A sampling method for detecting buried objects

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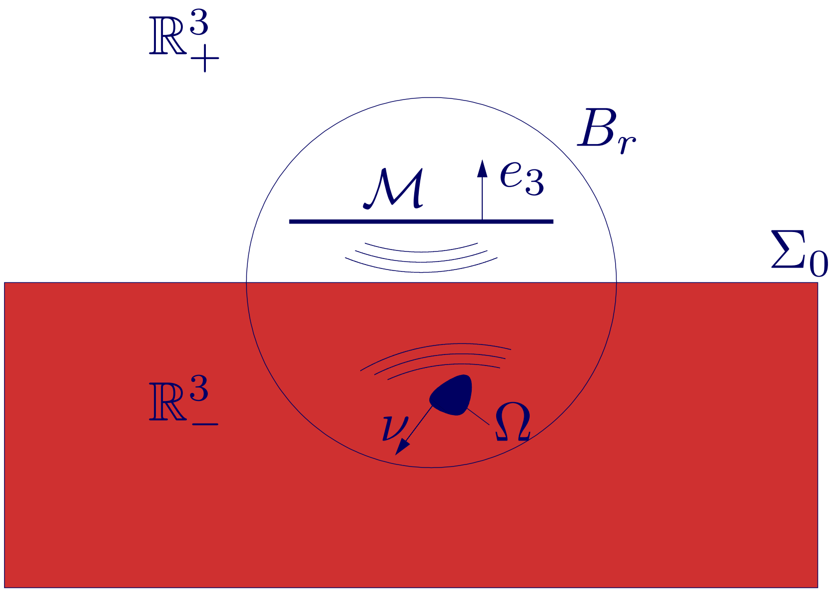
Mine detection



Dielectricity

$$\epsilon(x) = \begin{cases} \epsilon_+ > 0, & x \in \mathbb{R}^3_+ \\ \epsilon_-, & x \in \mathbb{R}^3_- \end{cases}$$

$$\operatorname{Re} \epsilon_- > 0, \quad \operatorname{Im} \epsilon_- \geq 0$$

Permeability

$$\mu(x) = \begin{cases} \mu_+ > 0, & x \in \mathbb{R}^3_+ \\ \mu_- > 0, & x \in \mathbb{R}^3_- \end{cases}$$

Perfect conductor ("the mine")

$$\overline{\Omega} \subset \mathbb{R}^3_-$$

Apply time-harmonic magnetic dipole density on $\mathcal{M}$

$$\varphi \in H^{-\frac{1}{2}}(\mathrm{div}, \mathcal{M})$$

$\rightsquigarrow$ Electromagnetic field (E, H) in $\mathbb{R}^3 \setminus \Omega$.

$\rightsquigarrow$ Measure tangential components $(\nu \wedge H|_{\mathcal{M}}) \wedge \nu$ on $\mathcal{M}$

Maxwells equations

No applied electric currents

$$\operatorname{curl} H + i\omega\epsilon E = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}.$$

Magnetic dipole density on $\mathcal{M}$

$$-\operatorname{curl} E + i\omega\mu H = i\omega\mu\, \varphi\delta_{\mathcal{M}} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega},$$

Perfectly conducting mine

$$\nu \wedge E|_{\partial\Omega} = 0.$$

Radiation condition (RC)

$$\int_{\partial B_r} \left|\nu \wedge \sqrt{\mu} H + \sqrt{\epsilon} E\right|^2 \mathrm{d}\sigma = o(1), \quad r \to \infty.$$

For every $\phi \in H^{-\frac{1}{2}}(\operatorname{div}, \mathcal{M})$, there exists a unique solution

$$H \in H_{loc}(\operatorname{curl}, \mathbb{R}^3 \setminus \Omega), \quad E \in H_{loc}(\operatorname{curl}, \mathbb{R}^3 \setminus (\Omega \cup \mathcal{M})).$$

(cf. e. g. Monk: Finite Element Methods for Maxwells Equations)

Interface conditions

$E, H \in L^2_{loc}(\mathbb{R}^3 \setminus \Omega)$ and

$$\begin{aligned} \operatorname{curl} H + i\omega\epsilon E &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ -\operatorname{curl} E + i\omega\mu H &= i\omega\mu\, \varphi\delta_{\mathcal{M}} && \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \end{aligned}$$

is equivalent to

$E, H \in L^2_{loc}(\mathbb{R}^3 \setminus \Omega)$ and

$$\begin{aligned} \operatorname{curl} H + i\omega\epsilon_+ E &= 0 \quad \text{in } \mathbb{R}^3_+ \setminus \overline{\mathcal{M}}, & [e_3 \wedge E]_{\mathcal{M}} &= i\omega\mu_+\varphi, \\ -\operatorname{curl} E + i\omega\mu_+ H &= 0 \quad \text{in } \mathbb{R}^3_+ \setminus \overline{\mathcal{M}}, & [e_3 \wedge H]_{\mathcal{M}} &= 0, \end{aligned}$$

$$\begin{aligned} \operatorname{curl} H + i\omega\epsilon_- E &= 0 \quad \text{in } \mathbb{R}^3_- \setminus \overline{\Omega}, & [e_3 \wedge E]_{\Sigma_0} &= 0, \\ -\operatorname{curl} E + i\omega\mu_- H &= 0 \quad \text{in } \mathbb{R}^3_- \setminus \overline{\Omega}, & [e_3 \wedge H]_{\Sigma_0} &= 0. \end{aligned}$$

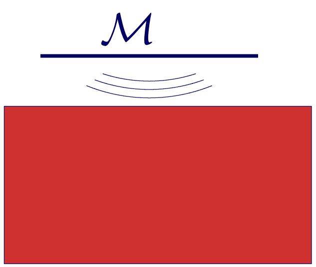
(cf. e. g. Cessenat: Mathematical Methods in Electromagnetism)

Primary / Secondary Field

Decompose (E, H)

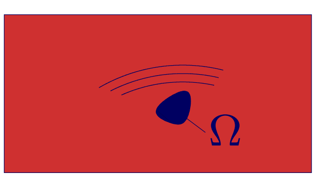
$$E = E^i + E^s, \qquad H = H^i + H^s$$

Primary electromagnetic field ("Excitation") (E^i, H^i)



$$\left.\begin{array}{rcll} \operatorname{curl} H^i + i\omega\epsilon E^i &=& 0 & \text{in } \mathbb{R}^3, \\ -\operatorname{curl} E^i + i\omega\mu H^i &=& i\omega\mu\varphi\delta_{\mathcal{M}} & \text{in } \mathbb{R}^3, \end{array}\right\} + \text{(RC)}$$

Secondary electromagnetic field (E^s, H^s) ("Scattered Field")



$$\left.\begin{array}{rcll} \operatorname{curl} H^s + i\omega\epsilon E^s &=& 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ -\operatorname{curl} E^s + i\omega\mu H^s &=& 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ \nu \wedge E^s|_{\partial\Omega} &=& -\nu \wedge E^i|_{\partial\Omega} & \end{array}\right\} + \text{(RC)}$$

Measurement Operator

Measurements:

$$M : H^{-\frac{1}{2}}(\mathrm{div}, \mathcal{M}) \to H^{-\frac{1}{2}}(\mathrm{curl}, \mathcal{M}),$$
$$\varphi \mapsto H_\tau^s|_{\mathcal{M}},$$

with $H_\tau^s := (e_3 \wedge H^s) \wedge e_3$.

Properties:

- M is compact
- $M = M^T$ using the identification

$$H^{-\frac{1}{2}}(\mathrm{curl}, \partial\Omega) = \left(H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega)\right)'$$ and vice versa.

- If $\omega^2 \epsilon_- \mu_-$ is not a resonance of Ω then M is injective.

Goal: Locate Ω from given M.

Virtual Measurements

$\psi \in H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega)$: given tangential electric field on $\partial\Omega$

$$L : H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega) \to H^{-\frac{1}{2}}(\mathrm{curl}, \mathcal{M}), \quad \psi \mapsto H_\tau^\psi|_\mathcal{M},$$

where

$$\left.\begin{array}{rcll} \mathrm{curl}\, H^\psi + i\omega\epsilon E^\psi &=& 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ -\,\mathrm{curl}\, E^\psi + i\omega\mu H^\psi &=& 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ \nu \wedge E^\psi|_{\partial\Omega} &=& \psi & \end{array}\right\} + \text{(RC)}$$

Obviously

$$M = LG, \text{ with } G : \varphi \mapsto -\nu \wedge E^i|_{\partial\Omega}.$$

$$\boxed{\phi \stackrel{G}{\longmapsto} -\nu \wedge E^i|_{\partial\Omega} \stackrel{L}{\longmapsto} H_\tau^s|_\mathcal{M}}$$

Factorization

$\chi \in H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega)$: given surface currents on $\partial\Omega$

$$\begin{aligned} F: \; H^{-\frac{1}{2}}(\mathrm{curl}, \partial\Omega) \quad &\to \quad H^{-\frac{1}{2}}(\mathrm{div}, \partial\Omega) \\ \chi \quad &\mapsto \quad \nu \wedge E^d|_{\partial\Omega}, \end{aligned}$$

where (E^d, H^d) solve the diffraction problem

$$\left.\begin{array}{rcll} \mathrm{curl}\, H^d + i\omega\epsilon E^d &=& 0 & \text{in } \mathbb{R}^3 \setminus \partial\Omega \\ -\,\mathrm{curl}\, E^d + i\omega\mu H^d &=& 0 & \text{in } \mathbb{R}^3 \setminus \partial\Omega \\ [H_\tau^d]_{\partial\Omega} &=& \chi & \\ [E_\tau^d]_{\partial\Omega} &=& 0 & \end{array}\right\} + \text{(RC)}$$

M can be written as

$$\boxed{M = -i\omega\mu_+ L F L^T,}$$

$$\phi \overset{-i\omega\mu_+ L^T}{\longmapsto} H_\tau|_{\partial\Omega} \overset{F}{\longmapsto} -\nu \wedge E^i|_{\partial\Omega} = \nu \wedge E^s|_{\partial\Omega} \overset{L}{\longmapsto} H_\tau^s|_{\mathcal{M}}$$

Range characterization

Dyadic Green's function $\mathbb{G} \in \mathcal{D}'(\mathbb{R}^3)^{3\times 3}$

$$\operatorname{curl} \frac{1}{\epsilon} \operatorname{curl} \mathbb{G}(x,y) - \omega^2 \mu \mathbb{G}(x,y) = \delta(x-y)\mathbb{I}$$

$+$(RC), columnwise

($\mathbb{I}$: 3×3 identity matrix)

Tangential component of a magnetic dipole in z with polarization p:

$$H_\tau(\cdot\,; z, p) = (e_3 \wedge \mathbb{G}(\cdot, z)p) \wedge e_3 \text{ on } \mathcal{M}$$

$\mathcal{R}(L)$ determines Ω:

> For every $z \in \mathbb{R}^3_-$ and every polarization p
>
> $z \in \Omega$ if and only if $H_\tau(\cdot\,; z, p) \in \mathcal{R}(L)$

$\rightsquigarrow H_\tau(\cdot\,; z, p) \in \mathcal{R}(M) \subseteq \mathcal{R}(L)$ implies $z \in \Omega$

Sketch of the Proof

If $z \in \Omega$ then

$$H_\tau(\cdot\,;z,p) = L\left(\nu \wedge \frac{1}{i\omega\epsilon}\operatorname{curl}(\mathbb{G}(\cdot,z)p)|_{\partial\Omega}\right) \in \mathcal{R}(L).$$

Assume that $H_\tau(\cdot\,;z,p) \in \mathcal{R}(L)$ but $z \in \mathbb{R}^3_- \setminus \Omega$

$\rightsquigarrow$ there exists solution H^ψ of Maxwells Equ. outside Ω with

$$(e_3 \wedge H^\psi) \wedge e_3 = (e_3 \wedge \mathbb{G}(\cdot,z)p) \wedge e_3 \text{ on } \mathcal{M}$$

$\rightsquigarrow$ $(e_3 \wedge H^\psi) \wedge e_3 = (e_3 \wedge \mathbb{G}(\cdot,z)p) \wedge e_3$ on the plane containing $\mathcal{M}$

$\rightsquigarrow$ $H^\psi = \mathbb{G}(\cdot,z)p$ in the halfspace above $\mathcal{M}$

$\rightsquigarrow$ $H^\psi = \mathbb{G}(\cdot,z)p$ in $\mathbb{R}^3_+$ and have the same Cauchy data on Σ_0

$\rightsquigarrow$ $H^\psi = \mathbb{G}(\cdot,z)p$ in $\mathbb{R}^3 \setminus (\Omega \cup \{z\})$, but $\mathbb{G}(\cdot,z)p \notin H_{loc}(\operatorname{curl},\mathbb{R}^3 \setminus \Omega)$ ↯

A sampling method

If $H_\tau(\cdot\,;z,p) \in \mathcal{R}(M)$ then $z \in \Omega$.

M injective, compact $\rightsquigarrow$ singular value decomposition (u_j, v_j, σ_j):

$$Mv_j = \sigma_j u_j, \qquad M^* u_j = \sigma_j v_j, \qquad \sigma_j > 0,\ j \in \mathbb{N}$$

Picard criterion:

$$H_\tau(\cdot\,;z,p) \in \mathcal{R}(M) \quad \text{iff} \quad \sum_{j\in\mathbb{N}} \frac{|\langle H_\tau(\cdot\,;z,p), \overline{u}_j\rangle|^2}{\sigma_j^2} < \infty$$

Numerical Realization (sketch):

Test whether $\displaystyle\sum_{j=1}^{n} \frac{|\langle H_\tau(\cdot\,;z,p), \overline{\tilde{u}_j}\rangle|^2}{\tilde{\sigma}_j^2}$ is small

for svd $(\tilde{u}_j, \tilde{v}_j, \tilde{\sigma}_j)$ of a (finite dimensional) approximation to M.

Numerical Realization

Given a finite dimensional approximation $\tilde{M} \in \mathbb{C}^{N \times N}$

- Compute svd $(\tilde{u}_j, \tilde{v}_j, \tilde{\sigma}_j)$ of $\tilde{M}$
- Truncate svd at a trust level δ, e. g. $\delta = \|\tilde{M} - \tilde{M}^T\|$

$$\sigma_1 > \sigma_2 > \ldots > \sigma_n > \delta > \sigma_{n+1}$$

- For every point z on a sampling grid
  - Calculate projection h_z of $H_\tau(\cdot\,; z, p)$ to the measurement space *(can be done in advance)*
  - With a threshold value C_∞, mark point as "inside" if

$$\frac{\|\tilde{M}^{-1} h_z\|^2}{\|h_z\|^2} \approx \sum_{j=1}^{n} \frac{|h_z \cdot \overline{\tilde{u}_j}|^2}{\tilde{\sigma}_j^2} \Big/ \sum_{j=1}^{n} |h_z \cdot \overline{\tilde{u}_j}|^2 < C_\infty$$

C_∞ has to be choosen empirically.

Numerical Results - Setup

Measurement device $\mathcal{M}$

$$32cm \times 32cm$$

Scatterer

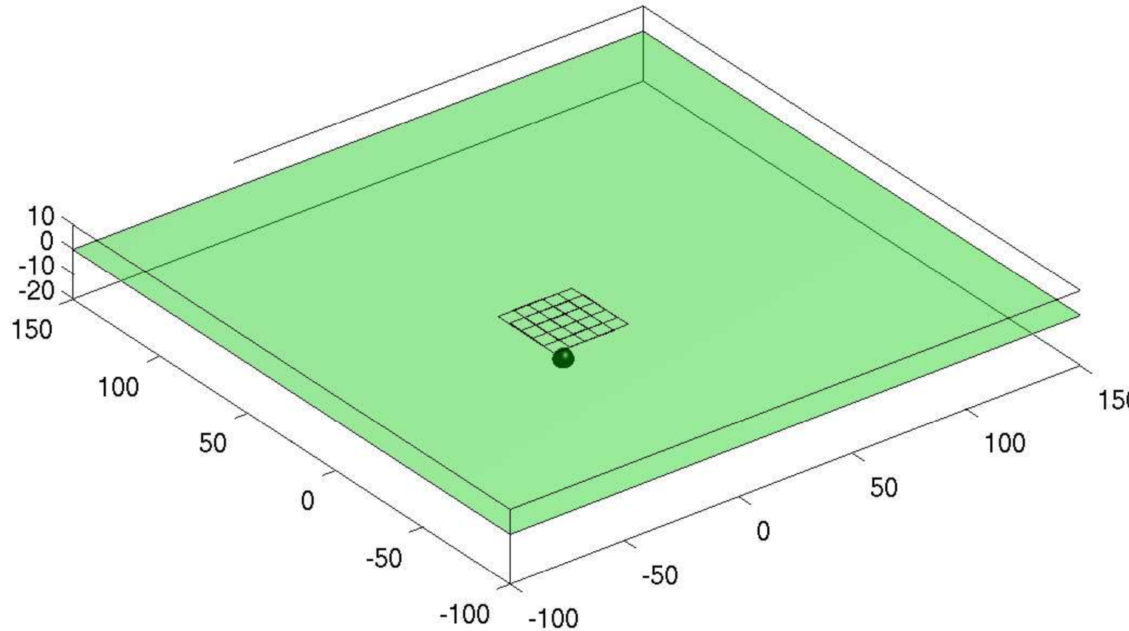
$10cm$–$15cm$ below $\mathcal{M}$

Size of the scatterer

$$6cm\text{–}12cm$$

Wavelength

$$300km$$

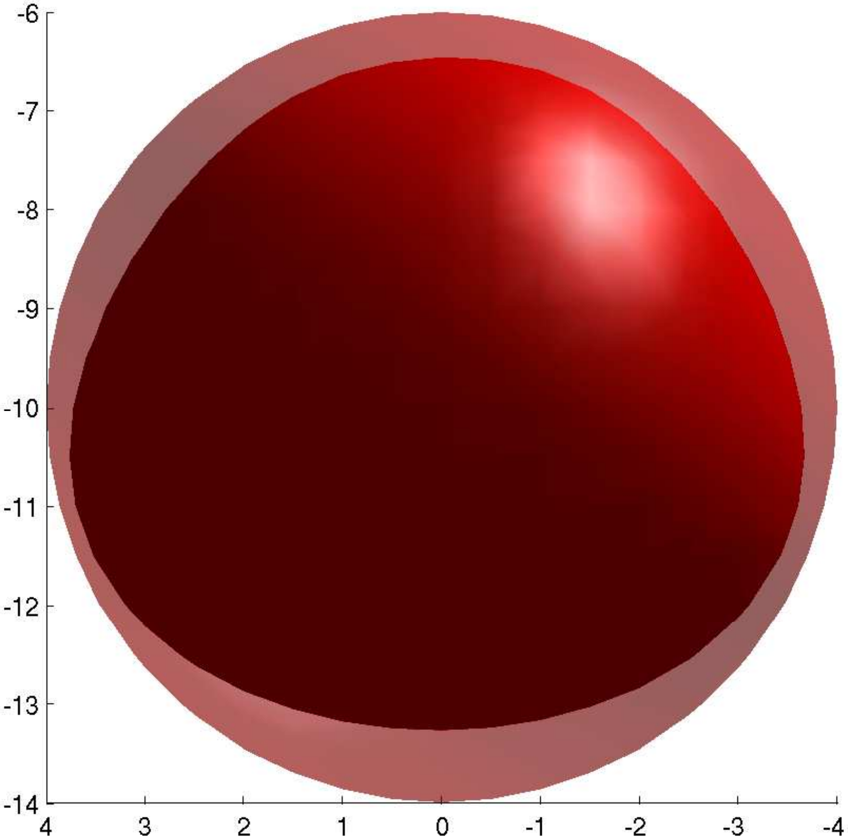


- Primary fields imposed / secondary fields measured on

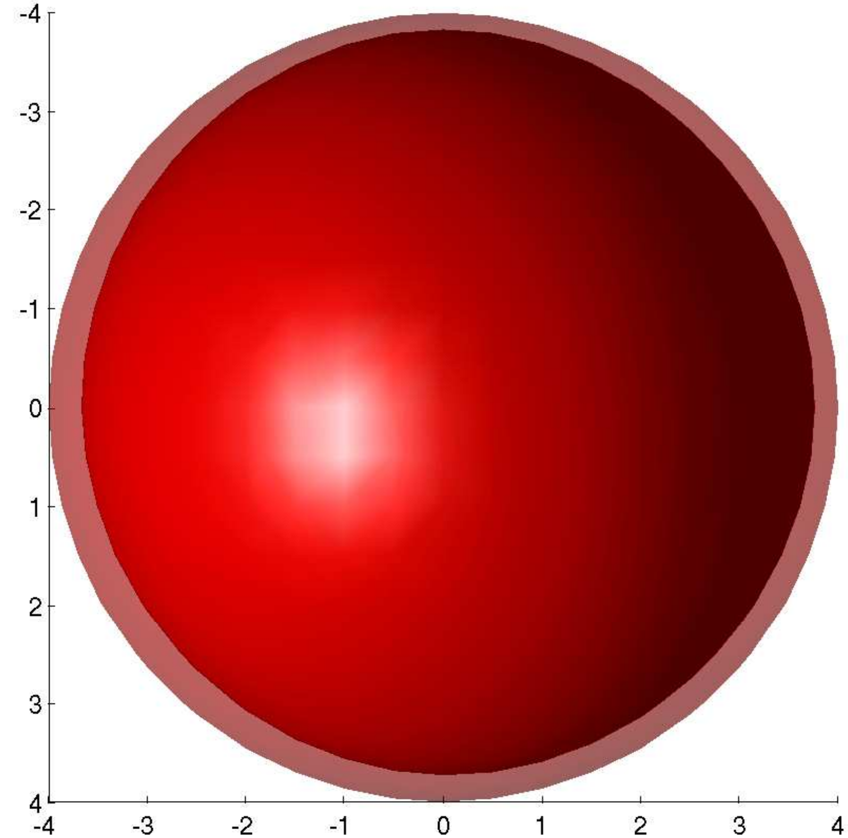
  6×6 grid on $\mathcal{M}$ (à 2 tangential components). $\rightsquigarrow \tilde{M} \in \mathbb{C}^{72\times 72}$

- Simulated data (BEM) provided by K. Erhard, Göttingen

Numerical Results



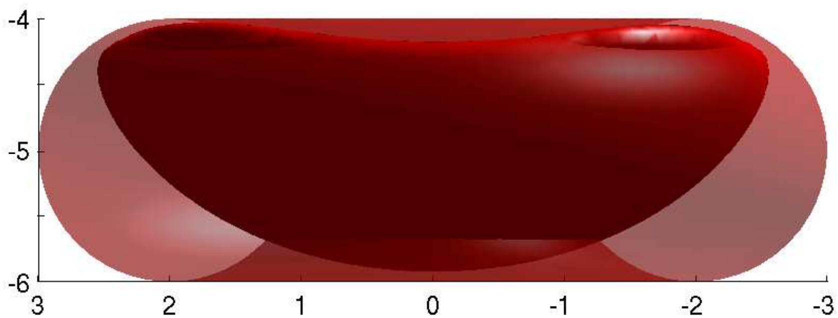
side view



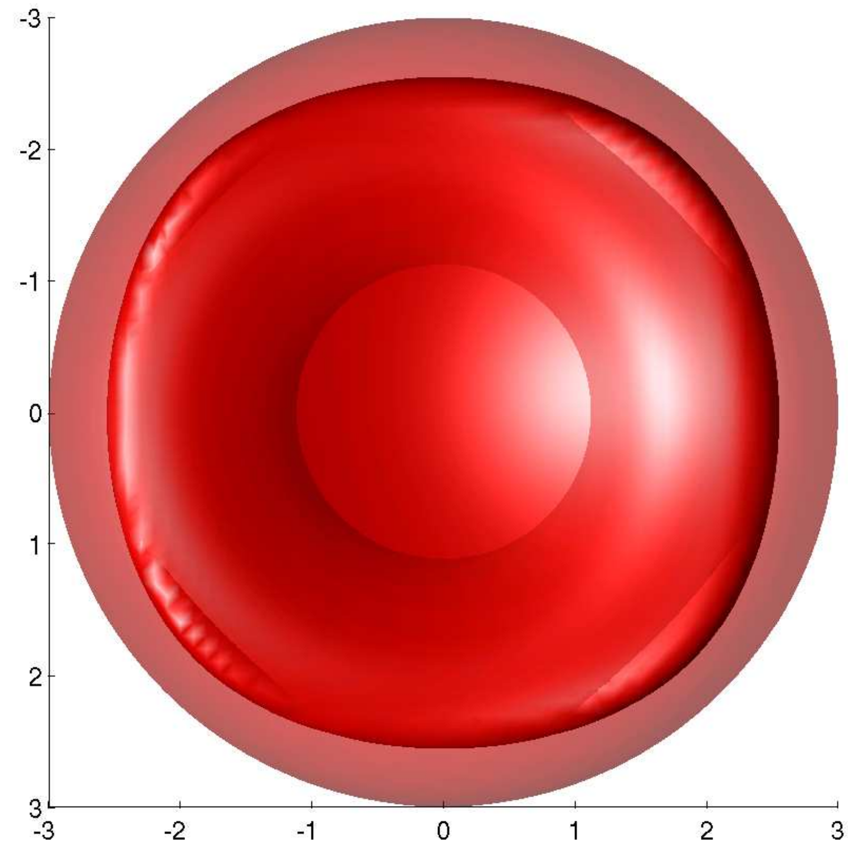
top view

Ball with radius $r = 4cm$

Numerical Results



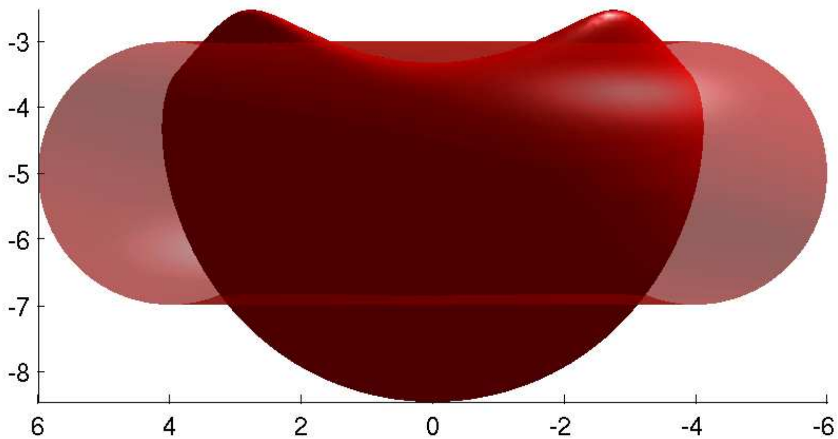
side view



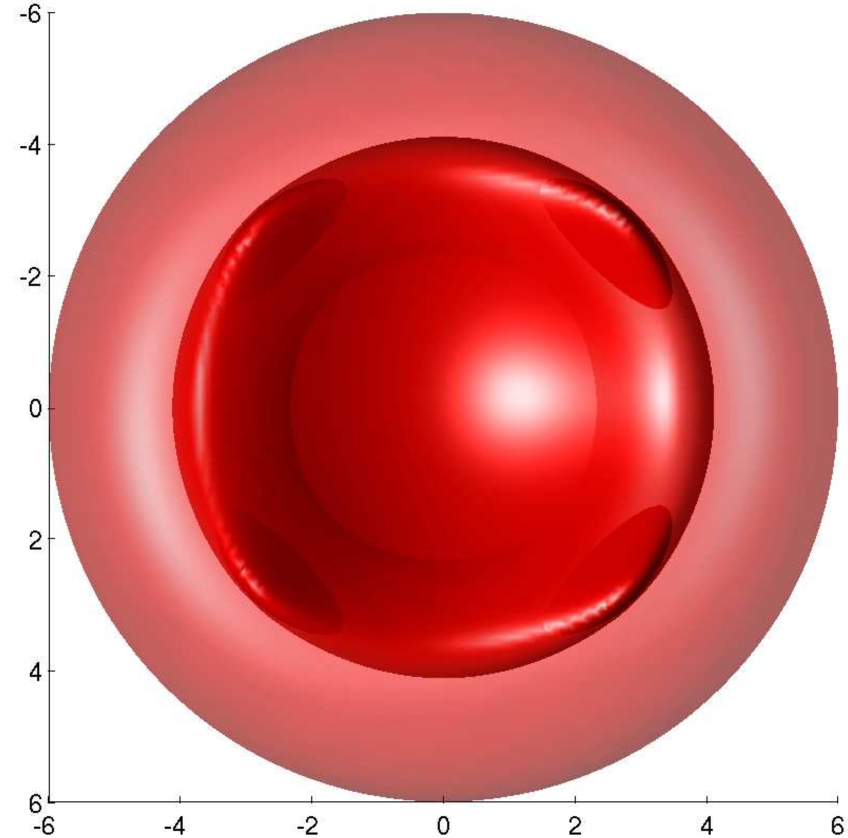
top view

Torus with inner radius $r_1 = 1cm$ and outer radius $r_2 = 3cm$

Numerical Results



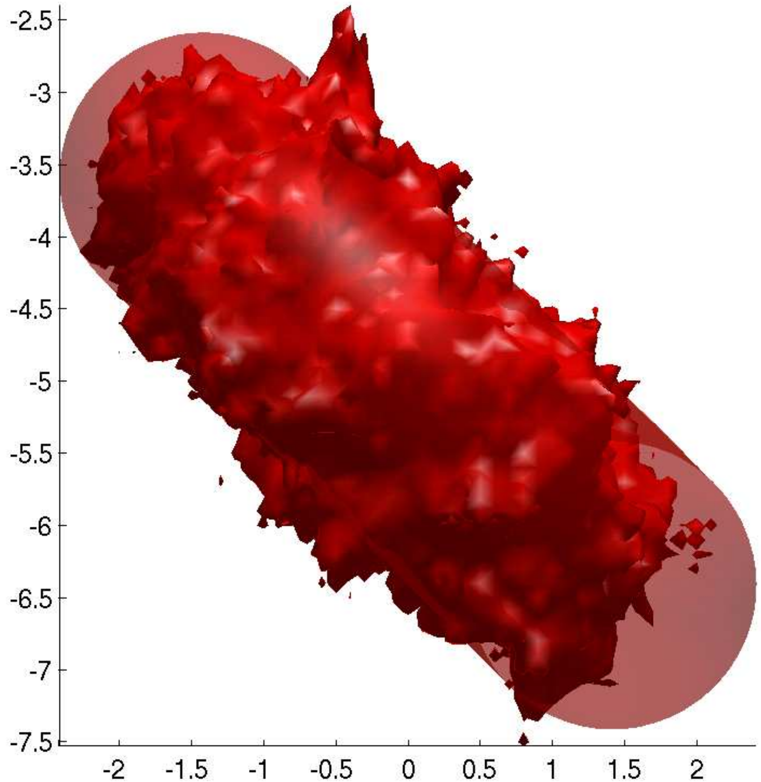
side view



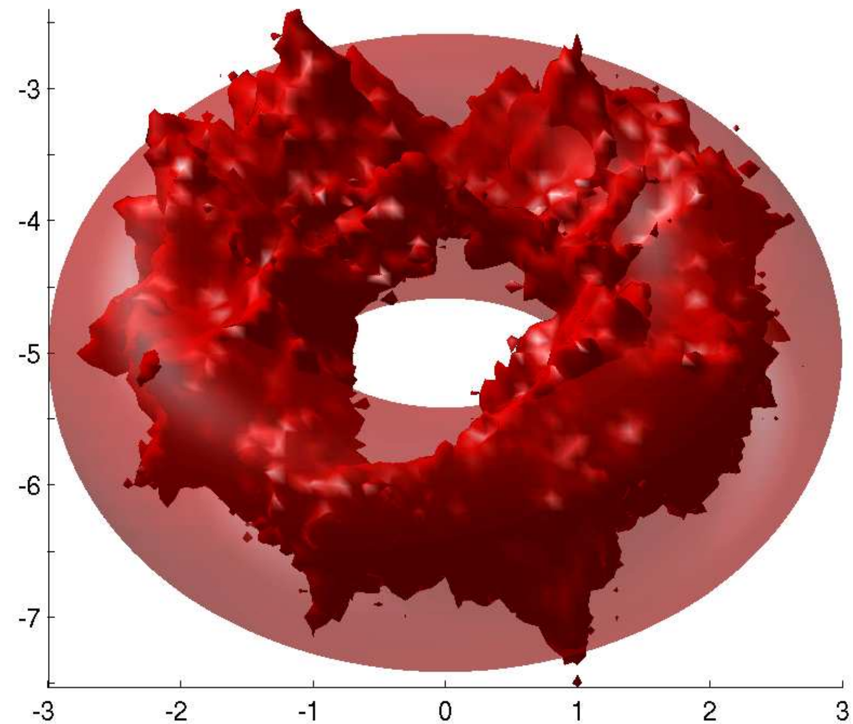
top view

Torus with inner radius $r_1 = 2cm$ and outer radius $r_2 = 6cm$

Numerical Results



side view



front view

Torus with inner radius $r_1 = 1cm$ and outer radius $r_2 = 3cm$

Alternative test: Test whether $(|h_z \cdot \overline{\tilde{u}_j}|^2)_j$ decays faster than $(\sigma_j^2)_j$ (using a line of best fit in semilogarithmic scale).

Outlook: An "upper" bound for Ω

So far: If $H_\tau(\cdot\,;z,p)|_{\mathcal{M}} \in \mathcal{R}(M)$ then $z \in \Omega$.

$\rightsquigarrow$ Only "lower" bound for Ω.

Actually (for $z \in \mathbb{R}^3_-$)

$$z \in \Omega \quad \text{iff} \quad \exists \psi : L\psi = H_\tau(\cdot\,;z,p)|_{\mathcal{M}}$$

$$\text{iff} \quad \exists (\varphi_n) : (-\nu \wedge E_n^i|_{\partial\Omega})_n \text{ converges}, \quad M\varphi_n \to H_\tau(\cdot\,;z,p),$$

With $R_z : H^{-\frac{1}{2}}(\mathrm{div}, \mathcal{M}) \to H^{-\frac{1}{2}}(\mathrm{div}, \partial B_\epsilon(z)), \quad \varphi \mapsto \nu \wedge E^i|_{\partial B_\epsilon(z)}$

If $z \in \Omega$, $B_\epsilon(z) \subseteq \Omega$ then $\exists (\varphi_n)_n \subseteq H^{-\frac{1}{2}}(\mathrm{div}, \mathcal{M})$ such that

$$R_z\varphi_n = \nu \wedge E_n^i|_{\partial B_\epsilon(z)} \text{ converge}, \quad M\varphi_n \to H_\tau(\cdot\,;z,p).$$

$\rightsquigarrow$ Same algorithm with MR_z^{-1} instead of M
(but now with one svd per sampling point).