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SIMULTANEOUS DETERMINATION OF THE DIFFUSION AND ABSORPTION COEFFICIENT FROM BOUNDARY DATA

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Abstract. We consider the inverse problem of determining both an unknown diffusion and an unknown absorption coefficient from knowledge of (partial) Cauchy data in an elliptic boundary value problem. For piecewise analytic coefficients, we prove a complete characterization of the reconstructible information. It is shown to consist of a combination of both coefficients together with the jumps in the leading order diffusion coefficient and its derivative.

1. **Introduction.** Let $B \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary ∂B and outer normal vector ν. We consider the following inverse problem: Determine *simultaneously* two unknown coefficients, the diffusivity $a(x)$ and the absorption $c(x)$, in the elliptic partial differential equation

$$-\nabla \cdot (a \nabla u) + cu = 0, \quad \text{in } B, \tag{1}$$

from knowledge of all possible Cauchy data on an arbitrarily small open boundary part $S \subseteq \partial B$,

$$\left\{ (a\partial_\nu u|_S, u|_S) \;:\; u \text{ solves (1) with } a\partial_\nu u|_{B\setminus \overline{S}} = 0 \right\}$$

This problem arises, e.g., in steady-state diffuse optical tomography, cf. the topical reviews of Gibson, Hebden and Arridge [17], and Arridge and Schotland [2].

For globally smooth coefficients, this and similar problems have been studied extensively. If a is smooth, then both unknown coefficients can be combined by setting $v := \sqrt{a}u$, which transforms (1) into

$$-\Delta v + \eta v = 0,$$

with the *effective absorption*

$$\eta := \frac{\Delta\sqrt{a}}{\sqrt{a}} + \frac{c}{a}. \tag{2}$$

If $a = 1$ in a neighborhood of S, then u and v have the same Cauchy boundary values on S. Hence, the Cauchy data can only contain information about the effective absorption η from which, generally, one cannot extract a and c. The consequence is that the inverse problem of steady-state diffuse optical tomography is *not uniquely solvable*, see Arridge and Lionheart [1].

These non-uniqueness arguments are, however, only valid for globally smooth a and c. In fact, the author [21] has shown that this inverse problem is *uniquely*

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solvable in the class of piecewise constant functions (c might even be piecewise analytic).

In this work we will close the gap between the non-uniqueness results in [1] and the uniqueness results in [21]. We derive a complete characterization of the reconstructible information for piecewise analytic a and c. Roughly speaking, the boundary data is shown to determine the effective absorption η wherever a and c are smooth, and also the jumps of a and its derivative on the discontinuity set of a. A formal motivation for this result is that jumps in a or its first derivatives lead to distributional-type singularities in η that can be distinguished from a regular function c.

The rigorous formulation of our result is given in theorem 2.2 below. Let us note that our characterization is complete in the sense that two pairs of coefficients (a, c) can only be distinguished by boundary data, if they differ in one of the properties given in theorem 2.2. Also, the result implies both the non-uniqueness result (for smooth a and c) and the uniqueness result (since a piecewise constant function is uniquely determined by its jumps).

The proof of our result follows the general approach in [21]. We first derive monotony results to relate the unknown coefficients to the Cauchy data, resp., the corresponding Neumann-to-Dirichlet operators. Then we separately control the terms in the monotony results using the technique of *localized potentials* developed by the author in [15]. Localized potentials are solutions of (3) that are large on some subsets of the domain while staying small somewhere else. Note that growth properties of special solutions are frequently being used in the study of coefficient determination problems. The specific advantage of localized potentials is that their construction relies on abstract, but simple, functional analytic arguments, which makes them particularly adaptable, and enables us to control the solutions' H^1- and L^2-norms on all kinds of different subsets and boundaries.

Let us give some more reference on related problems. The question whether η can be reconstructed from Cauchy data of v has mainly been studied in the context of the famous Calderón problem [8, 9], where $c = 0$. For full boundary data ($\partial B = S$) see Druskin [11, 12, 13], Kohn and Vogelius [31, 32], Sylvester and Uhlmann [38], Nachman [36] and Astala and Päivärinta [4] for seminal contributions and Uhlmann [39] for a recent overview. For some more related works, let us refer to [35, 14, 18, 19, 6, 30, 22, 3, 5].

Uniqueness results for partial boundary data were achieved in Bukhgeim and Uhlmann [7], Knudsen [29], Isakov [27], Kenig, Sjöstrand and Uhlmann [28] and the author's work [15]. For two-dimensional domains, recent breakthroughs have been made for the Calderón problem with partial data and also for general second-order elliptic equations by Imanuvilov, Uhlmann and Yamamoto [23, 24, 25]. If a is real but c has a known, non-zero imaginary part then one can reconstruct η in (2) and extract c and a from it, cf. Grinberg [20]. The detection of the combined support of diffusive and absorbing inclusions was studied by Hyvönen and the author in [16]. In three or higher dimensions, simultaneous identifiability of convection and absorption coefficients was achieved by Nakamura, Sun and Uhlmann [37], and in two dimensions, Cheng and Yamamoto [10] showed uniqueness for two convection coefficients. Result in the context of Maxwell's and elasticity equations are summarized in the book of Isakov [26].

We finish this introduction with some general comments on the technique of localized potentials. On the good side, the technique is independent of the dimension

$n \geq 2$, it immediately yields results for partial boundary data, and it can handle parameter jumps. Moreover, it is comparatively simple and seems to be extendable to several more complex problems. The major disadvantage is that (due to the monotony arguments) it can only distinguish two parameters if there is a neighborhood of the boundary in which one parameter is "for the first time larger than the other". This restricts the use of our technique to piecewise analytic parameters. Up to now, C^∞-parameters can not be handled as the difference of two such functions can have an infinite number of sign changes close to the boundary.

The outline of this work is as follows. We rigorously formulate our result in Section 2. In Section 3 we derive a general monotony lemma and some more specific corollaries. The existence of localized potentials is shown in Section 4. The monotony results and the localized potentials are then combined to prove our main result in Section 5.

2. **The main result.** We now rigorously state our main result. Let $B \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary ∂B and outer normal ν; cf. definition 2.1 below. Let $S \subseteq \partial B$ be an arbitrarily small open part of the boundary, $g \in L^2(S)$, and $a, c \in L^\infty_+(B)$, where the subscript "+" denotes positive essential infima. Then there exists a unique solution $u \in H^1(B)$ of the elliptic partial differential equation

$$-\nabla \cdot (a \nabla u) + cu = 0 \quad \text{in } B \tag{3}$$

with Neumann boundary values

$$a \partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{on } B \setminus \overline{S}. \end{cases} \tag{4}$$

The knowledge of all possible pairs of Neumann and Dirichlet boundary values $(a\partial_\nu u|_S, u|_S)$ is equivalent to knowing the local Neumann-to-Dirichlet operator,

$$\Lambda_{a,c}:\ L^2(S) \to L^2(S), \quad g \mapsto u|_S, \tag{5}$$

where u solves (3) and (4). $\Lambda_{a,c}$ is a linear, compact and selfadjoint operator.

Definition 2.1. (a) An open subset $\Gamma \subseteq \partial O$ of the boundary of a bounded domain $O \subset \mathbb{R}^n$ is called a *smooth piece* (resp., *Lipschitz piece*) if Γ is the graph of a C^∞- (resp., Lipschitz-) function, and O lies on one side of Γ.

(b) A bounded domain $O \subset \mathbb{R}^n$ is said to have *smooth boundary* (resp., *Lipschitz boundary*) if every point $x \in \partial O$ lies inside a smooth (resp. Lipschitz) piece.

It is said to have *piecewise smooth boundary* ∂O if ∂O is a countable union of closures of smooth pieces.

(c) A function $a \in L^\infty(B)$ is *piecewise analytic* (resp., *piecewise* C^∞) on a partition $(O_j, \Gamma)_{j=1}^J$ of pairwise disjoint domains $O_j \subseteq B$, if
   (i) ∂O_j is Lipschitz and piecewise smooth.
   (ii) $\overline{B} = \bigcup_{j=1}^J \overline{O}_j$,
   (iii) $\Gamma = \bigcup_{j=1}^J \partial O_j \setminus \partial B$,

and $a|_{O_j}$ has an extension to a (real-)analytic, (resp. C^∞) function on a neighborhood of $\overline{O}_j$.

Note that we require that the boundaries ∂O_j in definition 2.1(c) are not just piecewise smooth, but also Lipschitz. Hence, ∂O_j may have corners, but no cusps.

With a consistently oriented normal ν on Γ, we denote by $a^+|_\Gamma$, resp., $a^-|_\Gamma$, the traces taken from the side that the normal, resp., its negative, is oriented into.

$[a]_\Gamma := a^-|_\Gamma - a^+|_\Gamma$ is the jump in the direction of the normal. In the case of $a^+|_\Gamma = a^-|_\Gamma$ we also write $a|_\Gamma$ for the sake of brevity.

Our main result is the following theorem.

Theorem 2.2. *Let $a_1, a_2, c_1, c_2 \in L^\infty_+(B)$ be piecewise analytic coefficients on a joint partition $(O_j, \Gamma)_{j=1}^J$, and let Λ_{a_1,c_1}, Λ_{a_2,c_2} be the corresponding local Neumann-to-Dirichlet operators on the boundary part S.*

Then, $\Lambda_{a_1,c_1} = \Lambda_{a_2,c_2}$ if and only if

(a) on the boundary part S,

$$a_1|_S = a_2|_S, \quad \text{and} \quad \partial_\nu a_1|_S = \partial_\nu a_2|_S \quad \text{on } S,$$

(b) on the insulated remainder part $\partial B \setminus \overline{S}$,

$$\frac{\partial_\nu a_1}{a_1}|_{\partial B\setminus\overline{S}} = \frac{\partial_\nu a_2}{a_2}|_{\partial B\setminus\overline{S}} \quad \text{on } \partial B \setminus \overline{S},$$

(c) on the set where all coefficients are analytic,

$$\eta_1 := \frac{\Delta\sqrt{a_1}}{\sqrt{a_1}} + \frac{c_1}{a_1} = \frac{\Delta\sqrt{a_2}}{\sqrt{a_2}} + \frac{c_2}{a_2} =: \eta_2 \quad \text{on } B \setminus \Gamma,$$

(d) on the discontinuity set Γ,

$$\frac{a_1^+|_\Gamma}{a_1^-|_\Gamma} = \frac{a_2^+|_\Gamma}{a_2^-|_\Gamma}, \quad \text{and} \quad \frac{[\partial_\nu a_2]_\Gamma}{a_2^-|_\Gamma} = \frac{[\partial_\nu a_1]_\Gamma}{a_1^-|_\Gamma} \quad \text{on } \Gamma.$$

The theorem will be proven in Section 5 by combining monotony results and localized potentials derived in the following two sections.

3. Monotony results.

3.1. Motivation.

Let us start with a simple example of a monotony relation to motivate the elementary but somewhat technical results in the next subsection.

Let $a_1, a_2, c_1, c_2 \in L^\infty_+(B)$ and let Λ_{a_1,c_1}, Λ_{a_2,c_2} be the corresponding local Neumann-to-Dirichlet operators. It is easy to show, cf., e.g. [21, Lemma 4.1], that for all $g \in L^2(S)$,

$$\int_S g\,(\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2})\,g\,\mathrm{d}s \geq \int_B \left((a_2 - a_1)\,|\nabla u_2|^2 + (c_2 - c_1)\,|u_2|^2\right)\,\mathrm{d}x,$$

where u_2 is the solution of (3) and (4) for $(a, c) = (a_2, c_2)$.

Hence, if $a_2 > a_1$ and $c_2 > c_1$ then $\Lambda_{a_1,c_1} > \Lambda_{a_2,c_2}$ in the sense of quadratic forms, which is why we refer to such inequalities as monotony results.

Monotony relations can be used to prove uniqueness results: Assume that a_2 and a_1 are constant. If we can construct a solution u_2 for which $|\nabla u_2|^2$ is very large but $|u|^2$ is very small, then $a_2 > a_1$ must imply that $\Lambda_2 \neq \Lambda_1$. If a_2 and a_1 are constant in some neighbourhood of S then a similar argument holds if there exists a solution for which $|\nabla u_2|^2$ is very large only on this neighbourhood. In fact, such solutions (the so-called localized potentials) can be constructed. Together with the above monotony relation they can be used to prove that $\Lambda_{a,c}$ uniquely determines piecewise constant a and c, cf. [21].

In this work we require more technical monotony relations to study the uniqueness question for piecewise analytic coefficients. As explained in the introduction, setting $v = \sqrt{a}u$ transforms (3) into an equation where both a and c are combined into an effective absorption coefficient η. To be able to combine the coefficients only in a part of the domain, we will use this transformation with a replaced by a more general function α and derive a corresponding monotony relation. Three corollaries of this general monotony result will later be used in our uniqueness proof.

3.2. **A monotony lemma and three corollaries.** As in Theorem 2.2, let the coefficients $a_1, a_2, c_1, c_2 \in L^\infty_+(B)$ be piecewise analytic functions on a joint partition $(O_j, \Gamma)_{j=1}^J$, and let Λ_{a_1,c_1}, Λ_{a_2,c_2} be the corresponding local Neumann-to-Dirichlet operators.

Lemma 3.1. *For all functions $\alpha_1, \alpha_2 \in L^\infty_+(B)$ that are piecewise C^∞ on the partition (O_j, Γ), and for all $g \in L^2(S)$,*

$$\begin{aligned}
&\int_S g\left(\Lambda_{a_2,c_2} - \Lambda_{a_1,c_1}\right) g \,\mathrm{d}s \\
&\leq \int_S \left(1 - \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\right) g u_2 \,\mathrm{d}s - \int_{\partial B} \left(\frac{a_1}{\alpha_1}\frac{\partial_\nu \alpha_1}{2\alpha_1} - \frac{a_2}{\alpha_2}\frac{\partial_\nu \alpha_2}{2\alpha_2}\right) \alpha_2 |u_2|^2 \,\mathrm{d}s \\
&\quad + \int_{B\setminus\Gamma} \left(\eta_1^{(\alpha_1)} - \eta_2^{(\alpha_2)}\right) \alpha_2 |u_2|^2 \,\mathrm{d}x + \int_{B\setminus\Gamma} \left(\frac{a_1}{\alpha_1} - \frac{a_2}{\alpha_2}\right) \left|\nabla\left(\sqrt{\alpha_2} u_2\right)\right|^2 \,\mathrm{d}x \\
&\quad + \int_\Gamma \left\{\frac{1}{2}\left(\left[\frac{a_2}{\alpha_2}\partial_\nu \alpha_2\right]_\Gamma - \left[\frac{\alpha_2}{\alpha_1}\frac{a_1}{\alpha_1}\partial_\nu \alpha_1\right]_\Gamma\right)|u_2|^2 - 2\left[\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\right]_\Gamma a_1 \partial_\nu u_1 u_2\right\} \mathrm{d}s,
\end{aligned}$$

where

$$\eta_j^{(\alpha_j)} := \frac{\nabla \cdot \left(\frac{a_j}{\alpha_j}\nabla\sqrt{\alpha_j}\right)}{\sqrt{\alpha_j}} + \frac{c_j}{\alpha_j} \quad \text{on } B\setminus\Gamma, \tag{6}$$

$j = 1, 2$, and u_1 and u_2 are the solutions of (3), (4) for $(a, c) = (a_1, c_1)$, resp., $(a, c) = (a_2, c_2)$.

Note that we use two common, but somewhat sloppy notations here. First, the seemingly effectless removal of the Lebesgue null set Γ from the integration domain actually means that the derivatives in the integral are taken on $B \setminus \Gamma$. Second, the last term in the asserted inequality is in fact the dual pairing of

$$a_1 \partial_\nu u_1 \in H^{-1/2}(\Gamma) \quad \text{and} \quad 2\left[\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\right]_\Gamma u_2 \in H^{1/2}(\Gamma),$$

which, here and in the following, we write as an integral for ease of notation.

We will use this lemma with $\alpha_j = a_j$ on some part of B, and $\alpha_j = 1$ on another. We start with the cases where $\alpha_j = 1$ or $\alpha_j = a_j$ everywhere on B. In the first case we obtain the monotony result already stated in our motivation.

Corollary 1. *For all $g \in L^2(S)$,*

$$\int_S g\left(\Lambda_{a_2,c_2} - \Lambda_{a_1,c_1}\right) g \,\mathrm{d}s \leq \int_B \left((a_1 - a_2)|\nabla u_2|^2 + (c_1 - c_2)|u_2|^2\right) \mathrm{d}x,$$

where u_2 is the solution of (3) and (4) for $(a, c) = (a_2, c_2)$.

The second corollary follows from setting $\alpha_1 = a_1$ and $\alpha_2 = a_2$.

Corollary 2. *For all $g \in L^2(S)$,*

$$\begin{aligned}
&\int_S g\left(\Lambda_{a_2,c_2} - \Lambda_{a_1,c_1}\right) g \,\mathrm{d}s \\
&\leq \int_{B\setminus\Gamma} \left(\eta_1 - \eta_2\right) a_2 |u_2|^2 \,\mathrm{d}x \\
&\quad + \int_S \left(1 - \frac{\sqrt{a_2}}{\sqrt{a_1}}\right) g u_2 \,\mathrm{d}s - \int_{\partial B} \left(\frac{\partial_\nu a_1}{2a_1} - \frac{\partial_\nu a_2}{2a_2}\right) a_2 |u_2|^2 \,\mathrm{d}s \\
&\quad + \int_\Gamma \left\{ \frac{1}{2}\left([\partial_\nu a_2]_\Gamma - \left[\frac{a_2}{a_1}\partial_\nu a_1\right]_\Gamma \right) |u_2|^2 - 2\left[\frac{\sqrt{a_2}}{\sqrt{a_1}}\right]_\Gamma a_1 \partial_\nu u_1 u_2 \right\} \mathrm{d}s,
\end{aligned}$$

where u_1 and u_2 are the solutions of (3) and (4) for $(a,c) = (a_1, c_1)$, resp., $(a,c) = (a_2, c_2)$.

Note that by interchanging (a_1, c_1) and (a_2, c_2), and negation, analog estimates from above are obtained. In particular, Corollary 2 already implies the if-part in Theorem 2.2.

In the last corollary we eliminate the term containing u_1.

Corollary 3. *Let O be a subdomain of B whose boundary contains a smooth piece of S. Let $\alpha_1, \alpha_2 \in L^\infty_+(B)$ be piecewise C^∞ on the partition (O_j, Γ). Furthermore, let $\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}$ be continuous on a neighborhood of $\overline{O}$ and let $B \setminus \overline{O}$ have Lipschitz boundary.*

Then there exists $C > 0$ such that for all $g \in L^2(S)$,

$$\begin{aligned}
&\int_S g\left(\Lambda_{a_2,c_2} - \Lambda_{a_1,c_1}\right) g \,\mathrm{d}s \\
&\leq \int_{S\cap\partial O} \left(1 - \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\right) g u_2 \,\mathrm{d}s - \int_{\partial B\cap\partial O} \left(\frac{a_1}{\alpha_1}\frac{\partial_\nu \alpha_1}{2\alpha_1} - \frac{a_2}{\alpha_2}\frac{\partial_\nu \alpha_2}{2\alpha_2}\right) \alpha_2 |u_2|^2 \,\mathrm{d}s \\
&\quad + \int_{O\setminus\Gamma} \left(\eta_1^{(\alpha_1)} - \eta_2^{(\alpha_2)}\right) \alpha_2 |u_2|^2 \,\mathrm{d}x + \int_{O\setminus\Gamma} \left(\frac{a_1}{\alpha_1} - \frac{a_2}{\alpha_2}\right) \left|\nabla\left(\sqrt{\alpha_2} u_2\right)\right|^2 \,\mathrm{d}x \\
&\quad + \int_{O\cap\Gamma} \left\{ \frac{1}{2}\left(\left[\frac{a_2}{\alpha_2}\partial_\nu \alpha_2\right]_\Gamma - \left[\frac{\alpha_2}{\alpha_1}\frac{a_1}{\alpha_1}\partial_\nu \alpha_1\right]_\Gamma \right) |u_2|^2 \right\} + C\|u_2\|^2_{H^1(B\setminus\overline{O})},
\end{aligned}$$

where u_2 is the solution of (3) and (4) for $(a,c) = (a_2, c_2)$ and $\eta_j^{(\alpha_j)}$ is given by (6).

3.3. Proof of Lemma 3.1 and the corollaries.

Proof of Lemma 3.1. We define the space

$$H_\Delta(B \setminus \Gamma) := \{v \in H^1(B\setminus\Gamma) \;:\; \Delta v|_{B\setminus\Gamma} \in L^2(B\setminus\Gamma)\}.$$

Functions in $H_\Delta(B\setminus\Gamma)$ have well defined one-sided Neumann-boundary traces in $\Gamma \cup \partial B$. For $j = 1, 2$, it is easily checked that $u_j \in H_\Delta(O)$ and that u_j is a (possibly not unique) solution of

$$b_j(u_j, v) = l(v) \quad \text{ for all } v \in H_\Delta(B\setminus\Gamma),$$

with

$$\begin{aligned}
b_j(v,w) &:= \int_{B\setminus\Gamma} \left(a_j \nabla v \cdot \nabla w + c_j v w\right) \mathrm{d}x - \int_\Gamma [a_j w \partial_\nu v]_\Gamma \,\mathrm{d}s, \\
l(w) &:= \int_S g w \,\mathrm{d}s,
\end{aligned}$$

for all $v, w \in H_\Delta(B\setminus\Gamma)$.

Noting that $v \in H_\Delta(B \setminus \Gamma)$ if and only if $\sqrt{\alpha_j} v \in H_\Delta(B \setminus \Gamma)$, we also introduce

$$b_{\alpha_j}(v_{\alpha_j}, w_{\alpha_j}) := b_j\left(\frac{v_{\alpha_j}}{\sqrt{\alpha_j}}, \frac{w_{\alpha_j}}{\sqrt{\alpha_j}}\right) \quad \text{and} \quad l_{\alpha_j}(w_{\alpha_j}) := l\left(\frac{w_{\alpha_j}}{\sqrt{\alpha_j}}\right)$$

for all $v_{\alpha_j}, w_{\alpha_j} \in H_\Delta(B \setminus \Gamma)$. Thus, $u_{\alpha_j} := \sqrt{\alpha_j} u_j$ solves

$$b_{\alpha_j}(u_{\alpha_j}, v_{\alpha_j}) = l_{\alpha_j}(v_{\alpha_j}) \quad \text{for all } v_{\alpha_j} \in H_\Delta(B \setminus \Gamma),$$

Using the product rule and integration by parts on the subdomains where a_j is analytic, and α_j is C^∞, we obtain (omitting the index j)

$$\begin{aligned}\int_{B\setminus\Gamma} a\nabla v \cdot \nabla w \,\mathrm{d}x = &\int_{B\setminus\Gamma} \left(\frac{a}{\alpha}\nabla v_\alpha \cdot \nabla w_\alpha + \frac{\nabla\cdot\left(\frac{a}{\alpha}\nabla\sqrt{\alpha}\right)}{\sqrt{\alpha}} v_\alpha w_\alpha\right) \mathrm{d}x \\ &- \int_\Gamma \left[\frac{a}{\alpha}\frac{\partial_\nu \alpha}{2\alpha} v_\alpha w_\alpha\right]_\Gamma \mathrm{d}s - \int_{\partial B} \frac{a}{\alpha}\frac{\partial_\nu\alpha}{2\alpha} v_\alpha w_\alpha \,\mathrm{d}s,\end{aligned}$$

for all $v_\alpha = \sqrt{\alpha} v$ and $w_\alpha = \sqrt{\alpha} w$ in $H_\Delta(B \setminus \Gamma)$.

Hence,

$$\begin{aligned}b_{\alpha_j}(v_{\alpha_j}, w_{\alpha_j}) = &\int_{B\setminus\Gamma} \frac{a_j}{\alpha_j}\nabla v_{\alpha_j} \cdot \nabla w_{\alpha_j} \,\mathrm{d}x + \int_{B\setminus\Gamma} \eta_j^{(\alpha_j)} v_{\alpha_j} w_{\alpha_j} \,\mathrm{d}x \\ &- \int_{\partial B} \frac{a_j}{\alpha_j}\frac{\partial_\nu \alpha_j}{2\alpha_j} v_{\alpha_j} w_{\alpha_j} \,\mathrm{d}s - \int_\Gamma \left[\frac{a_j}{\alpha_j} w_{\alpha_j} \partial_\nu v_{\alpha_j}\right]_\Gamma \mathrm{d}s.\end{aligned}$$

Now we can write

$$\begin{aligned}&\int_S g\left(\Lambda_{a_2,c_2} - \Lambda_{a_1,c_1}\right) g \,\mathrm{d}s \\ &= b_2(u_2,u_2) - b_1(u_1,u_1) = b_{\alpha_2}(u_{\alpha_2}, u_{\alpha_2}) - b_{\alpha_1}(u_{\alpha_1}, u_{\alpha_1}) \\ &= \left(l_{\alpha_2} - l_{\alpha_1}\right)(u_{\alpha_2}) + b_{\alpha_1}(u_{\alpha_1}, u_{\alpha_2}) - b_{\alpha_1}(u_{\alpha_1}, u_{\alpha_1}) \\ &= \left(l_{\alpha_2} - l_{\alpha_1}\right)(u_{\alpha_2}) - b_{\alpha_1}(u_{\alpha_2} - u_{\alpha_1}, u_{\alpha_2} - u_{\alpha_1}) \\ &\quad + \left(b_{\alpha_1}(u_{\alpha_2}, u_{\alpha_2}) - b_{\alpha_2}(u_{\alpha_2}, u_{\alpha_2})\right) + \left(b_{\alpha_1}(u_{\alpha_1}, u_{\alpha_2}) - b_{\alpha_1}(u_{\alpha_2}, u_{\alpha_1})\right).\end{aligned}$$

The first summand in the final expression is

$$\left(l_{\alpha_2} - l_{\alpha_1}\right)(u_{\alpha_2}) = l\left(\frac{u_{\alpha_2}}{\alpha_2} - \frac{u_{\alpha_2}}{\alpha_1}\right) = \int_S \left(1 - \frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}\right) g u_2 \,\mathrm{d}s.$$

We estimate the second summand by

$$\begin{aligned}-b_{\alpha_1}(u_{\alpha_2} - u_{\alpha_1}, u_{\alpha_2} - u_{\alpha_1}) &= -b_1\left(\frac{u_{\alpha_2} - u_{\alpha_1}}{\sqrt{\alpha_1}}, \frac{u_{\alpha_2} - u_{\alpha_1}}{\sqrt{\alpha_1}}\right) \\ &\le \int_\Gamma \left[a_1 \frac{u_{\alpha_2} - u_{\alpha_1}}{\sqrt{\alpha_1}} \partial_\nu \left(\frac{u_{\alpha_2} - u_{\alpha_1}}{\sqrt{\alpha_1}}\right)\right]_\Gamma \mathrm{d}s.\end{aligned}$$

For the third and fourth summand we obtain

$$\begin{aligned}&b_{\alpha_1}(u_{\alpha_2}, u_{\alpha_2}) - b_{\alpha_2}(u_{\alpha_2}, u_{\alpha_2}) \\ &= \int_{B\setminus\Gamma} \left\{\left(\frac{a_1}{\alpha_1} - \frac{a_2}{\alpha_2}\right)|\nabla u_{\alpha_2}|^2 + \left(\eta_1^{(\alpha_1)} - \eta_2^{(\alpha_2)}\right)|u_{\alpha_2}|^2\right\} \mathrm{d}x \\ &\quad - \int_{\partial B} \left(\frac{a_1}{\alpha_1}\frac{\partial_\nu\alpha_1}{2\alpha_1} - \frac{a_2}{\alpha_2}\frac{\partial_\nu\alpha_2}{2\alpha_2}\right)|u_{\alpha_2}|^2 \,\mathrm{d}s \\ &\quad - \int_\Gamma \left(\left[\frac{a_1}{\alpha_1} u_{\alpha_2}\partial_\nu u_{\alpha_2}\right]_\Gamma - \left[\frac{a_2}{\alpha_2} u_{\alpha_2}\partial_\nu u_{\alpha_2}\right]_\Gamma\right) \mathrm{d}s\end{aligned}$$

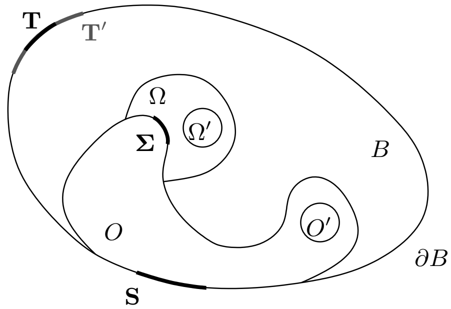

FIGURE 1. Sketch of the domains considered in Lemma 4.1.

and

$$b_{\alpha_1}(u_{\alpha_1}, u_{\alpha_2}) - b_{\alpha_1}(u_{\alpha_2}, u_{\alpha_1}) = \int_\Gamma \left(\left[\frac{a_1}{\alpha_1} u_{\alpha_1} \partial_\nu u_{\alpha_2} \right]_\Gamma - \left[\frac{a_1}{\alpha_1} u_{\alpha_2} \partial_\nu u_{\alpha_1} \right]_\Gamma \right) \mathrm{d}s$$

Combining these terms and using the interface conditions

$$[u_j]_\Gamma = 0 = [a_j \partial_\nu u_j]_\Gamma$$

yields the assertion of Lemma 3.1. □

Proof of Corollaries 1–3. The corollaries 1 and 2 immediately follow from Lemma 3.1 by setting $\alpha_1 = 1 = \alpha_2$, resp., $\alpha_1 = a_1$ and $\alpha_2 = a_2$.

To show Corollary 3 let O_ϵ be the neighborhood of $\overline{O}$ on which $\frac{\sqrt{\alpha_2}}{\sqrt{\alpha_1}}$ is continuous. Choose a $C^\infty(\overline{B})$-cutoff function $\varphi(x)$ with

$$0 < \varphi(x) < 1, \quad \text{and} \quad \varphi(x) = \begin{cases} 1 & \text{on } O, \\ 0 & \text{on } B \setminus \overline{O_\epsilon}, \end{cases}$$

and set

$$\tilde{\alpha_j} := \alpha_j^\varphi, \quad j = 1, 2.$$

Then, $\tilde{\alpha_j} = \alpha_j$ on O, $\tilde{\alpha_j} = 1$ on $B \setminus \overline{O}_\epsilon$, and, since $\frac{\alpha_1}{\alpha_2}$ has no jumps in O_ϵ, the fraction $\sqrt{\tilde{\alpha_2}}/\sqrt{\tilde{\alpha_1}}$ is continuous in B. Using Lemma 3.1 with $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ yields Corollary 3. □

4. **Localized potentials.** Now we show that we can independently control the terms in the monotony estimates. For the following lemma, see Figure 1 for a sketch of the considered domains.

4.1. Existence of localized potentials.

Lemma 4.1. *Let $a, c \in L^\infty_+(B)$ be piecewise analytic. Let $O \subseteq B$ be a subdomain with $S \subset \partial O$. In each of the following assertions, $u_k \in H^1(B)$ denotes the solution of (3) and (4), with Neumann boundary values $g = g_k$.*

(a) (i) There exists a sequence $(g_k)_{k\in\mathbb{N}} \subset L^2(S)$ such that

$$\|u_k\|^2_{H^1(O)} \to \infty, \quad \|u_k\|^2_{L^2(S)} + \|u_k\|^2_{L^2(O)} + \|u_k\|^2_{H^1(B\setminus\overline{O})} \to 0.$$

(ii) There exists a sequence $(g_k)_{k\in\mathbb{N}} \subset L^2(S)$ such that

$$\|u_k\|^2_{L^2(S)} \to \infty, \quad \|u_k\|^2_{L^2(O)} + \|u_k\|^2_{H^1(B\setminus\overline{O})} \to 0.$$

(iii) Let O' be an open subset of O. There exists $(g_k)_{k\in\mathbb{N}} \subset L^2(S)$ with

$$\|u_k\|^2_{L^2(O')} \to \infty, \quad \|u_k\|^2_{H^1(B\setminus\overline{O})} \to 0.$$

(b) Let Ω be another open subset of B, with $\Omega \cap O = \emptyset$, and let $\partial\Omega$ and ∂O contain a joint smooth piece Σ.

(i) There exists a sequence $(g_k)_{k\in\mathbb{N}} \subset L^2(S)$ such that

$$\|u_k\|^2_{H^1(\Omega)} \to \infty, \quad \|u_k\|^2_{L^2(\Sigma)} + \|u_k\|^2_{L^2(\Omega)} + \|u_k\|^2_{H^1(B\setminus\overline{O\cup\Omega})} \to 0.$$

(ii) There exists a sequence $(g_k)_{k\in\mathbb{N}} \subset L^2(S)$ such that

$$\|u_k\|^2_{L^2(\Sigma)} \to \infty, \quad \|u_k\|^2_{L^2(\Omega)} + \|u_k\|^2_{H^1(B\setminus\overline{O\cup\Omega})} \to 0.$$

(iii) Let Ω' be an open subset of Ω. There exists $(g_k)_{k\in\mathbb{N}} \subset L^2(S)$ with

$$\|u_k\|^2_{L^2(\Omega')} \to \infty, \quad \|u_k\|^2_{H^1(B\setminus\overline{O\cup\Omega})} \to 0.$$

(c) Let $T, T' \subset \partial B \setminus \overline{S}$ be open parts of ∂B with $\overline{T} \subset T'$. There exists a sequence $(g_k)_{k\in\mathbb{N}} \subset L^2(S)$ such that

$$\|u_k\|^2_{L^2(T)} \to \infty, \quad \|u_k\|^2_{L^2(\partial B\setminus\overline{S\cup T'})} \to 0.$$

4.2. **Proof of Lemma 4.1.** The main idea of localized potentials is to reformulate the desired growth properties as range inclusions. For the assertion (a), this is done in subsection 4.2.1. The range inclusions are then proved in subsection 4.2.2 by compactness and unique continuation arguments. We then derive assertion (b) from (a) and prove (c) again relying on unique continuation.

Before we start the reformulation, let us give some simplifying remarks. It suffices to show the assertions (a)(i) and (ii) for shrinked subsets S and O. For these two parts, we can therefore assume w.l.o.g. that O is also smoothly bounded and that a and c are analytic on a neighborhood of $\overline{O}$.

Also, instead of (a)(ii), it suffices to show

(a) (ii') there exists a sequence $(g_k)_{k\in\mathbb{N}} \subset L^2(S)$ such that

$$\|u_k\|_{L^2(\partial B)} \to \infty, \quad \|u_k\|^2_{L^2(O)} + \|u_k\|^2_{H^1(B\setminus\overline{O})} \to 0.$$

Then $\|u_k\|^2_{H^1(B\setminus\overline{O})} \to 0$ yields that $\|u_k\|_{L^2(S')} \to \infty$ for every neighborhood S' of $\overline{\partial B \cap \partial O}$, so that we obtain the original assertion (a)(ii) from choosing S and O small enough.

Furthermore, let us note the following unique continuation property. For every open connected subset $U \subseteq B$ only the trivial solution of

$$-\nabla\cdot(a\nabla u) + cu = 0 \quad \text{in } U,$$

vanishes on an open subset of U or possesses zero Cauchy data on a smooth, open part of ∂U. For Lipschitz continuous a and bounded c, this property is proven in Miranda [34, Thm. 19, II]. It can be extended to our case of piecewise analytic a and c by sequentially solving Cauchy problems (see also Druskin [13] for this argument).

4.2.1. *Reformulation of Lemma 4.1(a) as range inclusions.* In [21] the author has proven a similar assertion involving only the L^2- and H^1-terms on subdomains, and the reformulation arguments in [21] are easily extended to also include boundary terms. For the convenience of the reader, we summarize the main steps for the assertion (a)(i) here, the reformulations of (a)(ii') and (iii) follow analogously.

We first introduce the solution operator

$$G:\ H^1(B)' \to L^2(S), \quad f \mapsto u|_S,$$

where $u \in H^1(B)$ solves

$$b(u,v) := \int_B (a\nabla u \cdot \nabla v + cuv)\,\mathrm{d}x = \langle f, v\rangle \quad \text{for all } v \in H^1(B). \tag{7}$$

Here and in the following $\langle \cdot, \cdot \rangle$ denotes the dual pairing on $H^1(B)' \times H^1(B)$. For dual pairings on boundary pieces we continue our somewhat sloppy notation from the last section and write them as integrals.

The dual operator of G is given by

$$G':\ L^2(S) \to H^1(B), \quad g \mapsto u,$$

where $u \in H^1(B)$ solves

$$\int_B (a\nabla u \cdot \nabla v + cuv)\,\mathrm{d}x = \int_S g v|_S \,\mathrm{d}s \quad \text{for all } v \in H^1(B),$$

i.e., G' maps a given Neumann datum g to the solution $u \in H^1(B)$ of equations (3), (4).

We can interprete the assertion in terms of bounds on the solution operator. The assertion in (a)(i) is equivalent to the statement that there exists no $C > 0$ such that

$$\|G'g\|^2_{H^1(O)} \le C \left(\|G'g\|^2_{L^2(S)} + \|G'g\|^2_{L^2(O)} + \|G'g\|^2_{H^1(B\setminus\overline{O})} \right).$$

holds for all $g \in L^2(S)$.

To reformulate this as a range inclusion we use the following functional analytic lemma. For bounded linear operators $A_j : H_j \to H$, $j = 1, 2$, between Hilbert spaces H, H_1 and H_2 it holds that

$$\mathcal{R}(A_1) \subseteq \mathcal{R}(A_2), \quad \text{if and only if} \quad \exists C > 0 :\ \|A_1' g\| \le C\|A_2' g\| \ \forall g \in H',$$

cf, e.g., [15, Lemma 2.5].

We apply this equivalence using G' as an operator from $L^2(S)$ to $H^1(O)$, and as an operator from $L^2(S)$ to $L^2(S) \times L^2(O) \times H^1(B\setminus\overline{O})$. (For the rigorous formulation using inclusion and restriction operators we refer to [21].)

Note that the canonical restrictions from $H^1(B)$ to $L^2(S)$, $L^2(O)$, and $H^1(B\setminus\overline{O})$, yield, by duality, injections from $L^2(S)$, $L^2(O)$, and $H^1(B\setminus\overline{O})'$ to $H^1(B)'$. Hence, we obtain that the assertion (a)(i) is equivalent to the fact that

$$G\left(H^1(O)'\right) \not\subseteq G\left(L^2(S)\right) + G\left(L^2(O)\right) + G\left(H^1(B\setminus\overline{O})'\right). \tag{8}$$

Analogously, we obtain that (a)(ii') and (iii) are equivalent to

$$G\left(L^2(\partial B)\right) \not\subseteq G\left(L^2(O)\right) + G\left(H^1(B\setminus\overline{O})'\right), \tag{9}$$

$$G\left(L^2(O')\right) \not\subseteq G\left(H^1(B\setminus\overline{O})'\right). \tag{10}$$

4.2.2. *Proof of the range inclusions.* To show the assertion of Lemma 4.1, we now prove the equivalent range inclusions (or, rather, non-inclusions) (8)–(10).

Proof of (8). We first show that $G(H^1(B \setminus \overline{O})') \subseteq G(L^2(O))$. To that end let $u|_S = Gh$ with $h \in H^1(B \setminus \overline{O})'$. With a cutoff function $\chi \in C^\infty(\overline{B})$ which is one in a neighborhood of $\overline{S}$, vanishes outside O and fulfills $\partial_\nu \chi|_{\partial O} = 0$, we obtain

$$\int_B (a\nabla(\chi u) \cdot \nabla v + c(\chi u)v) \, \mathrm{d}x = -\int_O (\nabla \cdot (au\nabla\chi) + a\nabla u \cdot \nabla\chi) \, v \, \mathrm{d}x.$$

and thus $u|_S = (\chi u)|_S \in G(L^2(O))$.

In effect,

$$G\left(L^2(S) + L^2(O) + H^1(B \setminus \overline{O})'\right) \subseteq G(K),$$

where $K := L^2(S) + L^2(O)$ is a compact subset of $H^1(O)'$. Hence, $G(K)$ is a compact subset of $G\left(H^1(O)'\right)$ (equipped with the graph norm). Since the latter space is infinite dimensional, we deduce $G(K) \subsetneq G\left(H^1(O)'\right)$ and thus (8). ☐

Proof of (9). First note that replacing the piecewise-analytic a and c by (everywhere) C^∞ extensions of $a|_O$ and $c|_O$ (still satisfying $a, c \in L^\infty_+(B)$) changes the ranges in (9) only by a term in $G(H^1(B \setminus \overline{O})')$. For the proof of (9), we can therefore also assume w.l.o.g. that $a, c \in C^\infty(\overline{B})$.

From the proof of (8) we already know that $G(H^1(B \setminus \overline{O})') \subseteq G(L^2(O))$, so that the right hand side of (9) is simply $G(L^2(O))$.

We will show (9) by proving that $G(L^2(O)) \subsetneq G(L^2(\partial B))$. We apply the same compactness argument as in the proof of (8) and note that the first is a compact, and thus proper, subset of the infinite dimensional space $G(H^{1/2}(O)')$. It therefore suffices to show that $G(H^{1/2}(O)') \subseteq G\left(L^2(\partial B)\right)$.

To that end we introduce the solution operator of the Dirichlet problem,

$$\gamma^- : \; H^{1/2}(\partial B) \to H^1(B), \quad \gamma^- f = u,$$

where u solves (3) with Dirichlet boundary values $u|_{\partial B} = f$ on ∂B.

Due to our smoothness assumptions, γ^- can be extended to scales of Sobolev-spaces and their duals; cf. Lions and Magenes [33, Chp. 2, Theorem 7.4]. In particular, it extends by continuity to an operator from $L^2(\partial B)$ to $H^{1/2}(B)$, and so its dual $(\gamma^-)' : \; H^1(B)' \to H^{1/2}(\partial B)'$ fulfills

$$(\gamma^-)'(H^{1/2}(O)') \subseteq (\gamma^-)'(H^{1/2}(B)') \subseteq L^2(\partial B).$$

Now let $u|_S = Gf$, with $f \in H^{1/2}(O)'$, and let $\tilde{u} \in H^1(B)$ be the solution from the definition of $G((\gamma^-)'f)$. By construction,

$$b(u - \tilde{u}, v) = 0 \quad \text{ for all } v \in H^1(B) \text{ with } v = \gamma^- v|_{\partial B}.$$

Furthermore let $\tilde{u}_0 \in H^1_0(B)$ solve

$$b(\tilde{u}_0, v) = b(u - \tilde{u}, v) \quad \text{ for all } v \in H^1_0(B).$$

From the definition of γ^- it follows that $b(\tilde{u}_0, \gamma^- v|_{\partial B}) = 0$. Since

$$b(u - \tilde{u} - \tilde{u}_0, v) = b(u - \tilde{u} - \tilde{u}_0, v - \gamma^- v|_{\partial B}) + b(u - \tilde{u} - \tilde{u}_0, \gamma^- v|_{\partial B}) = 0,$$

holds for all $v \in H^1(B)$, we obtain $u = \tilde{u} + \tilde{u}_0$, so that $u|_S = \tilde{u}|_S \in G(L^2(\partial B))$. This shows that $G(H^{1/2}(O)') \subseteq G\left(L^2(\partial B)\right)$, and hence the assertion (9). ☐

Proof of (10). This follows from the unique continuation property as in [21, Thm. 3.1]; cf. also [15, Lemma 2.3]. ☐

4.2.3. *Proof of Lemma 4.1(b).* The assertions in (b) can be deduced from (a) in the following way. First, we can shrink O to assume w.l.o.g. that $B \setminus \overline{O}$ is smoothly bounded.

Then, analogously to subsection 4.2.1, we obtain that assertion (b)(i) is equivalent to the range inclusion

$$G\left(H^1(\Omega)'\right) \not\subseteq G\left(L^2(\Sigma)\right) + G\left(L^2(\Omega)\right) + G\left(H^1(B \setminus \overline{O \cup \Omega})'\right). \tag{11}$$

Now, we introduce the solution operator

$$\Gamma: \ H^1(B)' \to L^2(\Sigma), \quad f \mapsto u|_\Sigma,$$

which is defined in the same way as G but takes the trace on Σ instead of S.

By unique continuation on O, (11) is equivalent to

$$\Gamma\left(H^1(\Omega)'\right) \not\subseteq \Gamma\left(L^2(\Sigma)\right) + \Gamma\left(L^2(\Omega)\right) + \Gamma\left(H^1(B \setminus \overline{O \cup \Omega})'\right). \tag{12}$$

This is precisely what we showed for (a)(i) with Ω and Σ in place of O and S. Analogously, (b)(ii) and (iii) follow from (a)(ii) and (iii). □

4.2.4. *Proof of Lemma 4.1(c).* Analogously to subsection 4.2.1, we obtain that assertion (c) is equivalent to the range inclusion

$$G\left(L^2(T)\right) \not\subseteq G\left(L^2(\partial B \setminus \overline{S \cup T'})\right).$$

This immediately follows from unique continuation. □

5. **Proof of Theorem 2.2.** Now we can prove our main result. Let $a_1, a_2, c_1, c_2 \in L^\infty_+(B)$ be piecewise analytic coefficients on a joint partition $(O_j, \Gamma)_{j=1}^J$, and let Λ_{a_1,c_1}, Λ_{a_2,c_2} be the corresponding local Neumann-to-Dirichlet operators.

As we already remarked in section 3, the if-part of Theorem 2.2 follows from Corollary 2.

To show the only-if-part, we will proceed along the following steps. In subsections 5.1.1 and 5.1.2, we show that

$$a_1|_S = a_2|_S \quad \text{on } S, \tag{13}$$

$$\partial_\nu a_1|_S = \partial_\nu a_2|_S \quad \text{on } S. \tag{14}$$

We then prove in subsection 5.2 that

$$\eta_1 = \eta_2 \quad \text{on } B \setminus \Gamma, \tag{15}$$

$$\frac{a_1^+|_\Gamma}{a_1^-|_\Gamma} = \frac{a_2^+|_\Gamma}{a_2^-|_\Gamma} \quad \text{on } \Gamma, \tag{16}$$

$$\frac{[\partial_\nu a_2]_\Gamma}{a_2^-|_\Gamma} = \frac{[\partial_\nu a_1^-]_\Gamma}{a_1^-|_\Gamma} \quad \text{on } \Gamma. \tag{17}$$

Finally we conclude in subsection 5.3 that

$$\frac{\partial_\nu a_1}{a_1}|_{\partial B} = \frac{\partial_\nu a_2}{a_2}|_{\partial B} \quad \text{on } \partial B. \tag{18}$$

All of these steps will be proven by applying the localized potentials from section 4 to control the individual terms in our monotony estimates from section 3. More precisely, in each step, we will argue by contradiction and show that, if the respective assertion was not true, there would exist a sequence $(g^{(k)})_{k\in\mathbb{N}} \in L^2(S)$ with

$$\left| \int_S g^{(k)} \left(\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2}\right) g^{(k)} \, \mathrm{d}s \right| \to \infty, \tag{19}$$

thus contradicting $\Lambda_{a_1,c_1} = \Lambda_{a_2,c_2}$.

5.1. The boundary part S: Proof of (13), (14).

5.1.1. *Proof of (13).* To show (13), we assume the converse. Then, by continuity of a_1 and a_2, there exists a connected neighborhood O of a smooth piece S' of S on which all coefficients are analytic, and either $a_1 - a_2 \in L^\infty_+(O)$ or $a_2 - a_1 \in L^\infty_+(O)$. By possibly interchanging the roles of Λ_{a_1,c_1} and Λ_{a_2,c_2}, we can, w.l.o.g., assume the latter. We then apply Lemma 4.1(a)(i) to obtain a sequence $g^{(k)}$ such that the corresponding solutions $u_2^{(k)}$ of (3), (4) for $(a,c) = (a_2, c_2)$ and $g = g^{(k)}$ fulfill

$$\|u_2^{(k)}\|_{H^1(O)} \to \infty, \quad \|u_2^{(k)}\|_{L^2(O)} \to 0, \quad \text{and} \quad \|u_2^{(k)}\|_{H^1(B\setminus\overline{O})} \to 0.$$

Hence, we obtain from Corollary 1 that

$$\begin{aligned}
&\int_S g^{(k)}\left(\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2}\right) g^{(k)}\,\mathrm{d}s\\
&\geq \int_O (a_2 - a_1)\,|\nabla u_2^{(k)}|^2\,\mathrm{d}x - \|a_2 - a_1\|_{L^\infty(B\setminus\overline{O})}\|u_2^{(k)}\|^2_{H^1(B\setminus\overline{O})}\\
&\qquad - \|c_2 - c_1\|_{L^\infty(B)}\|u_2^{(k)}\|^2_{L^2(B)}\\
&\to \infty,
\end{aligned}$$

so that (19) holds, which contradicts $\Lambda_{a_1,c_1} = \Lambda_{a_2,c_2}$. □

5.1.2. *Proof of (14).* Similarly, we prove (14) by assuming the converse, which gives us a connected neighborhood O, such that $S' := \partial O \cap \partial B$ is a smooth piece of S, $B \setminus \overline{O}$ has Lipschitz boundary, all coefficients are analytic on O, and, w.l.o.g., $\partial_\nu a_2 - \partial_\nu a_1 \in L^\infty_+(S')$.

Now we set $\alpha_1 := a_1$ and $\alpha_2 = a_2$ on O and extend them smoothly to positive C^∞-functions on a neighborhood of $\overline{B}$. Using (13), we then obtain from Lemma 3.1 that there exists a constant $C > 0$ such that

$$\begin{aligned}
&\int_S g\left(\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2}\right) g\,\mathrm{d}s\\
&\geq \int_{S'} \left(\frac{\partial_\nu a_2}{2} - \frac{\partial_\nu a_1}{2}\right)|u_2|^2\,\mathrm{d}s - C\left(\|u_2\|^2_{L^2(B)} + \|u_2\|^2_{H^1(B\setminus\overline{O})}\right)
\end{aligned}$$

for all $g \in L^2(S)$ with support in S' and corresponding solutions u_2 of (3), (4) with $(a,c) = (a_2, c_2)$.

Hence, we can now apply Lemma 4.1(a)(ii) to obtain a sequence $g^{(k)} \subseteq L^2(S')$ such that the corresponding solutions $u_2^{(k)}$ fulfill

$$\|u_2^{(k)}\|_{L^2(S')} \to \infty, \quad \|u_2^{(k)}\|_{L^2(O)} \to 0, \quad \text{and} \quad \|u_2^{(k)}\|_{H^1(B\setminus\overline{O})} \to 0,$$

which again gives us (19) and thus the desired contradiction. □

5.2. **The interior B: Proof of (15)–(17).** Now we prove (15)–(17) by induction over the number of sets O_j. At least one of the boundaries ∂O_j must contain a smooth piece of S, w.l.o.g., let this be ∂O_1.

5.2.1. *Induction basis: Proof of (15) in O_1.* We first show that (15) holds in O_1. Assume that this is not the case. Since all coefficients have analytic extensions to a neighborhood of $\overline{O}_1$, there must be a smallest number $l \in \mathbb{N}_0$ such that the l-th normal derivative $\partial_\nu^l(\eta_1 - \eta_2)$ does not vanish everywhere on $\partial O_1 \cap \partial B$, cf. Kohn and Vogelius [31] for the origin of this argument.

By possibly shrinking S and O_1 we can therefore assume that, w.l.o.g.,

$$\eta_2 \geq \eta_1 \quad \text{in } O_1,$$

and that there exists an open subdomain $O_1' \subseteq O_1$ on which $\eta_2 - \eta_1 \in L_+^\infty(O_1')$.

Furthermore we can assume that $S' := \partial O_1 \cap \partial B \subset S$ is a smooth part of S and that $B \setminus \overline{O}$ has Lipschitz boundary.

As in the proof of (14), we set $\alpha_1 := a_1$ and $\alpha_2 = a_2$ on O_1 and extend them smoothly to positive functions in $C^\infty(\overline{B})$. Using (13) and (14), we now obtain from Lemma 3.1 that there exists a constant $C > 0$ with

$$\int_S g\,(\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2})\,g\,\mathrm{d}s \geq \int_{O_1} (\eta_2 - \eta_1) a_2 |u_2|^2 \,\mathrm{d}x - C \|u_2\|^2_{H^1(B\setminus\overline{O_1})}$$

for all $g \in L^2(S')$ and corresponding solutions u_2 of (3), (4) with $(a, c) = (a_2, c_2)$.

Hence, by applying the potentials from Lemma 4.1(a)(iii) we obtain (19) and thus the contradiction. □

5.2.2. *Induction step for (15)–(17).* Now assume that (15)–(17) holds on some number $k < J$ of the sets O_j, w.l.o.g. let these sets be $O_1, \ldots, O_k$. More precisely, we assume that (15) holds on $O_1 \cup \ldots \cup O_k$ and that (16) and (17) hold on the intersection of Γ with $O := \mathrm{int}(\overline{O}_1 \cup \ldots \cup \overline{O}_k)$.

At least one of the boundaries of the O_j, $j > k$, must contain a joint smooth piece with ∂O, w.l.o.g., let this be O_{k+1}.

Proof of (16). We first show that (16) holds on $\partial O \cap \partial O_{k+1}$. Assume that this is not the case. Then there exists a smooth piece $\Sigma \subset \partial O \cap \partial O_{k+1}$ on which, w.l.o.g.,

$$\frac{a_2^+|_\Sigma}{a_2^-|_\Sigma} - \frac{a_1^+|_\Sigma}{a_1^-|_\Sigma} \in L_+^\infty(\Sigma).$$

Now we set $\alpha_1 := a_1$ and $\alpha_2 := a_2$ on O and extend them to functions in B, so that they are C^∞ in a neighborhood of Σ. Again by continuity, there must exist a subdomain $\Omega \subseteq O_{k+1}$ with $\Sigma \subset \partial\Omega$ and

$$\frac{a_2}{\alpha_2} - \frac{a_1}{\alpha_1} \in L_+^\infty(\Omega),$$

where we assumed, w.l.o.g., that the "+"-direction points into Ω.

Note that by the induction assumption a_1/a_2 is continuous on O and thus α_1/α_2 is continuous on the interior of $O \cup \Omega \cup \Sigma$. Now, by shrinking O, Ω and Σ, we can assume that α_1/α_2 is continuous on a neighborhood of $\overline{O \cup \Omega}$, and, also by shrinking, assume that $S' := \partial O \cap \partial B$ is a smooth part of S, and that $B \setminus \overline{O \cup \Omega}$ has Lipschitz boundary.

Corollary 3 yields a $C > 0$ such that for all $g \in L^2(S)$ and corresponding solutions u_2 of (3), (4) with $(a, c) = (a_2, c_2)$,

$$\begin{aligned}\int_S g\,(\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2})\,g\,\mathrm{d}s \geq{}& \int_\Omega \left(\frac{a_2}{\alpha_2} - \frac{a_1}{\alpha_1}\right) \left|\nabla\left(\sqrt{\alpha_2}u_2\right)\right|^2 \mathrm{d}x \\ &- C\left(\|u_2\|^2_{H^1(B\setminus\overline{O\cup\Omega})} + \|u_2\|^2_{L^2(\Omega)} + \|u_2\|^2_{L^2(\Sigma)}\right).\end{aligned}$$

Applying the localized potentials from Lemma 4.1(b)(i) yields (19) and thus the desired contradiction. Hence, we have shown that (16) holds on $\partial O \cap \partial O_{k+1}$. □

Proof of (17). To show that (17) holds on $\partial O \cap \partial O_{k+1}$, we use our usual assumption of the contrary to obtain a smooth piece $\Sigma \subset \partial O \cap \partial O_{k+1}$ on which, w.l.o.g.,

$$\left[\frac{a_2}{a_1}\partial_\nu \alpha_1\right]_\Gamma - [\partial_\nu \alpha_2]_\Gamma \in L^\infty_+(\Sigma)$$

and a subdomain $\Omega \subset O_{k+1}$ with $\Sigma = \partial\Omega \cap \partial O$.

Then we shrink O, Ω and Σ so that $\partial O \cap \partial B$ is a smooth piece of S, a_1/a_2 is continuous on a neighborhood of $\overline{O \cup \Omega}$, and $B \setminus \overline{O \cup \Omega}$ has Lipschitz boundary. (Note that we already know that a_1/a_2 is continuous on O and across $\partial O \cap \partial O_{k+1}$.) After that we apply Corollary 3 with $\alpha_1 = a_1$, $\alpha_2 = a_2$ which gives us a $C > 0$ such that for all $g \in L^2(S)$ and corresponding solutions u_2 of (3), (4) with $(a, c) = (a_2, c_2)$,

$$\int_S g\,(\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2})\,g\,\mathrm{d}s \geq \int_\Sigma \left\{\frac{1}{2}\left(\left[\frac{a_2}{a_1}\partial_\nu a_1\right]_\Gamma - [\partial_\nu a_2]_\Gamma\right)|u_2|^2\right\} - C\left(\|u_2\|^2_{H^1(B\setminus\overline{O\cup\Omega})} + \|u_2\|^2_{L^2(\Omega)}\right).$$

Hence, using the localized potentials from Lemma Lemma 4.1(b)(ii) we obtain (19). Thus we have shown that (17) holds on $\partial O \cap \partial O_{k+1}$. □

Proof of (15). To finish the induction step it only remains to show (15) on O_{k+1}. We assume the contrary. By the same analyticity arguments as in the induction start in subsection 5.2.1, we can assume that (after shrinking O_{k+1})

$$\eta_2 \geq \eta_1 \quad \text{in } O_{k+1},$$

and that there exists an open subdomain $O'_{k+1} \subseteq O_{k+1}$ on which $\eta_2 - \eta_1 \in L^\infty_+(O'_{k+1})$.

Again, we also shrink O to obtain that $\partial O \cap \partial B$ is a smooth piece of S and, again by shrinking, we can assume that a_1/a_2 has no jumps on a neighborhood of $\overline{O} \cup \overline{O}_{k+1}$, and that $B \setminus \overline{O \cup O_{k+1}}$ has Lipschitz boundary. We then obtain from Corollary 3 that there exists $C > 0$ such that for all $g \in L^2(S)$ and corresponding solutions u_2 of (3), (4) with $(a, c) = (a_2, c_2)$,

$$\int_S g\,(\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2})\,g\,\mathrm{d}s \geq \int_{O_{k+1}} (\eta_2 - \eta_1) a_2 |u_2|^2\,\mathrm{d}x - C\|u_2\|^2_{H^1(B\setminus\overline{O\cup O_{k+1}})}.$$

Using the localized potentials with the third property in Lemma 4.1(b) we obtain (19) and thus the desired contradiction. Hence, (15) holds on O_{k+1}. This finishes our induction step and finally yields that (15)–(17) holds in B. □

5.3. **The insulated part $\partial B \setminus S$: Proof of (18).** With all that we have shown so far, it follows from Corollary 2 that for all $g \in L^2(S)$ and corresponding solutions u_2 of (3), (4) with $(a, c) = (a_2, c_2)$,

$$\int_S g\,(\Lambda_{a_1,c_1} - \Lambda_{a_2,c_2})\,g\,\mathrm{d}s \geq \int_{\partial B}\left(\frac{\partial_\nu a_1}{2a_1} - \frac{\partial_\nu a_2}{2a_2}\right) a_2 |u_2|^2\,\mathrm{d}s.$$

So, if

$$\frac{\partial_\nu a_1}{2a_1} - \frac{\partial_\nu a_2}{2a_2} = 0.$$

was not true on some part of ∂B, then we would obtain (19) using appropriate localized potentials from Lemma 4.1(c).

Hence, (18) must hold true, and this finishes the proof of Theorem 2.2. □

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