SIAM J. MATH. ANAL.
Vol. 45, No. 6, pp. 3382–3403

MONOTONICITY-BASED SHAPE RECONSTRUCTION IN ELECTRICAL IMPEDANCE TOMOGRAPHY*

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Abstract. Current-voltage measurements in electrical impedance tomography (EIT) can be partially ordered with respect to definiteness of the associated self-adjoint Neumann-to-Dirichlet operators. With this ordering, a pointwise larger conductivity leads to smaller current-voltage measurements, and smaller conductivities lead to larger measurements. We present a converse of this simple monotonicity relation and use it to solve the shape reconstruction (a.k.a. inclusion detection) problem in EIT. The outer shape of a region where the conductivity differs from a known background conductivity can be found by simply comparing the measurements to that of smaller or larger test regions.

Key words. inverse problems, electrical impedance tomography, shape reconstruction, monotonicity

AMS subject classifications. 35R30, 35Q60, 35J25, 35R05

DOI. 10.1137/120886984

1. Introduction.

We consider the shape reconstruction (a.k.a. inclusion detection) problem in electrical impedance tomography (EIT). Let Ω describe an electrically conducting object which contains inclusions in which the conductivity $\sigma(x)$ differs from an otherwise known background conductivity. Our aim is to detect these inclusions from current/voltage measurements on the boundary $\partial\Omega$.

We assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a domain with smooth boundary $\partial\Omega$ and outer normal vector ν. For ease of presentation we also assume that Ω is bounded, the background conductivity is equal to 1, and we are given measurements on the complete boundary $\partial\Omega$. Our results easily extend to inhomogeneous (but known) backgrounds and partial boundary measurements; cf. section 4.3.

With these assumptions, our goal is to determine the inclusions shape, i.e., the set $\operatorname{supp}(\sigma - 1)$, from knowledge of the Neumann-to-Dirichlet (NtD) operator

$$\Lambda(\sigma) : \; g \mapsto u_\sigma^g|_{\partial\Omega},$$

where u_σ^g is the solution of

$$\nabla \cdot \sigma \nabla u_\sigma^g = 0 \text{ in } \Omega, \quad \sigma \partial_\nu u_\sigma^g|_{\partial\Omega} = g \text{ on } \partial\Omega;$$

cf. section 2.1 for the precise mathematical setting.

In this work, we show that $\operatorname{supp}(\sigma - 1)$ can be reconstructed by so-called *monotonicity tests*, which simply compare $\Lambda(\sigma)$ (in the sense of quadratic forms) to NtD operators $\Lambda(\tau)$ of test conductivities τ. To be more precise, the support of $\sigma - 1$ can be reconstructed under the assumption that $\operatorname{supp}(\sigma - 1) \subset \Omega$ has a connected complement. Otherwise, what we can reconstruct is essentially the support together with all holes that have no connection to the boundary $\partial\Omega$.

*Received by the editors August 3, 2012; accepted for publication (in revised form) August 16, 2013; published electronically November 12, 2013.
http://www.siam.org/journals/sima/45-6/88698.html
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Moreover, we show that the test NtDs $\Lambda(\tau)$ can be replaced (without losing any information) by their linear approximations using the Fréchet derivative $\Lambda'(1)$ of $\Lambda(\sigma)$ around the background conductivity. Let us stress that the linearized tests still exactly recover the inclusion, which is in accordance with the general principle that the linearized EIT problem still contains the exact shape information; cf. [16].

The term *monotonicity tests* is used because our test criteria are motivated and partly follow from the simple and well-known monotonicity relation

$$\sigma \leq \tau \quad \text{implies} \quad \Lambda(\sigma) \geq \Lambda(\tau). \tag{1.1}$$

It seems quite natural and intuitive to probe the domain with test inclusions using the implication (1.1), and this idea has been worked out and numerically tested in the works of Tamburrino and Rubinacci [47, 46]. The main new part of this work is to rigorously justify this natural idea by proving a nontrivial converse of the implication (1.1). Our proofs are based on the theory of localized potentials [6].

For a quick impression of our result let us state it for two frequently considered special cases (see Examples 4.2, 4.4, 4.8, and 4.10). (Note that throughout the paper we use the relation symbol "$\subset$" instead of "$\subseteq$" if nonequality of the two related sets is obvious.)

(a) Let $\sigma = 1 + \chi_D$, where D is open, and $\overline{D} \subset \Omega$ has a connected complement. Then for every open ball $B \subseteq \Omega$

$$\begin{aligned} B \subseteq D \quad & \text{if and only if} \quad \Lambda(1 + \chi_B) \geq \Lambda(\sigma) \\ & \text{if and only if} \quad \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma). \end{aligned}$$

(b) Let $\sigma = 1 + \chi_{D^+} - \frac{1}{2}\chi_{D^-}$, where $D^+, D^- \subseteq \Omega$ are open, $\overline{D^+} \cap \overline{D^-} = \emptyset$, and $\overline{D^+} \cup \overline{D^-} \subset \Omega$ has a connected complement. Then for every closed $C \subset \Omega$ with a connected complement

$$\begin{aligned} D^+ \cup D^- \subseteq C \quad & \text{if and only if} \quad \Lambda(1 + \chi_C) \leq \Lambda(\sigma) \leq \Lambda\left(1 - \frac{1}{2}\chi_C\right) \\ & \text{if and only if} \quad \Lambda(1) + \Lambda'(1)\chi_C \leq \Lambda(\sigma) \leq \Lambda(1) - \Lambda'(1)\chi_C. \end{aligned}$$

(a) is a special case of the *definite case* in which all inclusions have either a higher conductivity or a lower conductivity than the background. (a) shows how to test whether a small ball B lies inside the inclusion or not. The inclusion can thus be obtained as the union of all balls that fulfill the test.

(b) is a special case of the more general *indefinite case* in which the conductivity may differ in both directions from the background. Using the result in (b) we can test whether a large set C contains the inclusions or not. The inclusion can thus be obtained as the intersection of all of these large sets.

Our results show that (under quite general assumptions) monotonicity tests determine $\operatorname{supp}(\sigma - 1)$ up to holes that have no connection to the boundary $\partial\Omega$.

Noniterative methods for shape reconstruction problems have been studied intensively in the last 25 years; cf., e.g., the overview of Potthast [43]. In the context of EIT, the inclusion detection problem was first considered by Friedman and Isakov [3, 4]. For the following brief overview, we restrict ourselves to the two most prominent and elaborated methods for detecting inclusions of unknown conductivity from the full Neumann-to-Dirichlet (or Dirichlet-to-Neumann) operator on all or part of the boundary: the factorization method and the enclosure method.

The factorization method was introduced by Kirsch [33, 34] for inverse scattering problems and extended to impedance tomography by Brühl and Hanke [2, 1]. For its further developments in the context of EIT see [35, 10, 5, 11, 20, 40, 42, 7, 36, 9, 15, 44, 17, 45] and the recent review [14]. The factorization method reconstructs the shape of inclusions (up to holes that have no connection to the boundary), but two major problems have not been solved so far. First of all, the method relies on a range test (or infinity test) for which there is no known convergent implementation (see, however, Lechleiter [39] for a first step in this direction). Second, the method has been justified only for the *definite case* (or cases in which the domain can be split into two a priori known regions with the definiteness property; cf. Schmitt [44] and the review [14]).

The enclosure method was introduced by Ikehata [25, 26]. Further extensions including the use of the Sylvester–Uhlmann complex geometrical optics solutions have been worked out in [2, 30, 27, 31, 22, 48, 21]. The method yields a stable testing criterion, and it does not require the definiteness assumption (see [22]). However, it does require the construction of special, strongly oscillating probe functions and only reconstructs the convex hull of the inclusions (plus some nonconvex features depending on the probe functions).

The monotonicity tests presented herein seem to be a particularly simple and intuitively appealing solution to the long-studied inclusion detection problem. They characterize the outer shape of the inclusions and not just the convex hull. They work for the general indefinite case (though the implementation is simpler in the definite case). Also, they allow a stable implementation (see Remark 3.5), and their linearized versions do not require solving inhomogeneous forward problems.

The paper is organized as follows. Section 2 introduces the mathematical setting and the concept of inner and outer support. In section 3 we derive the main theoretical tools for our proofs: monotonicity estimates and localized potentials. Section 4 then contains our main results: the characterization of inclusion by simple and stable monotonicity tests. The paper also contains an appendix on local definiteness properties of piecewise analytic functions.

2. Basic notation and support definitions.

2.1. Basic notation and the mathematical setting. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$ and outer normal vector ν. $L^\infty_+(\Omega)$ denotes the subspace of $L^\infty(\Omega)$-functions with positive essential infima. $H^1_\diamond(\Omega)$ and $L^2_\diamond(\partial\Omega)$ denote the spaces of H^1- and L^2-functions with vanishing integral mean on $\partial\Omega$.

The $L^2_\diamond(\partial\Omega)$-inner product is denoted by $\langle \cdot, \cdot \rangle$. For two bounded self-adjoint operators $A, B:\ L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega)$ we write

$$A \geq B$$

if it holds in the sense of quadratic forms, i.e.,

$$\langle g, (A - B)g \rangle \geq 0 \quad \text{ for all } g \in L^2_\diamond(\partial\Omega).$$

For $\sigma_1, \sigma_2 \in L^\infty(\Omega)$ we write $\sigma_1 \geq \sigma_2$ if it holds pointwise (a.e.) on Ω.

For $\sigma \in L^\infty_+(\Omega)$, the NtD operator $\Lambda(\sigma)$ is defined by

$$\Lambda(\sigma):\ L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u^g_\sigma|_{\partial\Omega},$$

where $u_\sigma^g \in H_\diamond^1(\Omega)$ is the unique solution of

$$\nabla \cdot \sigma \nabla u_\sigma^g = 0 \text{ in } \Omega, \quad \sigma \partial_\nu u_\sigma^g|_{\partial\Omega} = g \text{ on } \partial\Omega, \tag{2.1}$$

which is equivalent to

$$\int_\Omega \sigma \nabla u_\sigma^g \cdot \nabla v \,\mathrm{d}x = \int_{\partial\Omega} g v|_{\partial\Omega} \,\mathrm{d}s \quad \text{ for all } v \in H_\diamond^1(\Omega). \tag{2.2}$$

It is well known $\Lambda(\sigma)$ is a self-adjoint compact linear operator, and that the associated bilinear form is given by

$$\langle g, \Lambda(\sigma) h\rangle = \int_\Omega \sigma \nabla u_\sigma^g \cdot \nabla u_\sigma^h \,\mathrm{d}x.$$

Λ is Fréchet-differentiable; cf., e.g., Lechleiter and Rieder [41] for a recent proof that uses only the abstract variational formulation (see also [23] for similar results). Given some direction $\kappa \in L^\infty(\Omega)$, the derivative

$$\Lambda'(\sigma)\kappa : \; L_\diamond^2(\partial\Omega) \to L_\diamond^2(\partial\Omega)$$

is the self-adjoint compact linear operator associated to the bilinear form

$$\langle (\Lambda'(\sigma)\kappa)\, g, h\rangle = -\int_\Omega \kappa \nabla u_\sigma^g \cdot \nabla u_\sigma^h \,\mathrm{d}x.$$

Note that for $\kappa_1, \kappa_2 \in L^\infty(\Omega)$ we obviously have that

$$\kappa_1 \leq \kappa_2 \quad \text{ implies } \quad \Lambda'(\sigma)\kappa_1 \geq \Lambda'(\sigma)\kappa_2. \tag{2.3}$$

The terms *piecewise continuous* and *piecewise analytic* are understood in the following sense.

Definition 2.1.

(a) *A subset $\Gamma \subseteq \partial O$ of the boundary of an open set $O \subseteq \mathbb{R}^n$ is called a* smooth boundary piece *if it is a C^∞-surface and O lies on one side of it, i.e., if for each $z \in \Gamma$ there exists a ball $B_\epsilon(z)$ and a function $\gamma \in C^\infty(\mathbb{R}^{n-1}, \mathbb{R})$ such that upon relabeling and reorienting*

$$\begin{aligned} \Gamma = \partial O \cap B_\epsilon(z) &= \{x \in B_\epsilon(z) \mid x_n = \gamma(x_1, \ldots, x_{n-1})\}, \\ O \cap B_\epsilon(z) &= \{x \in B_\epsilon(z) \mid x_n > \gamma(x_1, \ldots, x_{n-1})\}. \end{aligned}$$

(b) *O is said to have* smooth boundary *if ∂O is a union of smooth boundary pieces. O is said to have* piecewise smooth boundary *if ∂O is a countable union of the closures of smooth boundary pieces.*

(c) *A function $\kappa \in L^\infty(\Omega)$ is called* piecewise analytic *if there exist finitely many pairwise disjoint subdomains $O_1, \ldots, O_M \subset \Omega$ with piecewise smooth boundaries, such that $\overline{\Omega} = \overline{O_1 \cup \cdots \cup O_M}$, and $\kappa|_{O_m}$ has an extension which is (real)analytic in a neighborhood of $\overline{O_m}$, $m = 1, \ldots, M$.*

(d) *A function $\kappa \in L^\infty(\Omega)$ is called* piecewise continuous *if κ is continuous on an open set $O \subset \Omega$ and $\Omega \setminus O$ is a set of zero measure.*

2.2. Inner and outer support. We will show that our method reconstructs $\operatorname{supp}(\sigma - 1)$ (the *inclusion*) up to holes that cannot be connected to the boundary $\partial\Omega$ without crossing the support. For the precise formulation, we will now introduce the concept of the *inner* and the *outer support* of a measurable function. For the frequently considered case that the inclusion has a connected complement and the conductivity is piecewise continuous, the inner and the outer support differ only by the boundary of the support; cf. Corollary 2.5. The following has been inspired by the use of the infinity support of Kusiak and Sylvester [38]; cf. also [8, 16].

Definition 2.2. *A relatively open set $U \subseteq \overline{\Omega}$ is called* connected to $\partial\Omega$ *if $U \cap \Omega$ is connected and $U \cap \partial\Omega \neq \emptyset$.*

Definition 2.3. *For a measurable function $\kappa : \Omega \to \mathbb{R}$ we define*

(a) *the* support $\operatorname{supp} \kappa$ *as the complement (in $\overline{\Omega}$) of the union of those relatively open $U \subseteq \overline{\Omega}$, for which $\kappa|_U \equiv 0$;*

(b) *the* inner support $\operatorname{inn\,supp} \kappa$ *as the union of those open sets $U \subseteq \Omega$, for which $\operatorname{ess\,inf}_{x \in U} |\kappa(x)| > 0$;*

(c) *the* outer support $\operatorname{out}_{\partial\Omega} \operatorname{supp} \kappa$ *as the complement (in $\overline{\Omega}$) of the union of those relatively open $U \subseteq \overline{\Omega}$ that are connected to $\partial\Omega$ and for which $\kappa|_U \equiv 0$.*

The interior of a set $M \subseteq \Omega$ is denoted by $\operatorname{int} M$ and its closure (with respect to $\mathbb{R}^n$) by $\overline{M}$. If M is measurable, we also define

(d) $\operatorname{out}_{\partial\Omega} M = \operatorname{out}_{\partial\Omega} \operatorname{supp} \chi_M$,

where χ_M is the characteristic function of M.

Lemma 2.4. *For every measurable function $\kappa : \Omega \to \mathbb{R}$ and every measurable set M the following properties hold:*

(a) $\operatorname{supp} \kappa, \operatorname{out}_{\partial\Omega} \operatorname{supp} \kappa, \operatorname{out}_{\partial\Omega} M \subseteq \overline{\Omega}$ *are closed.*

(b) $\operatorname{inn\,supp} \kappa \subseteq \Omega$ *is open.*

(c) $\operatorname{inn\,supp} \kappa \subseteq \operatorname{supp} \kappa \subseteq \operatorname{out}_{\partial\Omega} \operatorname{supp} \kappa$.

(d) $\operatorname{out}_{\partial\Omega} (\operatorname{supp} \kappa) = \operatorname{out}_{\partial\Omega} \operatorname{supp} \kappa$.

(e) *If $\operatorname{supp} \kappa \subseteq \Omega$ and $\Omega \setminus \operatorname{supp} \kappa$ is connected, then $\operatorname{supp} \kappa = \operatorname{out}_{\partial\Omega} \operatorname{supp} \kappa$.*

(f) *If κ is piecewise continuous, then $\operatorname{supp} \kappa = \overline{\operatorname{inn\,supp} \kappa}$.*

Proof.

(a) and (b) immediately follow from Definition 2.3.

(c) If $\kappa = 0$ (a.e.) on a relatively open set $U \subset \overline{\Omega}$, then $\kappa = 0$ (a.e.) on the open set $U \cap \operatorname{inn\,supp} \kappa$. From the definition of the inner support, it follows that $U \cap \operatorname{inn\,supp} \kappa = \emptyset$. This shows the first inclusion in (c). The second inclusion is obvious.

(d) follows from the fact that for every relatively open set $U \subseteq \overline{\Omega}$ we have

$$\begin{aligned} \kappa = 0 \text{ (a.e.) on } U \quad &\text{if and only if} \quad U \subseteq \overline{\Omega} \setminus \operatorname{supp} \kappa \\ &\text{if and only if} \quad \chi_{\operatorname{supp} \kappa} = 0 \text{ (a.e.) on } U. \end{aligned}$$

(e) Since $\operatorname{supp} \kappa \subseteq \Omega$ implies that $\overline{\Omega} \setminus \operatorname{supp} \kappa$ contains $\partial\Omega$, (e) immediately follows from (c) and Definition 2.3.

(f) Let κ be continuous on an open set $O \subset \Omega$ where $\Omega \setminus O$ has zero measure. The assertion follows from (a) and (c) if we can show that for every $x \in \overline{\Omega}$

$$x \notin \overline{\operatorname{inn\,supp} \kappa} \quad \text{implies} \quad x \notin \operatorname{supp} \kappa.$$

Let $x \notin \overline{\operatorname{inn\,supp} \kappa}$. Then there exists a relatively open set $B \subset \overline{\Omega}$ with $x \in B$ and $B \cap \overline{\operatorname{inn\,supp} \kappa} = \emptyset$. Obviously, $\{\xi \in O : \kappa(\xi) \neq 0\} \subseteq \operatorname{inn\,supp} \kappa$, so that $\kappa = 0$ on $O \cap B$. Since $\Omega \setminus O$ has zero measure, we have that $\kappa = 0$ (a.e.) on B and thus $B \cap \operatorname{supp} \kappa = \emptyset$, which shows the assertion. $\square$

As a consequence of Lemma 2.4(e) and (f) we have the following corollary.

COROLLARY 2.5. *If κ is piecewise continuous,* $\operatorname{supp}\kappa \subseteq \Omega$, *and* $\Omega \setminus \operatorname{supp}\kappa$ *is connected, then*

$$\overline{\operatorname{inn}\operatorname{supp}\kappa} = \operatorname{supp}\kappa = \operatorname{out}_{\partial\Omega}\operatorname{supp}\kappa.$$

3. Monotonicity and localized potentials.

3.1. A monotonicity principle.

Our main theoretical tools are a monotonicity estimate and the theory of localized potentials. The following estimate goes back to Ikehata, Kang, Seo, and Sheen [32, 24]; cf. also the similar results in Ide et al. [22], Kirsch [35], and [15, 16]. For the convenience of the reader we state the estimate together with a short proof that we copy from [16, Lemma 2.1].

LEMMA 3.1. *Let $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$ be two conductivities, let $g \in L^2_\diamond(\Omega)$ be an applied boundary current, and let $u_2 := u^g_{\sigma_2} \in H^1_\diamond(\Omega)$. Then*

$$\int_\Omega (\sigma_1 - \sigma_2)|\nabla u_2|^2 \,\mathrm{d}x \geq \langle g, (\Lambda(\sigma_2) - \Lambda(\sigma_1))\, g\rangle \geq \int_\Omega \frac{\sigma_2}{\sigma_1}(\sigma_1 - \sigma_2)|\nabla u_2|^2 \,\mathrm{d}x. \tag{3.1}$$

Proof. Let $u_1 := u^g_{\sigma_1} \in H^1_\diamond(\Omega)$. From (2.2) we deduce

$$\int_\Omega \sigma_1 \nabla u_1 \cdot \nabla u_2 \,\mathrm{d}x = \langle g, \Lambda(\sigma_2) g\rangle = \int_\Omega \sigma_2 \nabla u_2 \cdot \nabla u_2 \,\mathrm{d}x$$

and thus

$$\begin{aligned}\int_\Omega \sigma_1 |\nabla(u_1 - u_2)|^2 \,\mathrm{d}x &= \int_\Omega \sigma_1 |\nabla u_1|^2 \,\mathrm{d}x - 2\int_\Omega \sigma_2 |\nabla u_2|^2 \,\mathrm{d}x + \int_\Omega \sigma_1 |\nabla u_2|^2 \,\mathrm{d}x \\ &= \langle g, \Lambda(\sigma_1) g\rangle - \langle g, \Lambda(\sigma_2) g\rangle + \int_\Omega (\sigma_1 - \sigma_2)|\nabla u_2|^2 \,\mathrm{d}x.\end{aligned}$$

Since the left-hand side is nonnegative, the first asserted inequality follows.

Interchanging σ_1 and σ_2, we obtain

$$\begin{aligned}&\langle g, (\Lambda(\sigma_2) - \Lambda(\sigma_1))\, g\rangle \\ &= \int_\Omega (\sigma_1 - \sigma_2)|\nabla u_1|^2 \,\mathrm{d}x + \int_\Omega \sigma_2 |\nabla(u_2 - u_1)|^2 \,\mathrm{d}x \\ &= \int_\Omega \left(\sigma_1 |\nabla u_1|^2 + \sigma_2 |\nabla u_2|^2 - 2\sigma_2 \nabla u_1 \cdot \nabla u_2\right) \,\mathrm{d}x \\ &= \int_\Omega \sigma_1 \left|\nabla u_1 - \frac{\sigma_2}{\sigma_1}\nabla u_2\right|^2 \,\mathrm{d}x + \int_\Omega \left(\sigma_2 - \frac{{\sigma_2}^2}{\sigma_1}\right) |\nabla u_2|^2 \,\mathrm{d}x.\end{aligned}$$

Since the first integral on the right-hand side is nonnegative, the second asserted inequality follows. ∎

We call Lemma 3.1 a *monotonicity estimate* because of the following corollary.

COROLLARY 3.2. *For two conductivities $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$*

$$\sigma_1 \leq \sigma_2 \quad \textit{implies} \quad \Lambda(\sigma_1) \geq \Lambda(\sigma_2). \tag{3.2}$$

Remark 3.3. Corollary 3.2 already yields a simple monotonicity-based reconstruction algorithm. Assume that the conductivity in the investigated object is $\sigma = 1 + \chi_D$, where the measurable set $D \subseteq \Omega$ describes the unknown inclusion. Then for all other measurable sets $B \subseteq \Omega$

$$B \subseteq D \quad \text{implies} \quad \Lambda(1+\chi_B) \geq \Lambda(\sigma), \tag{3.3}$$

so that the set

$$R := \bigcup \{B \subseteq \Omega \,:\, B \text{ measurable, and } \Lambda(1+\chi_B) \geq \Lambda(\sigma)\}$$

is an upper bound of D.

A numerical approximation of (this upper bound of) D can be calculated by choosing a number of small balls $B = B_\epsilon(z) \subseteq \Omega$ (with center $z \in \Omega$ and radius $\epsilon > 0$) and marking all balls where the *monotonicity test* $\Lambda(1+\chi_B) \geq \Lambda(\sigma)$ holds true. Algorithms based on this idea have been worked out and numerically tested in the works of Tamburrino and Rubinacci [47, 46].

Also, Lemma 3.1 gives an estimate for the Fréchet derivative of Λ that will be the basis for linearizing our monotonicity tests without losing shape information (cf. [16] for the origin of this idea).

COROLLARY 3.4. *Let $\sigma \in L^\infty_+(\Omega)$. Let $\Lambda(1)$ be the NtD operator corresponding to the background conductivity* 1, *and let $\Lambda'(1)$ be its Fréchet derivative (see subsection* 2.1*). Then*

$$\Lambda'(1)(1-\sigma) \geq \Lambda(1) - \Lambda(\sigma) \geq \Lambda'(1)\left(\frac{1}{\sigma}(1-\sigma)\right).$$

Of course, in practical EIT applications, it is not possible to measure boundary data with infinite precision. Moreover, with a limited number of electrodes on the boundary of an imaging subject (and limited accuracy), we can only obtain a finite-dimensional approximation to the true NtD. Also, we can only calculate finite-dimensional approximations of the NtD for test conductivities (and their linearized counterparts). Hence, let us comment on the stability of monotonicity tests with respect to such errors.

Remark 3.5. Monotonicity/definiteness tests can be stably implemented in the following sense. Let $A \in \mathcal{L}(H)$ be a self-adjoint compact operator on a Hilbert space H, and let $(A^\delta)_{\delta>0} \subseteq \mathcal{L}(H)$ be a family of compact (e.g., finite-dimensional) approximations with

$$\|A^\delta - A\|_{\mathcal{L}(\mathcal{H})} < \delta.$$

Possibly replacing A^δ by its symmetric part, we can assume that A^δ is self-adjoint.

For $\alpha > 0$, we define the regularized definiteness test

$$R_\alpha(A^\delta) := \begin{cases} 1 & \text{if } \langle A^\delta g, g\rangle \geq -\alpha\|g\|^2 \text{ for all } g \in H, \\ 0 & \text{otherwise,} \end{cases}$$

which is equivalent to checking whether the smallest eigenvalue of A^δ is not below $-\alpha$.

If $A \geq 0$, then $\langle A^\delta g, g\rangle \geq -\delta\|g\|^2$ for all $g \in H$. If $A \not\geq 0$, then A has a negative eigenvalue $\lambda < 0$ so that $\langle A^\delta g, g\rangle \geq -\delta\|g\|^2$ cannot hold for all $g \in H$ for $\delta < |\lambda|/2$. Hence,

$$R_\delta(A^\delta) = \begin{cases} 1 & \text{if } A \geq 0, \\ 0 & \text{if } A \not\geq 0 \text{ and } \delta \text{ is sufficiently small.} \end{cases}$$

3.2. Localized potentials. We will show that a certain converse of the monotonicity relation (3.2), respectively, (3.3), holds true. The main theoretical tool for this result is to use the theory of localized potentials by one of the authors [6] to control the energy terms $|\nabla u|^2$ in the monotonicity estimate in Lemma 3.1.

Roughly speaking, [6] shows that there exist electric potentials which have arbitrarily large energy $|\nabla u|^2$ in some region and arbitrarily small energy in another region, as long as the high-energy region can be reached from the boundary without crossing the low-energy region.

We will make use of the following variant of the result in [6].

THEOREM 3.6. *Let $D_1, D_2 \subseteq \overline{\Omega}$ be two measurable sets with*

$$\operatorname{int} D_1 \not\subseteq \operatorname{out}_{\partial\Omega} D_2.$$

Furthermore let $\sigma \in L^\infty_+(\Omega)$ be piecewise analytic.

Then there exists $(g_m)_{m\in\mathbb{N}} \subset L^2_\diamond(\partial\Omega)$ such that the solutions $(u_m)_{m\in\mathbb{N}} \subset H^1_\diamond(\Omega)$ of

$$\nabla \cdot \sigma \nabla u_m = 0 \quad \text{in } \Omega, \qquad \sigma \partial_\nu u_m|_{\partial\Omega} = g_m,$$

fulfill

$$\lim_{m\to\infty} \int_{D_1} |\nabla u_m|^2 \,\mathrm{d}x = \infty \quad \text{and} \quad \lim_{m\to\infty} \int_{D_2} |\nabla u_m|^2 \,\mathrm{d}x = 0.$$

Proof. The proof is a slight adaptation of the one in [6, section 2.2]; see also [13] for the general approach.

(a) *Reformulation as range (non)inclusion.* We define the *virtual measurement* operators L_j $(j = 1, 2)$ by

$$L_j : \; L^2(D_j)^n \to L^2_\diamond(\partial\Omega), \qquad F \mapsto u|_{\partial\Omega},$$

where $u \in H^1_\diamond(\Omega)$ solves

$$\int_\Omega \sigma \nabla u \cdot \nabla w \,\mathrm{d}x = \int_{D_j} F \cdot \nabla w \,\mathrm{d}x \quad \text{for all } w \in H^1_\diamond(\Omega).$$

Note that this implies $\nabla \cdot \sigma \nabla u = 0$ in $\Omega \setminus \overline{D}_j$, and if $\overline{D}_j \subseteq \Omega$, then it also implies the homogeneous Neumann boundary condition $\sigma \partial_\nu u|_{\partial\Omega} = 0$.
It is easily checked that the dual operators

$$L_j' : \; L^2_\diamond(\partial\Omega) \to L^2(D_j)^n, \quad j = 1, 2,$$

are given by $L_j' g = \nabla v|_{D_j}$, where $v \in H^1_\diamond(B)$ solves

$$\nabla \cdot \sigma \nabla v = 0 \quad \text{and} \quad \sigma \partial_\nu v|_{\partial\Omega} = g.$$

Now the assertion is equivalent to the statement

$$\nexists C > 0 \; : \; \|L_1' g\| \le C \|L_2' g\| \quad \text{for all } g \in L^2_\diamond(\partial\Omega),$$

which is (see, e.g., [6, Lemma 2.5]) equivalent to the range (non)inclusion

$$\mathcal{R}(L_1) \not\subseteq \mathcal{R}(L_2). \tag{3.4}$$

(b) *Proof of the range (non)inclusion* (3.4). Since $\operatorname{int} D_1 \not\subseteq \operatorname{out}_{\partial\Omega} D_2$, the set $\operatorname{int} D_1$ must intersect one of the sets U in the definition of the outer support of χ_{D_2}. Hence, there exists a set $U \subset \overline{\Omega} \setminus \operatorname{out}_{\partial\Omega} D_2$ with U (relatively) open in $\overline{\Omega}$, U connected to $\partial\Omega$, and $U \cap D_1$ contains an open ball B. Possibly shrinking the ball we can assume that $\overline{B} \subset \Omega$ and that $(U \cap \Omega) \setminus \overline{B}$ is connected.

Let L_B denote the virtual measurement operator corresponding to the ball B. Obviously, $B \subseteq D_1$ implies $\mathcal{R}(L_B) \subseteq \mathcal{R}(L_1)$, so that it suffices to prove that

$$\mathcal{R}(L_B) \not\subseteq \mathcal{R}(L_2). \tag{3.5}$$

To that end let $\varphi \in \mathcal{R}(L_B) \cap \mathcal{R}(L_2)$. Then there exist $u_B, u_2 \in H^1_\diamond(\Omega)$ with $u_B|_{\partial\Omega} = \varphi = u_2|_{\partial\Omega}$, $\sigma\partial_\nu u_B|_{U\cap\partial\Omega} = 0 = \sigma\partial_\nu u_2|_{U\cap\partial\Omega}$, and

$$\begin{aligned} \nabla\cdot\sigma\nabla u_B &= 0 \quad \text{in } \Omega\setminus\overline{B},\\ \nabla\cdot\sigma\nabla u_2 &= 0 \quad \text{in } U. \end{aligned}$$

By unique continuation $u_B = u_2$ in $U \setminus \overline{B}$. Hence

$$u := \begin{cases} u_B & \text{in } \Omega\setminus\overline{B},\\ u_2 & \text{in } \overline{B} \end{cases}$$

defines a function $u \in H^1_\diamond(\Omega)$ with $\nabla\cdot\sigma\nabla u = 0$ in Ω and homogeneous Neumann boundary data $\sigma\partial_\nu u|_{\partial\Omega} = 0$. It follows that $\varphi = u|_{\partial\Omega} = 0$, and thus we have shown that $\mathcal{R}(L_B) \cap \mathcal{R}(L_2) = \{0\}$.

Finally, using unique continuation again, we obtain that L_B' is injective, so that $\mathcal{R}(L_B)$ is dense in $L^2_\diamond(\partial\Omega)$. A fortiori, $\mathcal{R}(L_B) \neq \{0\}$, which, together with $\mathcal{R}(L_B) \cap \mathcal{R}(L_2) = \{0\}$, proves (3.5) and thus the assertion. ∎

Note that Theorem 3.6 also holds for less regular conductivities as long as a unique continuation property is fulfilled, and that localized potentials can be constructed by solving regularized operator equations; cf. [6].

We now show that (regardless of regularity) the properties of the localized potentials do not depend on the conductivity in the low-energy region.

LEMMA 3.7. *Let $D_1, D_2 \subseteq \overline{\Omega}$ be two measurable sets. Let $\sigma, \tau \in L^\infty_+(\Omega)$ and $u_m^\sigma, u_m^\tau \in H^1_\diamond(\Omega)$ denote the corresponding solutions of*

$$\begin{aligned} \nabla\cdot\sigma\nabla u_m^\sigma &= 0 \quad \text{in } \Omega, \qquad \sigma\partial_\nu u_m^\sigma|_{\partial\Omega} = g_m,\\ \nabla\cdot\tau\nabla u_m^\tau &= 0 \quad \text{in } \Omega, \qquad \tau\partial_\nu u_m^\tau|_{\partial\Omega} = g_m, \end{aligned}$$

for a sequence of boundary currents $(g_m)_{m\in\mathbb{N}} \subset L^2_\diamond(\partial\Omega)$.

If $\operatorname{supp}(\sigma-\tau) \subseteq D_2$, *then*

$$\lim_{m\to\infty}\int_{D_1} |\nabla u_m^\sigma|^2 \,\mathrm{d}x = \infty \quad \text{and} \quad \lim_{m\to\infty}\int_{D_2} |\nabla u_m^\sigma|^2 \,\mathrm{d}x = 0$$

holds if and only if

$$\lim_{m\to\infty}\int_{D_1} |\nabla u_m^\tau|^2 \,\mathrm{d}x = \infty \quad \text{and} \quad \lim_{m\to\infty}\int_{D_2} |\nabla u_m^\tau|^2 \,\mathrm{d}x = 0.$$

Proof. For both conductivities σ and τ, we define the virtual measurement operators

$$L_2^\sigma, L_2^\tau : \ L^2(D_2)^n \to L^2_\diamond(\partial\Omega)$$

as in the proof of Theorem 3.6. If $u^\sigma|_{\partial\Omega} = L_2^\sigma F$ with $F \in L^2(D_2)$ and a solution $u^\sigma \in H_\diamond^1(\Omega)$ of

$$\int_\Omega \sigma \nabla u^\sigma \cdot \nabla w \, \mathrm{d}x = \int_{D_2} F \cdot \nabla w \, \mathrm{d}x \quad \text{ for all } w \in H_\diamond^1(\Omega),$$

then u^σ also solves

$$\int_\Omega \tau \nabla u^\sigma \cdot \nabla w \, \mathrm{d}x = \int_{D_2} (F + (\tau - \sigma)\nabla u^\sigma) \cdot \nabla w \, \mathrm{d}x \quad \text{ for all } w \in H_\diamond^1(\Omega).$$

This shows that $\mathcal{R}(L_2^\sigma) \subseteq \mathcal{R}(L_2^\tau)$. As in the proof of Theorem 3.6, this implies that

$$\exists C > 0: \int_{D_2} |\nabla u_m^\sigma|^2 \mathrm{d}x \le C \int_{D_2} |\nabla u_m^\tau|^2 \, \mathrm{d}x \quad \text{ for all } m \in \mathbb{N}.$$

By interchanging σ and τ, we obtain that

$$\lim_{m\to\infty} \int_{D_2} |\nabla u_m^\sigma|^2 \, \mathrm{d}x = 0 \quad \Longleftrightarrow \quad \lim_{m\to\infty} \int_{D_2} |\nabla u_m^\tau|^2 \, \mathrm{d}x = 0.$$

Using the same argument on $D_1 \cup D_2$ it follows that also

$$\lim_{m\to\infty} \int_{D_1\cup D_2} |\nabla u_m^\sigma|^2 \, \mathrm{d}x = \infty \quad \Longleftrightarrow \quad \lim_{m\to\infty} \int_{D_1\cup D_2} |\nabla u_m^\tau|^2 \, \mathrm{d}x = \infty,$$

so that the assertion follows. ∎

Remark 3.8. Localized potentials can be numerically constructed by solving regularized operator equations (see [6]), and they can be used to probe for an unknown inclusion in the spirit of the probe or needle method; cf., e.g., [28, 29]. We briefly sketch the idea on a simple test example. Assume that the conductivity is $\sigma = 1 + \chi_D$ and that $(g_m)_{m\in\mathbb{N}}$ is a sequence such that the solutions $(u_m)_{m\in\mathbb{N}} \subset H_\diamond^1(\Omega)$ of $\Delta u_m = 0$ and $\partial_\nu u_m|_{\partial\Omega} = g_m$ fulfill

$$\lim_{m\to\infty} \int_{D_1} |\nabla u_m|^2 \, \mathrm{d}x = \infty \quad \text{and} \quad \lim_{m\to\infty} \int_{D_2} |\nabla u_m|^2 \, \mathrm{d}x = 0.$$

Then the monotonicity estimate in Lemma 3.1 yields that

$$\begin{aligned} D \subseteq D_2 \quad &\text{implies} \quad |\langle g_m, (\Lambda(1) - \Lambda(\sigma)) g_m \rangle| \to 0, \\ D_1 \subseteq D \quad &\text{implies} \quad |\langle g_m, (\Lambda(1) - \Lambda(\sigma)) g_m \rangle| \to \infty. \end{aligned}$$

Choosing D_2 to cover most of Ω and D_1 to be, e.g., a small ball inside $\Omega \setminus D_2$, one may thus estimate the shape of D by slowly shrinking D_2.

Such an algorithm would, however, suffer from high computational cost (to construct a high number of localized potentials), and it is not clear how to check the limit of $\langle g_m, (\Lambda(1) - \Lambda(\sigma)) g_m \rangle$ in a numerically stable way. Furthermore, the choice of the sets D_1 and D_2 would certainly impose some geometrical restrictions on the shapes of inclusions that can be recovered.

In the following, we take a different approach. The monotonicity methods derived in the next section do not require the numerical construction of localized potentials. We will require only the above abstract existence results for localized potentials in order to show that simple monotonicity tests recover the true (outer) shape of an inclusion.

4. Monotonicity-based shape reconstruction.

4.1. The definite case. We will now show how the shape reconstruction problem can be solved via simple monotonicity tests. We start with the definite case, in which the inclusions conductivity is everywhere higher or everywhere lower than the background. We treat this case separately since it allows a particularly simple reconstruction strategy. Given a small ball, the following theorems show how to check whether the ball belongs to the inclusion or not. The proofs of the theorems are postponed until the end of this subsection. The main idea of this subsection has previously been summarized in the extended conference abstract [18].

THEOREM 4.1. *Let $\sigma \in L^\infty_+(\Omega)$ and $\sigma \geq 1$.*
For every open ball $B := B_\epsilon(z)$ and every $\alpha > 0$,

$$\begin{aligned} \alpha\chi_B \leq \sigma - 1 \qquad & \textit{implies} \quad \Lambda(1+\alpha\chi_B) \geq \Lambda(\sigma), \\ B \not\subseteq \operatorname{out}_{\partial\Omega} \operatorname{supp}(\sigma - 1) \qquad & \textit{implies} \quad \Lambda(1+\alpha\chi_B) \not\geq \Lambda(\sigma). \end{aligned}$$

Hence, the set

$$R := \bigcup_{\alpha>0} \{B = B_\epsilon(z) \subseteq \Omega \ : \ \Lambda(1+\alpha\chi_B) \geq \Lambda(\sigma)\}$$

fulfills

$$\operatorname{inn} \operatorname{supp}(\sigma - 1) \subseteq R \subseteq \operatorname{out}_{\partial\Omega} \operatorname{supp}(\sigma - 1).$$

Example 4.2. Let $\sigma = 1 + \chi_D$, where the inclusion D is open, and $\overline{D} \subset \Omega$ has a connected complement. Then for every open ball $B \subseteq \Omega$

$$B \subseteq D \quad \text{if and only if} \quad \Lambda(1+\chi_B) \geq \Lambda(\sigma).$$

Note that implementing the monotonicity tests in Theorem 4.1 or Example 4.2 would be computationally expensive since for each ball B (and possibly also for each test level α) we would have to solve the EIT equation with a new inhomogeneous conductivity in order to calculate $\Lambda(1+\alpha\chi_B)$. The following theorem shows that we can replace the tests by linearized versions, which do not require such inhomogeneous forward solutions. Since this is a bit counterintuitive, let us stress that the following result is not affected by the linearization error, no matter how large that may be. The linearized inverse problem in EIT still contains the exact shape information; cf. [16].

THEOREM 4.3. *Let $\sigma \in L^\infty_+(\Omega)$ and $\sigma \geq 1$.*
For every open ball $B := B_\epsilon(z)$ and every $\alpha > 0$,

$$\begin{aligned} \alpha\chi_B \leq \frac{1}{\sigma}(\sigma - 1) \qquad & \textit{implies} \quad \Lambda(1) + \alpha\Lambda'(1)\chi_B \geq \Lambda(\sigma), \\ B \not\subseteq \operatorname{out}_{\partial\Omega} \operatorname{supp}(\sigma - 1) \qquad & \textit{implies} \quad \Lambda(1) + \alpha\Lambda'(1)\chi_B \not\geq \Lambda(\sigma). \end{aligned}$$

Hence, the set

$$R := \bigcup_{\alpha>0} \{B = B_\epsilon(z) \subseteq \Omega \ : \ \Lambda(1) + \alpha\Lambda'(1)\chi_B \geq \Lambda(\sigma)\}$$

fulfills

$$\operatorname{inn}\operatorname{supp}(\sigma - 1) \subseteq R \subseteq \operatorname{out}_{\partial\Omega}\operatorname{supp}(\sigma - 1).$$

Example 4.4. Let $\sigma = 1 + \chi_D$, where the inclusion D is open, and $\overline{D} \subset \Omega$ has a connected complement. Then for every ball $B = B_\epsilon(z)$

$$B \subseteq D \quad \text{if and only if} \quad \Lambda(1) + \frac{1}{2}\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

Proof of Theorem 4.1. Let $\sigma \in L^\infty_+(\Omega)$, $\sigma \geq 1$. Let $B = B_\epsilon(z)$ and $\alpha > 0$. Corollary 3.2 yields that

$$\alpha\chi_B \leq \sigma - 1 \quad \text{implies} \quad \Lambda(1 + \alpha\chi_B) \geq \Lambda(\sigma).$$

It remains to show that

$$B \not\subseteq \operatorname{out}_{\partial\Omega}\operatorname{supp}(\sigma - 1) \quad \text{implies} \quad \Lambda(1 + \alpha\chi_B) \not\geq \Lambda(\sigma).$$

Let $B \not\subseteq \operatorname{out}_{\partial\Omega}\operatorname{supp}(\sigma - 1)$. Corollary 3.2 yields that shrinking the ball B only makes $\Lambda(1 + \alpha\chi_B)$ larger, so that we can assume without loss of generality (w.l.o.g.) that

$$B \subseteq \Omega \setminus \operatorname{out}_{\partial\Omega}\operatorname{supp}(\sigma - 1).$$

We have that $1 + \alpha\chi_B$ is piecewise analytic,

$$B = \operatorname{int} B \quad \text{and} \quad \operatorname{out}_{\partial\Omega}\operatorname{supp}(\sigma - 1) = \operatorname{out}_{\partial\Omega}(\operatorname{supp}(\sigma - 1))$$

(see Lemma 2.4(d)). Hence, we can apply Theorem 3.6 and obtain a sequence of currents $(g_m)_{m\in\mathbb{N}} \subset L^2_\diamond(\partial\Omega)$ so that the solutions $(u_m)_{m\in\mathbb{N}} \subset H^1_\diamond(\Omega)$ of

$$\nabla \cdot (1 + \alpha\chi_B)\nabla u_m = 0 \quad \text{in } \Omega, \qquad (1 + \alpha\chi_B)\partial_\nu u_m|_{\partial\Omega} = g_m,$$

fulfill

$$\lim_{m\to\infty} \int_B |\nabla u_m|^2 \,\mathrm{d}x = \infty \quad \text{and} \quad \lim_{m\to\infty} \int_{\operatorname{supp}(\sigma-1)} |\nabla u_m|^2 \,\mathrm{d}x = 0.$$

From Lemma 3.1 it follows that

$$\begin{aligned}\langle g_m, (\Lambda(1 + \alpha\chi_B) - \Lambda(\sigma))\, g_m\rangle &\leq \int_\Omega (\sigma - 1 - \alpha\chi_B)|\nabla u_m|^2 \,\mathrm{d}x\\ &= -\alpha\int_B |\nabla u_m|^2 \,\mathrm{d}x + \int_{\operatorname{supp}(\sigma-1)} (\sigma - 1)|\nabla u_m|^2 \,\mathrm{d}x\\ &\to -\infty,\end{aligned}$$

and hence $\Lambda(1 + \alpha\chi_B) \not\geq \Lambda(\sigma)$. ∎

Proof of Theorem 4.3. Let $\sigma \in L^\infty_+(\Omega)$, $\sigma \geq 1$. Let $B = B_\epsilon(z)$ and $\alpha > 0$.

For every $g \in L^2_\diamond(\partial\Omega)$ and solution $u \in H^1_\diamond(\Omega)$ of

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \partial_\nu u|_{\partial\Omega} = g,$$

we obtain from Lemma 3.1

$$\langle g, (\Lambda(1) + \alpha\Lambda'(1)\chi_B - \Lambda(\sigma))\, g\rangle \geq \int_\Omega \left(\frac{1}{\sigma}(\sigma - 1) - \alpha\chi_B\right)|\nabla u|^2 \,\mathrm{d}x.$$

This shows that

$$\alpha\chi_B \leq \frac{1}{\sigma}(\sigma - 1) \quad \text{implies} \quad \Lambda(1) + \alpha\Lambda'(1)\chi_B \geq \Lambda(\sigma).$$

It remains to show that

$$B \not\subseteq \mathrm{out}_{\partial\Omega}\,\mathrm{supp}\,(\sigma - 1) \quad \text{implies} \quad \Lambda(1) + \alpha\Lambda'(1)\chi_B \not\geq \Lambda(\sigma).$$

To show this let $B \not\subseteq \mathrm{out}_{\partial\Omega}\,\mathrm{supp}\,(\sigma - 1)$. The linearized monotonicity relation (2.3) yields that shrinking the ball B only makes $\Lambda(1) + \alpha\Lambda'(1)\chi_B$ larger, so that we can assume w.l.o.g. that $B \subseteq \Omega \setminus \mathrm{out}_{\partial\Omega}\,\mathrm{supp}\,(\sigma - 1)$. Then

$$\begin{aligned}&\langle g, (\Lambda(1) + \alpha\Lambda'(1)\chi_B - \Lambda(\sigma))\, g\rangle \\ &\leq \int_\Omega (\sigma - 1 - \alpha\chi_B)\, |\nabla u|^2 \,\mathrm{d}x = -\alpha \int_B |\nabla u|^2 \,\mathrm{d}x + \int_{\mathrm{supp}(\sigma-1)} (\sigma - 1)|\nabla u|^2 \,\mathrm{d}x,\end{aligned}$$

so that the assertion follows using localized potentials for the background conductivity 1 and the same sets as in Theorem 3.6. ∎

Remark 4.5. If $\sigma \in L^\infty_+(\Omega)$ and $\sigma \leq 1$, then we obtain with the same arguments that for every open ball $B \subseteq \Omega$ and every $0 < \alpha < 1$,

$$\begin{aligned}\alpha\chi_B &\leq 1 - \sigma && \text{implies} \quad \Lambda(1 - \alpha\chi_B) \leq \Lambda(\sigma),\\ B &\not\subseteq \mathrm{out}_{\partial\Omega}\,\mathrm{supp}\,(\sigma - 1) && \text{implies} \quad \Lambda(1 - \alpha\chi_B) \not\leq \Lambda(\sigma),\end{aligned}$$

and for every open ball $B \subseteq \Omega$ and every $\alpha > 0$

$$\begin{aligned}\alpha\chi_B &\leq 1 - \sigma && \text{implies} \quad \Lambda(1) - \alpha\Lambda'(1)\chi_B \leq \Lambda(\sigma),\\ B &\not\subseteq \mathrm{out}_{\partial\Omega}\,\mathrm{supp}\,(\sigma - 1) && \text{implies} \quad \Lambda(1) - \alpha\Lambda'(1)\chi_B \not\leq \Lambda(\sigma).\end{aligned}$$

Remark 4.6. An inspection of the proofs shows that the balls can be replaced by arbitrary measurable sets B with nonempty interior in Theorem 4.3 (and the second part of Remark 4.5). For Theorem 4.1 (and the first part of Remark 4.5) the sets B must additionally possess a piecewise smooth boundary (so that $1 + \alpha\chi_B$ remains piecewise analytic). We comment on further generalizations in section 4.3.

4.2. The indefinite case. We now consider the general indefinite case where σ is no longer required to be everywhere larger or everywhere smaller than the background conductivity 1. Instead of testing whether a small test region is part of the unknown inclusion, we will now test whether a large test region contains the unknown inclusions.

The main idea is the following. Consider a large test region C with connected complement. If C overlaps the inclusions, then a large enough, respectively, small enough, test conductivity on C will make the corresponding test NtD smaller, respectively, larger, than the measured NtD. Hence if C overlaps the inclusions, then two monotonicity tests (one with a large and one with a small test level on C) hold true. On the other hand, if C does not overlap the inclusions, then we can connect the nonoverlapped part with the boundary, and construct a localized potential with large energy in the nonoverlapped part and small energy in C. Depending on whether the conductivity is larger, respectively, smaller, than the background in the nonoverlapped part, this localized potential shows that one of the monotonicity tests cannot hold true.

However, for this argument we need a *local definiteness property.* If a conductivity differs from the background, then there must either be a neighborhood of the boundary where it differs from the background in the positive direction, or a neighborhood where it differs in the negative direction. Note that even C^∞-conductivities might oscillate infinitely and thus violate this property. This property holds, however, if the conductivity is either piecewise analytic or if the higher-conductivity and lower-conductivity parts have some distance from each other, and the inner support does not deviate too much from the true support (which already holds, e.g., for piecewise continuous functions; see Corollary 2.5).

More precisely, we assume that $\sigma \in L^\infty_+(\Omega)$ is either piecewise analytic or

$$\operatorname{supp}(\sigma-1)^+ \cap \operatorname{supp}(\sigma-1)^- = \emptyset, \quad \overline{\operatorname{inn}\operatorname{supp}(\sigma-1)} = \operatorname{supp}(\sigma-1), \tag{4.1}$$

where $(\sigma-1)^+ := \max\{\sigma-1,0\}$, $(\sigma-1)^- := \min\{\sigma-1,0\}$.

THEOREM 4.7. *Let $\sigma \in L^\infty_+(\Omega)$ either be piecewise analytic or fulfill* (4.1). *Then, for every set $C \subseteq \overline{\Omega}$ with $C = \operatorname{out}_{\partial\Omega} C$ and every $\alpha > 1$,*

$$\begin{aligned} 1-\frac{1}{\alpha}\chi_C \le \sigma \quad &\textit{implies} \quad \Lambda\left(1-\frac{1}{\alpha}\chi_C\right) \ge \Lambda(\sigma),\\ 1+\alpha\chi_C \ge \sigma \quad &\textit{implies} \quad \Lambda(1+\alpha\chi_C) \le \Lambda(\sigma) \end{aligned}$$

and

$$\Lambda(1+\alpha\chi_C) \le \Lambda(\sigma) \le \Lambda\left(1-\frac{1}{\alpha}\chi_C\right) \quad \textit{implies} \quad \operatorname{out}_{\partial\Omega}\operatorname{supp}(\sigma-1) \subseteq C.$$

Hence,

$$R := \bigcap \left\{ C = \operatorname{out}_{\partial\Omega} C \subseteq \overline{\Omega},\ \exists \alpha > 1:\ \Lambda(1+\alpha\chi_C) \le \Lambda(\sigma) \le \Lambda\left(1-\frac{1}{\alpha}\chi_C\right)\right\}$$

fulfills $R = \operatorname{out}_{\partial\Omega}\operatorname{supp}(\sigma-1)$.

We postpone the proof until the end of this subsection and first give an example and formulate the linearized version.

Example 4.8. Let $\sigma = 1 + \chi_{D^+} - \frac{1}{2}\chi_{D^-}$, where $D^+, D^- \subseteq \Omega$ are open sets with $\overline{D^+} \cap \overline{D^-} = \emptyset$, and $\overline{D^+} \cup \overline{D^-} \subset \Omega$ has a connected complement.

Then for every closed set $C \subset \Omega$ with connected complement $\Omega \setminus C$

$$D^+ \cup D^- \subseteq C \quad \text{if and only if} \quad \Lambda(1+\chi_C) \le \Lambda(\sigma) \le \Lambda\left(1-\frac{1}{2}\chi_C\right).$$

THEOREM 4.9. *Under the assumptions of Theorem* 4.7 *we have that for every set $C \subseteq \overline{\Omega}$ with $C = \operatorname{out}_{\partial\Omega} C$ and every $\alpha > 0$*

$$\begin{aligned} 1-\alpha\chi_C \le 2-\frac{1}{\sigma} \quad &\textit{implies} \quad \Lambda(1)-\alpha\Lambda'(1)\chi_C \ge \Lambda(\sigma),\\ 1+\alpha\chi_C \ge \sigma \quad &\textit{implies} \quad \Lambda(1)+\alpha\Lambda'(1)\chi_C \le \Lambda(\sigma) \end{aligned}$$

and

$$\Lambda(1)+\alpha\Lambda'(1)\chi_C \le \Lambda(\sigma) \le \Lambda(1)-\alpha\Lambda'(1)\chi_C \quad \textit{implies} \quad \operatorname{out}_{\partial\Omega}\operatorname{supp}(\sigma-1) \subseteq C.$$

Hence,

$$R := \bigcap \{C = \mathrm{out}_{\partial\Omega}\, C \subseteq \overline{\Omega},\ \exists \alpha > 0 : \\ \Lambda(1) + \alpha\Lambda'(1)\chi_C \leq \Lambda(\sigma) \leq \Lambda(1) - \alpha\Lambda'(1)\chi_C\}$$

fulfills $R = \mathrm{out}_{\partial\Omega}\, \mathrm{supp}\,(\sigma - 1)$.

Example 4.10. Let $\sigma = 1 + \chi_{D^+} - \frac{1}{2}\chi_{D^-}$, where $D^+, D^- \subseteq \Omega$ are open sets with $\overline{D^+} \cap \overline{D^-} = \emptyset$, and $\overline{D^+} \cup \overline{D^-} \subset \Omega$ has a connected complement.

Then for every closed set $C \subset \Omega$ with connected complement $\Omega \setminus C$

$$D^+ \cup D^- \subseteq C \quad \text{if and only if} \quad \Lambda(1) + \Lambda'(1)\chi_C \leq \Lambda(\sigma) \leq \Lambda(1) - \Lambda'(1)\chi_C.$$

Proof of Theorem 4.7. Let $\alpha > 1$ and $C = \mathrm{out}_{\partial\Omega}\, C \subseteq \overline{\Omega}$. Then C is closed and thus measurable, so that $1 - \frac{1}{\alpha}\chi_C, 1 + \alpha\chi_C \in L^\infty_+(\Omega)$.

Corollary 3.2 yields the first two assertions

$$\begin{aligned} 1 - \frac{1}{\alpha}\chi_C \leq \sigma \quad &\text{implies} \quad \Lambda\left(1 - \frac{1}{\alpha}\chi_C\right) \geq \Lambda(\sigma), \\ 1 + \alpha\chi_C \geq \sigma \quad &\text{implies} \quad \Lambda(1 + \alpha\chi_C) \leq \Lambda(\sigma). \end{aligned}$$

It remains to show that $\mathrm{out}_{\partial\Omega}\, \mathrm{supp}\,(\sigma - 1) \not\subseteq C$ implies that either

$$\Lambda\left(1 - \frac{1}{\alpha}\chi_C\right) \not\geq \Lambda(\sigma) \quad \text{or} \quad \Lambda(1 + \alpha\chi_C) \not\leq \Lambda(\sigma).$$

Let $\mathrm{out}_{\partial\Omega}\, \mathrm{supp}\,(\sigma - 1) \not\subseteq C = \mathrm{out}_{\partial\Omega}\, C$. Then there exists a relatively open set $U \subseteq \overline{\Omega}$ that is connected to $\partial\Omega$ where $\sigma|_U \not\equiv 1$ and $C \cap U = \emptyset$.

We first prove the assertion for the case that σ is piecewise analytic. Using the local definiteness property derived in Corollary A.2 in the appendix, we can choose (note that $\Omega \setminus D_2 \subseteq U$ implies $C \subseteq D_2$)

$$D_1, D_2 \subseteq \overline{\Omega}, \quad \text{with} \quad D_1 = \mathrm{int}\, D_1 \not\subseteq \mathrm{out}_{\partial\Omega}\, D_2 = D_2, \quad C \subseteq D_2,$$

so that either

(a) $\sigma \geq 1$ on $\Omega \setminus D_2$, $\sigma - 1 \in L^\infty_+(D_1)$, or

(b) $\sigma \leq 1$ on $\Omega \setminus D_2$, $1 - \sigma \in L^\infty_+(D_1)$.

Replacing D_1 with $D_1 \setminus \mathrm{out}_{\partial\Omega}\, D_2$, we can also assume that $D_1 \cap D_2 = \emptyset$.

We then use the localized potentials Theorem 3.6 for the homogeneous conductivity $\tau = 1$ and obtain a sequence $(g_m)_{m\in\mathbb{N}} \subset L^2_\diamond(\partial\Omega)$ so that the solutions $(u_m^\tau)_{m\in\mathbb{N}} \subseteq H^1_\diamond(\Omega)$ of

$$\nabla \cdot \tau \nabla u_m^\tau = 0 \quad \text{in } \Omega, \qquad \tau\partial_\nu u_m^\tau|_{\partial\Omega} = g_m,$$

fulfill

$$\lim_{m\to\infty} \int_{D_1} |\nabla u_m|^2 \,\mathrm{d}x = \infty \quad \text{and} \quad \lim_{m\to\infty} \int_{D_2} |\nabla u_m|^2 \,\mathrm{d}x = 0.$$

Since $C \subseteq D_2$ it follows from Lemma 3.7 that the solutions u_m^τ for the conductivities $\tau = 1 - \frac{1}{\alpha}\chi_C$ and $\tau = 1 + \alpha\chi_C$ have the same property.

Hence, in case (a), we apply Lemma 3.1 with $\tau = 1+\alpha\chi_C$ and obtain (using that $\sigma \geq 1$ on $\Omega \setminus (D_1 \cup D_2)$, and that $C \subseteq D_2$)

$$\begin{aligned}
&\langle g_m, (\Lambda(1+\alpha\chi_C) - \Lambda(\sigma))\, g_m\rangle \\
&\geq \int_\Omega \frac{1+\alpha\chi_C}{\sigma}(\sigma - (1+\alpha\chi_C))|\nabla u_m^\tau|^2 \,\mathrm{d}x \\
&= \int_{\Omega\setminus(D_1\cup D_2)} \frac{\sigma-1}{\sigma}|\nabla u_m^\tau|^2 \,\mathrm{d}x + \int_{D_1} \frac{\sigma-1}{\sigma}|\nabla u_m^\tau|^2 \,\mathrm{d}x \\
&\quad + \int_{D_2} \frac{1+\alpha\chi_C}{\sigma}(\sigma - (1+\alpha\chi_C))|\nabla u_m^\tau|^2 \,\mathrm{d}x \\
&\geq \int_{D_1} \frac{\sigma-1}{\sigma}|\nabla u_m^\tau|^2 \,\mathrm{d}x + \int_{D_2} \frac{1+\alpha\chi_C}{\sigma}(\sigma - (1+\alpha\chi_C))|\nabla u_m^\tau|^2 \,\mathrm{d}x \to \infty.
\end{aligned}$$

In case (b), we apply Lemma 3.1 with $\tau = 1 - \frac{1}{\alpha}\chi_C$ and obtain (using that $\sigma \leq 1$ on $\Omega \setminus (D_1 \cup D_2)$, and that $C \subseteq D_2$)

$$\begin{aligned}
&\left\langle g_m, \left(\Lambda\left(1 - \frac{1}{\alpha}\chi_C\right) - \Lambda(\sigma)\right) g_m\right\rangle \\
&\leq \int_\Omega \left(\sigma - \left(1 - \frac{1}{\alpha}\chi_C\right)\right)|\nabla u_m^\tau|^2 \,\mathrm{d}x \\
&= \int_{\Omega\setminus(D_1\cup D_2)} (\sigma-1)|\nabla u_m^\tau|^2 \,\mathrm{d}x + \int_{D_1} (\sigma-1)|\nabla u_m^\tau|^2 \,\mathrm{d}x \\
&\quad + \int_{D_2} \left(\sigma - \left(1 - \frac{1}{\alpha}\chi_C\right)\right)|\nabla u_m^\tau|^2 \,\mathrm{d}x \\
&\leq \int_{D_1} (\sigma-1)|\nabla u_m^\tau|^2 \,\mathrm{d}x + \int_{D_2} \left(\sigma - \left(1 - \frac{1}{\alpha}\chi_C\right)\right)|\nabla u_m^\tau|^2 \,\mathrm{d}x \to -\infty,
\end{aligned}$$

which proves the assertion for piecewise analytic conductivities.

Now we prove that the assertion also holds for (not necessarily piecewise analytic) conductivities fulfilling (4.1). It suffices to show that also in this case there exist

$$D_1, D_2 \subseteq \overline{\Omega}, \quad \text{with} \quad D_1 = \operatorname{int} D_1 \not\subseteq \operatorname{out}_{\partial\Omega} D_2 = D_2, \quad C \subseteq D_2,$$

such that either (a) or (b) from above holds.

First note that if $\operatorname{supp}(\sigma-1)^+$ and $\operatorname{supp}(\sigma-1)^-$ are disjoint compact sets, then

$$\delta := \operatorname{dist}\left(\operatorname{supp}(\sigma-1)^+, \operatorname{supp}(\sigma-1)^-\right) > 0.$$

$\sigma|_U \not\equiv 1$ implies that there exists a point $y \in U \cap \operatorname{supp}(\sigma - 1)$. Let $x \in \partial\Omega \cap U$. Since $\partial\Omega$ is a smooth boundary and $U \cap \Omega$ is open and connected, we can connect x and y with a continuous path

$$\gamma : \ [0,1] \to U, \quad \gamma(0) = x, \ \gamma(1) = y.$$

Using that U is relatively open, there exists, for each $t \in [0,1]$, a ball $B_t := B_{\epsilon(t)}(\gamma(t))$ with radius $\epsilon(t) < \delta/2$ and $B_t \cap \overline{\Omega} \subseteq U$.

By compactness of $\gamma([0,1])$ we can choose a finite number $0 \leq t_1 < \cdots < t_N \leq 1$, so that

$$\gamma([0,1]) \subset (B_{t_1} \cup \cdots \cup B_{t_N}) \cap \overline{\Omega}.$$

Since $\gamma(1) = y \in \operatorname{supp}(\sigma - 1)$, there exists a smallest index J for which

$$B_{t_J} \cap \overline{\operatorname{inn}\operatorname{supp}(\sigma - 1)} = B_{t_J} \cap \operatorname{supp}(\sigma - 1) \neq \emptyset$$

so that there exists an open set $D_1 \subseteq B_{t_J}$ with $|\sigma - 1| \in L^\infty_+(D_1)$.

We define $D_2 := \Omega \setminus (B_{t_1} \cup \cdots \cup B_{t_J})$. Then

$$D_1, D_2 \subseteq \overline{\Omega}, \quad \text{with} \quad D_1 = \operatorname{int} D_1 \not\subseteq \operatorname{out}_{\partial\Omega} D_2 = D_2, \quad C \subseteq D_2.$$

Furthermore, since B_{t_J} has diameter less than δ, it cannot intersect both $\operatorname{supp}(\sigma-1)^+$ and $\operatorname{supp}(\sigma - 1)^-$, so that either

(a) $\sigma \geq 1$ on $\Omega \setminus D_2$, $\sigma - 1 \in L^\infty_+(D_1)$, or

(b) $\sigma \leq 1$ on $\Omega \setminus D_2$, $1 - \sigma \in L^\infty_+(D_1)$,

which finishes the proof. ∎

Proof of Theorem 4.9. If $1 - \alpha\chi_C \leq 2 - \frac{1}{\sigma}$, then $\alpha\chi_C \geq \frac{1}{\sigma}(1 - \sigma)$, so that (2.3) and Corollary 3.4 imply that

$$\Lambda(\sigma) \leq \Lambda(1) - \Lambda'(1)\left(\frac{1}{\sigma}(1 - \sigma)\right) \leq \Lambda(1) - \alpha\Lambda'(1)\chi_C.$$

Likewise, if $1 + \alpha\chi_C \geq \sigma$, then (2.3) and Corollary 3.4 imply that

$$\Lambda(\sigma) \geq \Lambda(1) - \Lambda'(1)(1 - \sigma) \geq \Lambda(1) + \alpha\Lambda'(1)\chi_C.$$

This shows the first two assertions.

Moreover, Lemma 3.1 yields that for all $\alpha \in \mathbb{R}$

$$\langle(\Lambda(1) + \alpha\Lambda'(1)\chi_C - \Lambda(\sigma))g, g\rangle \geq \int_\Omega \left(\frac{1}{\sigma}(\sigma - 1) - \alpha\chi_C\right)|\nabla u_g|^2 \,\mathrm{d}x$$

and

$$\langle(\Lambda(1) - \alpha\Lambda'(1)\chi_C - \Lambda(\sigma))g, g\rangle \leq \int_\Omega (\sigma - 1 + \alpha\chi_C)|\nabla u_g|^2 \,\mathrm{d}x,$$

where $u_g \in H^1_\diamond(\Omega)$ solves $\Delta u_g = 0$ and $\partial_\nu u_g|_{\partial\Omega} = g$. Hence, the third assertion follows by using localized potentials for the homogeneous conductivity and the same sets D_1, D_2 as in Theorem 4.7. ∎

4.3. Remarks and extensions. Let us comment on some extensions and generalizations of our results. Our assumption that the background conductivity is equal to 1 and that we are given measurements on the complete boundary $\partial\Omega$ have been merely for the ease of presentation. All our results and proofs remain valid if $\partial\Omega$ is replaced by an arbitrarily small open piece $S \subset \partial\Omega$ and we are given the partial NtD operator

$$\Lambda(\sigma):\ L^2_\diamond(S) \to L^2_\diamond(S), \quad g \mapsto u^g_\sigma|_S,$$

where $u^g_\sigma \in H^1_\diamond(\Omega)$ is the unique solution of

$$\nabla \cdot \sigma\nabla u^g_\sigma = 0 \text{ in } \Omega, \quad \sigma\partial_\nu u^g_\sigma|_{\partial\Omega} = \begin{cases} g & \text{on } S, \\ 0 & \text{on } \partial\Omega \setminus S. \end{cases}$$

Also, all the results still hold when the background conductivity 1 is replaced by a known piecewise analytic function.

Let us also note that our results require piecewise analyticity for only two purposes: the existence of localized potentials and the local definiteness property. Localized potentials exist for less regular conductivities; it is only required that the solutions of the corresponding elliptic EIT equations satisfy a unique continuation property; cf. [6]. Local definiteness can hold for quite general functions if additional assumption are made (e.g., that the positive and negative parts are separated as in (4.1)). However, the authors are not aware of any natural function classes beyond piecewise analytic functions in which a property in the spirit of Theorem A.1 holds without further assumptions.

Appendix A. Local definiteness of piecewise analytic functions. In this appendix, we show that piecewise analytic functions have a *local definiteness property*. If they do not vanish identically, then there is either a neighborhood of the boundary where they differ from zero in the positive direction, or a neighborhood where they differ in the negative direction.

The property follows from the arguments used in the proofs of [12, Theorem 4.2] and [16, Lemma 3.7]. However, some subtle and not entirely trivial topological details were omitted in [12, 16], which is why we give the proof here in full detail.

THEOREM A.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a smoothly bounded domain, and let $\sigma \in L^\infty_+(\Omega)$ be piecewise analytic. Let $U \subseteq \overline{\Omega}$ be relatively open and connected to $\partial\Omega$, and let $\sigma|_U \not\equiv 0$.*

Then we can find a subset $V \subseteq U$ with the same properties, on which σ does not change sign, i.e.,

(a) *$V \subseteq \overline{\Omega}$ is relatively open, V is connected to $\partial\Omega$, $V \subseteq U$;*

(b) *$\sigma|_V \not\equiv 0$, and either $\sigma|_V \geq 0$ or $\sigma|_V \leq 0$.*

Obviously, if a piecewise analytic function is not identically zero, we can find a neighborhood where it is bounded away from zero. Hence, choosing $D_2 := \Omega \setminus V$, we obtain the following corollary.

COROLLARY A.2. *Under the assumptions of Theorem* A.1 *we can choose*

$$D_1, D_2 \subseteq \overline{\Omega}, \quad \textit{with} \quad D_1 = \operatorname{int} D_1 \not\subseteq \operatorname{out}_{\partial\Omega} D_2 = D_2, \quad \Omega \setminus D_2 \subseteq U,$$

and either

$$\begin{aligned} &\sigma|_{\Omega\setminus D_2} \geq 0, \quad \sigma|_{D_1} \in L^\infty_+(D_1), \quad \textit{or} \\ &\sigma|_{\Omega\setminus D_2} \leq 0, \quad -\sigma|_{D_1} \in L^\infty_+(D_1). \end{aligned}$$

In the following, let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a smoothly bounded domain, and let $\sigma \in L^\infty_+(\Omega)$ be piecewise analytic with respect to

$$\overline{\Omega} = \overline{O_1 \cup \cdots \cup O_M}, \quad \partial O_m = \bigcup_{k\in\mathbb{N}} \overline{\Gamma_m^k},$$

where, w.l.o.g., we assume that every ∂O_m consists of infinitely many pieces. Furthermore, let $U \subseteq \overline{\Omega}$ be relatively open and connected to $\partial\Omega$.

LEMMA A.3. *There exists an open ball $B \subseteq \mathbb{R}^n$ such that*

$$B \cap \overline{\Omega} \subseteq U, \quad \textit{and} \quad B \cap \overline{\Omega} \textit{ is connected to } \partial\Omega,$$

and for one of the O_m and one of its smooth boundary pieces $\Gamma_m^k \subseteq \partial O_m$,

$$B \cap \Omega = B \cap O_m \quad \textit{and} \quad B \cap \partial\Omega \subseteq \Gamma_m^k.$$

Proof. Since U is relatively open and $U \cap \partial\Omega \neq \emptyset$, there exists an open ball B with $\emptyset \neq B \cap \partial\Omega$, $B \cap \overline{\Omega} \subseteq U$, and by shrinking B we can assume that

$$\emptyset \neq S := \overline{B} \cap \partial\Omega \subseteq U.$$

$\overline{\Omega} = \overline{O_1 \cup \cdots \cup O_M}$ implies that

$$\partial\Omega \subseteq \bigcup_{m=1}^{M} \partial O_m = \bigcup_{m=1}^{M} \bigcup_{k \in \mathbb{N}} \overline{\Gamma_m^k} \quad \text{and thus} \quad S = \bigcup_{m,k} \overline{\Gamma_m^k} \cap S.$$

By Baire's theorem, one of the countably many closed sets $\overline{\Gamma_m^k} \cap S$ must have a nonempty interior in S. Hence, for one of the $\overline{\Gamma_m^k}$, there exists an open ball B with

$$\emptyset \neq B \cap \partial\Omega \subseteq \overline{\Gamma_m^k} \cap U.$$

Moreover, $B \cap \partial\Omega$ must intersect Γ_m^k because of the following dimension theoretical argument; cf., e.g., the classical book of Hurewicz and Wallman [19, Chapter IV, section 4]. $\Omega \cap B$ is an open (neither empty nor dense) subset of the $(n-1)$-dimensional ball B. As a subset of B, the boundary of $\Omega \cap B$ is $\partial\Omega \cap B$, which shows that $\partial\Omega \cap B$ is $(n-1)$-dimensional (and not of lesser dimension). Γ_m^k is a (neither empty nor dense) open subset of a set that is homeomorphic to $\mathbb{R}^{n-1}$. Hence, Γ_m^k is $(n-1)$-dimensional and $\overline{\Gamma_m^k} \setminus \Gamma_m^k$ is $(n-2)$-dimensional. This shows that $B \cap \partial\Omega \not\subseteq \overline{\Gamma_m^k} \setminus \Gamma_m^k$, so that by shrinking B we can assume that

$$\emptyset \neq B \cap \partial\Omega \subseteq \Gamma_m^k \cap U.$$

Finally, we can shrink B so that $B \cap \Omega = B \cap O_m$ and that $B \cap \Omega$ is connected. ◻

LEMMA A.4. *Every open ball $B \subseteq \mathbb{R}^n$ that intersects a smooth boundary piece Γ_m^k contains an open subball $B' \subseteq B$ intersecting Γ_m^k, where either*

$$\sigma|_{B' \cap O_m} \geq 0 \quad \textit{or} \quad \sigma|_{B' \cap O_m} \leq 0.$$

Proof. We use an argument of Kohn and Vogelius [37]. If $\sigma|_{B \cap O_m} \equiv 0$, then the assumption is trivial. Otherwise, by analyticity, there must be a smallest $k \in \mathbb{N}$, so that the normal derivative $\partial_{\nu(z)}^k \sigma(z)$ is not identically zero for all $z \in \Gamma_m^k \cap B$. Hence there is a neighborhood of a point $z \in \Gamma_m^k \cap B$ on which either $\sigma \geq 1$ or $\sigma \leq 1$. ◻

Now we are ready to prove the local definiteness property.

Proof of Theorem A.1. From Lemma A.3 we obtain an open ball $B \subseteq \mathbb{R}^n$ with

$$B \cap \overline{\Omega} \subseteq U, \quad B \cap \overline{\Omega} \text{ is connected to } \partial\Omega,$$

and (w.l.o.g.)

$$B \cap \Omega = B \cap O_1, \quad B \cap \partial\Omega \subseteq \Gamma_1^1.$$

If σ is not identically zero on O_1, then the assertion follows from Lemma A.4.

Otherwise, $M > 1$, and the set $V := B \cap \overline{\Omega}$ has the following properties:

(i) V is a relatively open subset of $\overline{\Omega}$ that is connected to $\partial\Omega$.

(ii) V fulfills $B \cap \overline{\Omega} \subseteq V \subseteq U$.

(iii) $\sigma|_V = 0$.

Obviously these properties are closed under union, so that we can choose V to be the maximal set fulfilling (i)–(iii).

Now we show that

$$\emptyset \neq \partial V \cap U \cap \Omega \subseteq \bigcup_{m=1}^{M} \partial O_m. \tag{A.1}$$

Since V is relatively open in Ω and $V \subseteq U$ it follows that $V \cap \Omega$ is relatively open in $U \cap \Omega$. If ∂V has no intersection with $U \cap \Omega$, then $V \cap \Omega$ is relatively closed, so that $U \cap \Omega = V \cap \Omega$, which contradicts $\sigma|_U \not\equiv 0$. Hence, the $\partial V \cap U \cap \Omega \neq \emptyset$. To show the second assertion in (A.1), assume that there exists $z \in \partial V \cap \Omega \cap U$ with $z \in O_m$ for one O_m. Then we can choose an open ball $B \subseteq O_m \cap U$ containing z. Since σ is analytic on B and $B \cap V$ has nonempty interior, it follows that $\sigma|_B \equiv 0$, and hence $V \cup B$ has the properties (i)–(iii). This contradicts the maximality of V, so that also the second assertion in (A.1) must hold.

Because of (A.1) we can choose an open ball B with $\overline{B} \subseteq U \cap \Omega$ and

$$\emptyset \neq S := \overline{B} \cap \partial V = \bigcup_{m,k} \overline{\Gamma_m^k} \cap S. \tag{A.2}$$

Using the same arguments as in the proof of Lemma A.3 it follows that by shrinking B we can assume that

$$\emptyset \neq B \cap \partial V \subseteq \Gamma_m^k$$

with a smooth boundary piece Γ_m^k of one O_m.

Since $\partial O_m \subseteq \bigcup_{m' \neq m} \partial O_{m'} \cup \partial\Omega$, (A.2) still holds if we restrict the union to all $m' \in \{1, \ldots, M\} \setminus \{m\}$. By repeating the above argument (and possibly shrinking B again) we obtain $m' \neq m$ with

$$\emptyset \neq \overline{B} \cap \partial V \subseteq \Gamma_m^k \cap \Gamma_{m'}^{k'}.$$

From the definition of smooth boundary pieces it follows that (if we choose B small enough)

$$B \subseteq \overline{(B \cap O_m) \cup B \cap O_{m'}},$$

so that either $B \cap O_m$ or $B \cap O_{m'}$, but not both, intersects V. W.l.o.g., let $B \cap O_m$ intersect V. Then $\sigma|_{O_m} \equiv 0$, and using Lemma A.4 we can shrink B so that $\sigma|_{B \cap O_{m'}}$ is either nonnegative or nonpositive. Hence $B \cup V$ fulfills the above properties (i) and (ii) and it is a proper superset of V. Hence, σ cannot identically vanish on $B \cup V$, which shows that $B \cup V$ fulfills the assertion of Theorem A.1. $\square$

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