  - ⇝ If there is a solution, then it is unique and satisfies (6).

Measurements



- Apply surface currents J on $\mathcal{M}$
- $\rightsquigarrow$ electromagnetic field (E^ω, H^ω)
- Measure field on $\mathcal{M}$

"Full set of measurements" corresponds to measurement operator

$$\Lambda^\omega : \begin{cases} TL^2_\diamond(\mathcal{M};\mathbb{C}^3) & \to & TL^2(\mathcal{M};\mathbb{C}^3), \\ J & \mapsto & E^\omega_\tau|_\mathcal{M}, \end{cases}$$

where E^ω solves Maxwell's equations.

Magnetostatic measurements would be

$$\Lambda : \begin{cases} TL^2_\diamond(\mathcal{M};\mathbb{C}^3) & \to & TL^2(\mathcal{M};\mathbb{C}^3), \\ J & \mapsto & E_\tau|_\mathcal{M}, \end{cases}$$

where E solves the magnetostatic equations.

Measurements

"We measure the magnetostatic potential of steady currents."

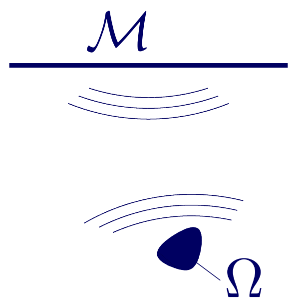
$$\Lambda = \frac{1}{i\omega}\Lambda^\omega + O(\omega^2) \quad \text{in } \mathcal{L}(TL^2_\diamond(\mathcal{M};\mathbb{C}^3), TL^2(\mathcal{M};\mathbb{C}^3))$$

- Magnetostatic equations are real differential equations.
  - ⇝ Consider Λ to be an operator between real Hilbert spaces of real-valued functions.
- Factor out functions of the form $\nabla\phi$:

$$TL^2(\mathcal{M};\mathbb{R}^3) = TL^2_\diamond(\mathcal{M};\mathbb{R}^3) \perp \nabla_{\mathcal{M}} H^1(\mathcal{M};\mathbb{R})$$

⇝ $\Lambda \in \mathcal{L}(TL^2_\diamond(\mathcal{M};\mathbb{R}^3), TL^2_\diamond(\mathcal{M};\mathbb{R}^3))$ independent from ϵ (as long as $\epsilon = 1$ around $\mathcal{M}$ and $\epsilon = 1$ outside some B_r).

The inverse problem



- Suppose there is a magnetic object Ω
- $\rightsquigarrow$ Permeability:

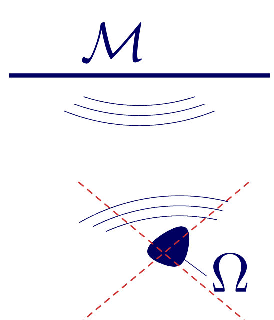
$$\mu(x) = 1 + \mu_1\, \chi_\Omega(x), \qquad \mu_1 > 0$$

- Goal: Reconstruct Ω from Λ

Factorization Method:

Find Ω by comparing Λ with reference measurements Λ_0

(reference = without object Ω).



$$\Lambda_0 : \begin{cases} TL^2_\diamond(\mathcal{M};\mathbb{R}^3) & \rightarrow & TL^2_\diamond(\mathcal{M};\mathbb{R}^3), \\ J & \mapsto & (E_0)_\tau|_\mathcal{M}, \end{cases}$$

E_0 solves the magnetostatic equations with

$$\mu_0(x) = 1.$$

Factorization Method

Factorization Method

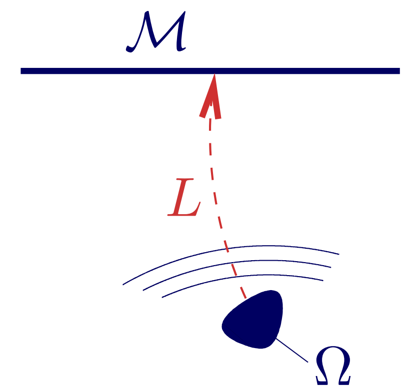
- originally developed by Kirsch (1998) for far-field measurements in inverse scattering (Helmholtz equation).
- generalized to EIT by Brühl and Hanke (1999).
- works for far-field measurements for Maxwell's equations (Kirsch, 2004)
- works for harmonic vector fields (Kress, 2002)
- works for general real elliptic equations (G, 2005)

Linear Sampling Method (similar, but with less theoretical justification)

- works for this near-field problem for Maxwell's equations (G, Hanke, Kirsch, Muniz, Schneider, 2005).

Factorization Method

Factorization Method relies on two facts:



- Range identity:

$$\mathcal{R}((\Lambda - \Lambda_0)^{1/2}) = \mathcal{R}(L),$$

with some auxiliary operator L.

$\rightsquigarrow$ $\mathcal{R}(L)$ is determined by the measurements Λ, Λ_0.

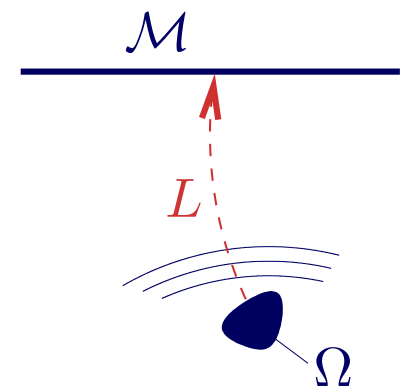
- Test functions:

$$z \in \Omega \quad \text{if and only if} \quad (v_z)_\tau|_\mathcal{M} \in \mathcal{R}(L)$$

with some functions v_z having a singularity in z.

$\rightsquigarrow$ Object Ω can be located from $\mathcal{R}(L)$.

Range identity



Auxiliary operator $L: \ g \mapsto E_\tau|_{\mathcal{M}}$, where E solves

$$\begin{aligned} &\text{magnetostatic equations} && \text{in } \mathbb{R}^3 \setminus \overline{\Omega} \\ \nu \wedge \operatorname{curl} E|_{\partial\Omega} &= g && \text{on } \partial\Omega, \end{aligned}$$

$\rightsquigarrow L$ contains information about $\mathbb{R}^3 \setminus \overline{\Omega}$ and thus about Ω.

Obviously $(\Lambda - \Lambda_0)J = L(\nu \wedge \operatorname{curl}(E - E_0)|_{\partial\Omega}) \rightsquigarrow \mathcal{R}(\Lambda_1 - \Lambda_0) \subseteq \mathcal{R}(L)$.

Factorization Method for real elliptic problems:

If "$\operatorname{curl}\operatorname{curl} - \operatorname{curl} \frac{1}{1+\mu_1} \operatorname{curl}$ is coercive on Ω", then

(more precisely: if the corresponding bilinear form is coercive on a space of functions on Ω)

$$\mathcal{R}((\Lambda - \Lambda_0)^{1/2}) = \mathcal{R}(L).$$

(holds for $\mu_1 > 0$ or even $\mu_1 = \mu_1(x) \in L^\infty_+(\Omega)$)

Test functions

Test functions $v_z(x) := \operatorname{grad}\operatorname{div}(\Phi_z(x)p)$,

$\Phi_z(x) = \frac{1}{x-z}$: fundamental solution of Laplace equation,

$p \in \mathbb{R}^3$, $|p| = 1$: arbitrary direction

$\rightsquigarrow$ v_z solves magnetostatic equations in $\mathbb{R}^3 \setminus \{z\}$.

$\rightsquigarrow$ If $z \in \Omega$ then $(v_z)_\tau|_\mathcal{M} = L(\nu \wedge \operatorname{curl} v_z|_{\partial\Omega}) \in \mathcal{R}(L)$.

For points below $\mathcal{M}$ the converse can be shown by analytic continuation.

For every point z below $\mathcal{M}$ and every direction p

$z \in \Omega$ if and only if $(v_z)_\tau|_\mathcal{M} \in \mathcal{R}(L) = \mathcal{R}((\Lambda - \Lambda_0)^{1/2})$.

Detection algorithm: For every point z on a sampling grid:

- Test whether $(v_z)_\tau|_\mathcal{M} \in \mathcal{R}((\Lambda - \Lambda_0)^{1/2})$.
- If yes, mark point as "inside object Ω".

Numerical results - setup

Christoph Schneider tested this method with his code from the BMBF project "HuMin/MD – Metal detectors for humanitarian demining".

Measurement device $\mathcal{M}$

$$32cm \times 32cm$$

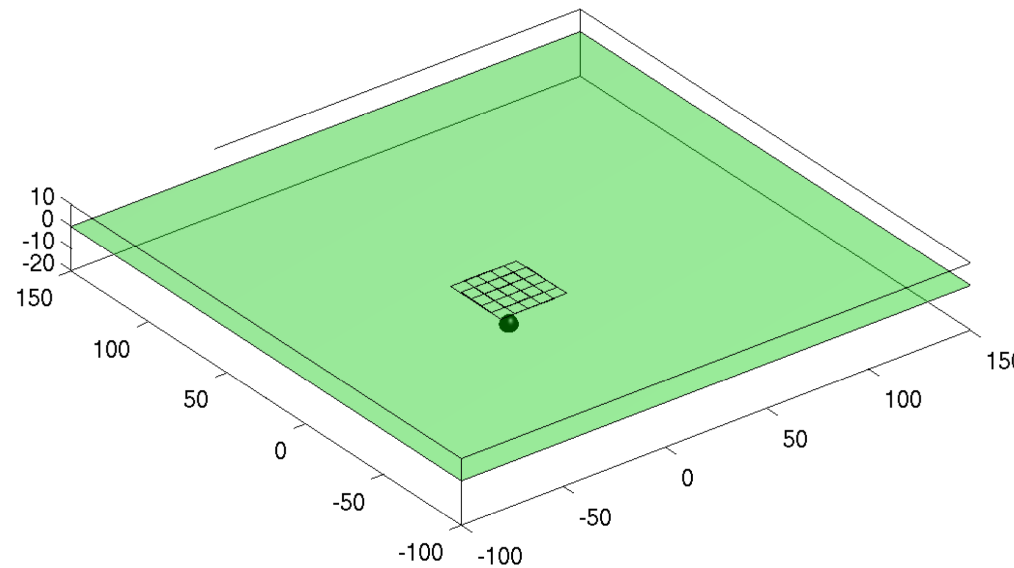
Scatterer ("the mine")

$6cm$–$8cm$ (diameter)

$10cm$–$15cm$ below $\mathcal{M}$

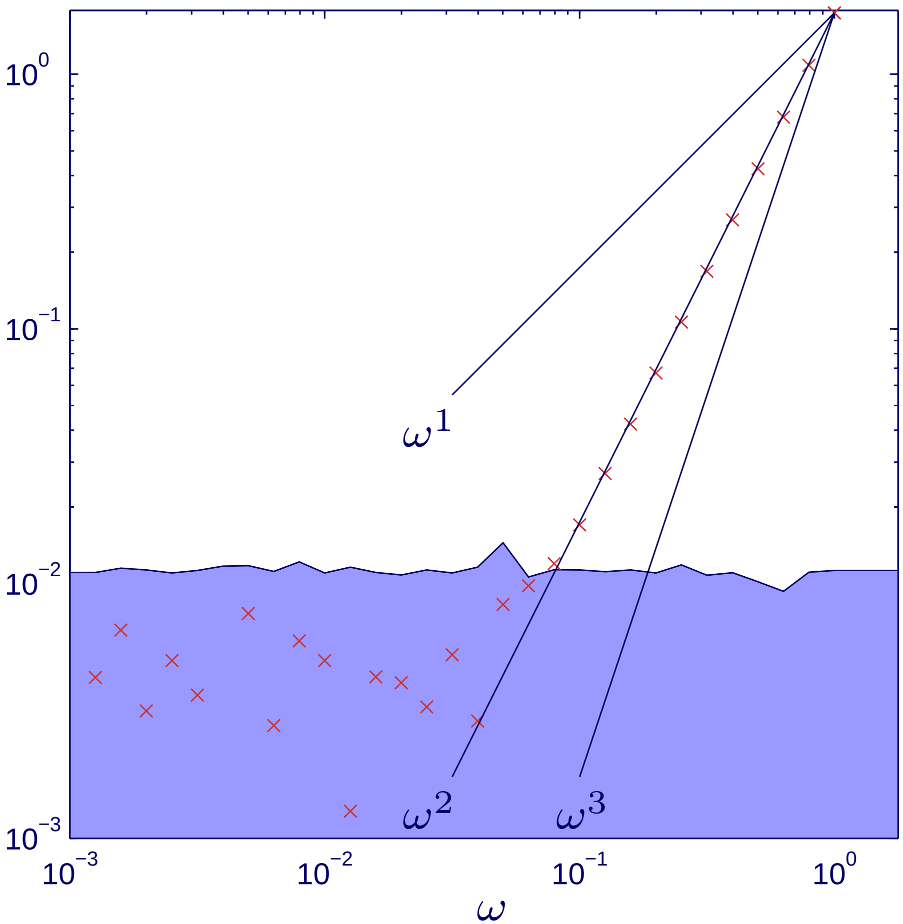
Wavelength $300km$

Permeability "$\mu = \infty$" in Ω



- Currents imposed / electric fields measured on a 6×6 grid on $\mathcal{M}$
- Simulated data (BEM) using code from K. Erhard, Göttingen

Numerical results - asymptotics



Numerical test for convergence

$$\omega \mapsto \| \tfrac{1}{i\omega} \tilde{\Lambda}^\omega - \tilde{\Lambda} \| \, / \, \| \tilde{\Lambda} \| \, ,$$

where

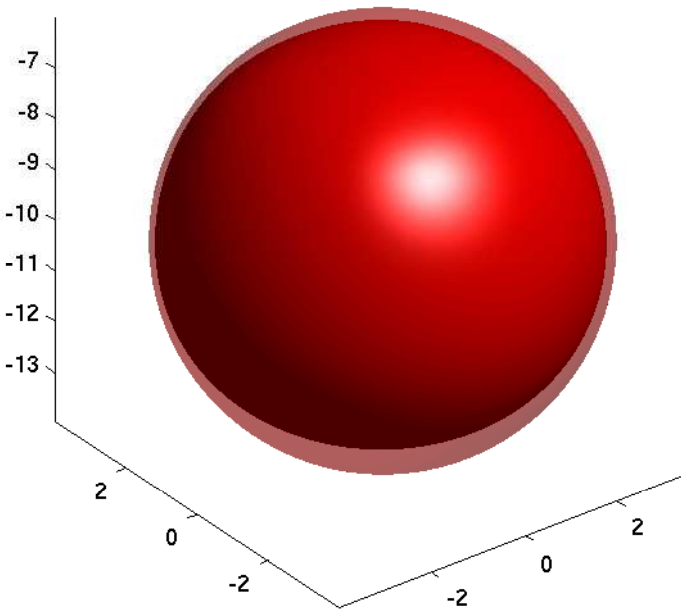
$$\begin{aligned} \tilde{\Lambda}^\omega &\approx \Lambda^\omega - \Lambda_0^\omega \\ \tilde{\Lambda} &\approx \Lambda^{10^{-7}} - \Lambda_0^{10^{-7}} \end{aligned}$$

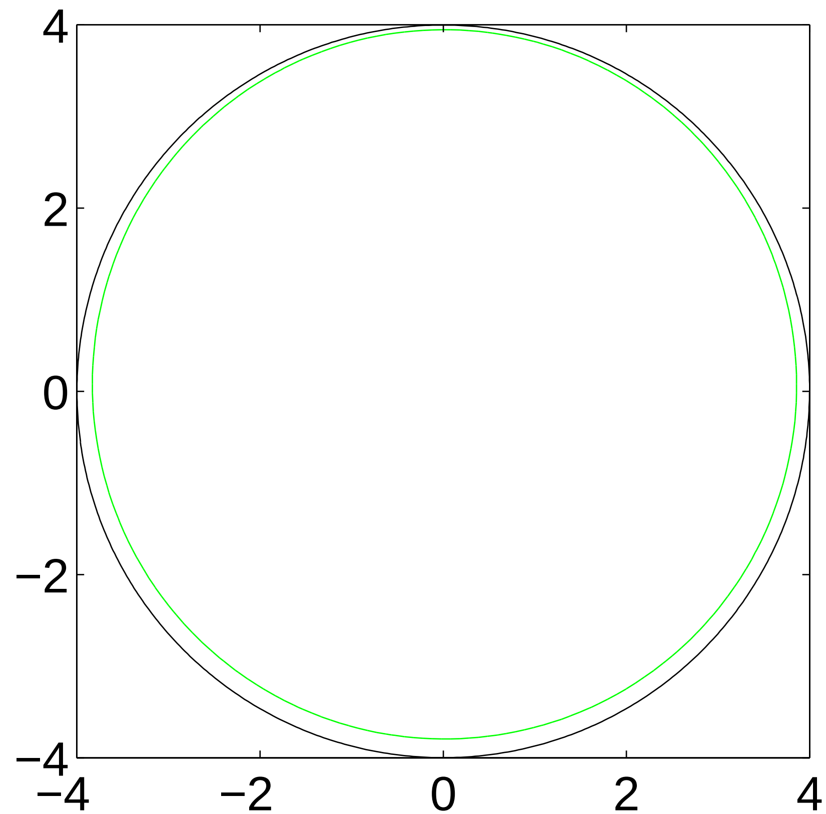
are calculated with the forward solver from Göttingen.

$$\rightsquigarrow \quad \Lambda = \tfrac{1}{i\omega} \tilde{\Lambda}^\omega + O(\omega^2)$$

light blue: estimated error of solver

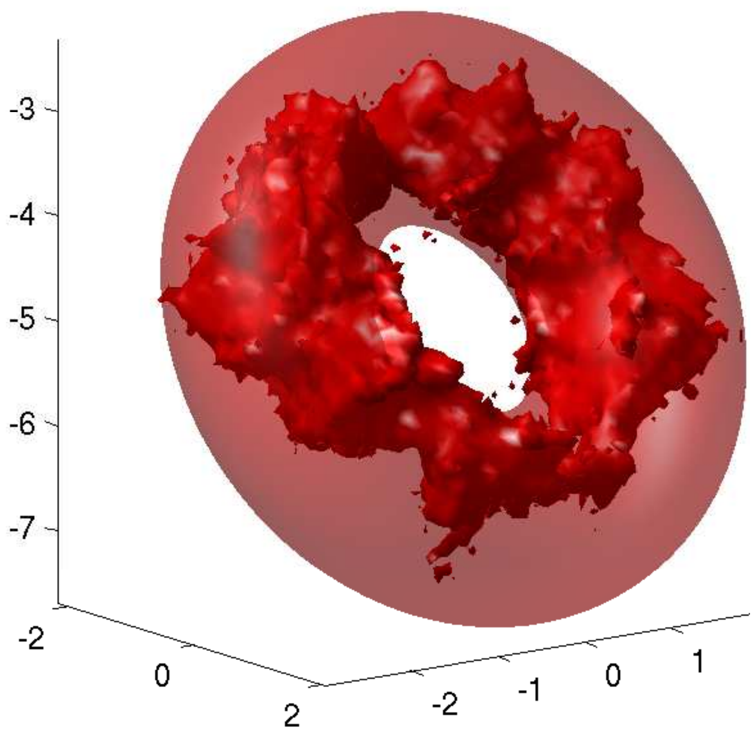
Numerical results - reconstruction

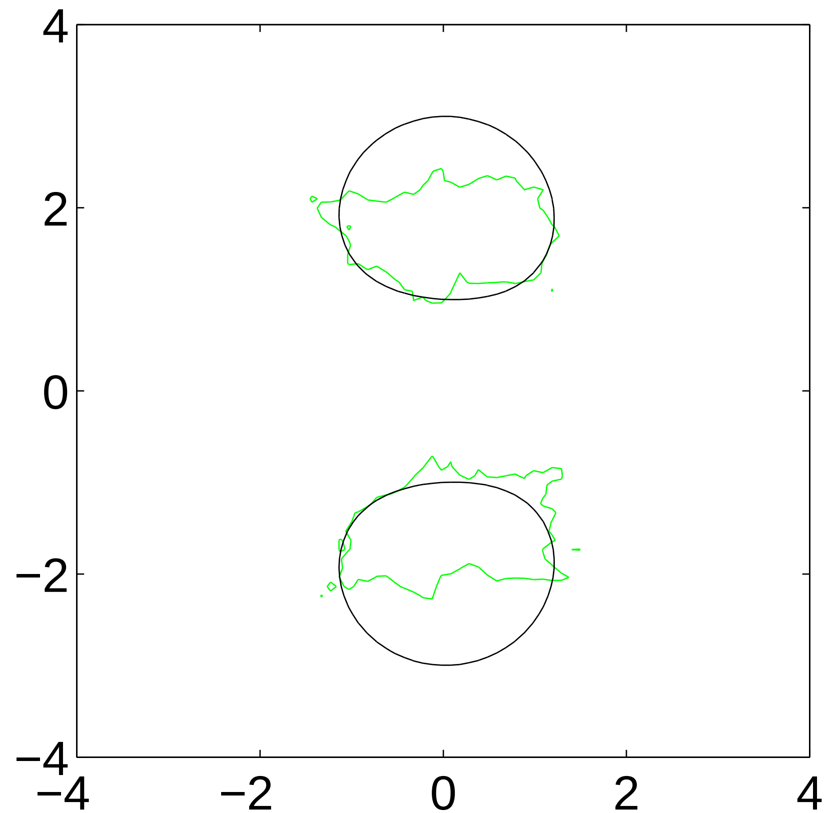




Ball with radius $r = 4cm$ located $15cm$ below $\mathcal{M}$

Numerical results - reconstruction





Torus with inner radius $r = 1cm$, outer radius $r = 3cm$, $10cm$ below $\mathcal{M}$