First published in Inverse Problems and Imaging in Vol. 2, No.2, 2008,
published by the American Institute of Mathematical Sciences

Inverse Problems and Imaging
Volume 2, No. 2, 2008, 251–269

Web site: http://www.aimSciences.org

LOCALIZED POTENTIALS IN ELECTRICAL IMPEDANCE TOMOGRAPHY

Bastian Gebauer

Institut für Mathematik
Johannes Gutenberg-Universität, 55099 Mainz, Germany

(Communicated by Otmar Scherzer)

Abstract. In this work we study localized electric potentials that have an arbitrarily high energy on some given subset of a domain and low energy on another. We show that such potentials exist for general L^∞_+-conductivities in almost arbitrarily shaped subregions of a domain, as long as these regions are connected to the boundary and a unique continuation principle is satisfied. From this we deduce a simple, but new, theoretical identifiability result for the famous Calderón problem with partial data. We also show how to construct such potentials numerically and use a connection with the factorization method to derive a new non-iterative algorithm for the detection of inclusions in electrical impedance tomography.

1. Introduction.

Consider a steady electric current g that is applied to an open part $S \subseteq \partial B$ of the otherwise insulated surface ∂B of a body $B \subset \mathbb{R}^n$, $n \geq 2$. Denoting by $\sigma(x) > 0$ the spatially dependent conductivity distribution inside the body, the applied currents give rise to an electric potential u, which solves, in the state of equilibrium, the boundary value problem

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } B,$$

with Neumann boundary data $\sigma \partial_\nu u = g$ on S and $\sigma \partial_\nu u = 0$ on $\partial B \setminus S$.

The steady flow of electric currents through the body leads to a permanent absorption of electrical power with the density $\sigma |\nabla u|^2$ heating up the body. In addition to the obvious dependance of the conductivity σ, this energy distribution also strongly depends on the applied currents g. Roughly speaking, one would expect fast spatial variations in the applied currents on S to lead to higher electric currents (and thus higher heating effects) close to the surface, while the heating caused by slowly spatially varying currents penetrates deeper into the body.

We want to study the question whether the shape of this energy distribution can be controlled by the applied currents, i.e., whether we can create *localized potentials* with a high energy in some given part of the body and a low energy in some other part. For the case that $\partial B \setminus S$ is grounded rather than insulated, and the conductivity is smooth, an answer to this question can be found in the classical papers of Kohn and Vogelius [27, 28]. Under the assumption that $\sigma \in C^r$, $r > \frac{n}{2}$, in some neighbourhood of a boundary point z, they construct a series of potentials with rapidly oscillating boundary values whose energy tends to zero outside every neighbourhood of z but not on neighbourhoods containing z. These potentials are

2000 *Mathematics Subject Classification.* 35J25, 35R05, 35R30.
Key words and phrases. Electrical impedance tomography, Calderon problem, factorization method.

First published in Inverse Problems and Imaging in Vol. 2, No.2, 2008,
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easily scaled to have diverging energy around z and Kohn and Vogelius also show how the high energy part can be "shifted" inside the domain B using Runge's approximation property.

Using these localized potentials Kohn and Vogelius show that the set of all current-voltage-pairs measured on S determine, for smooth conductivities, the boundary trace of σ and its derivatives and, for piecewise analytic conductivities, the conductivity also in the interior of the domain B. Using singular potentials, Isakov shows in [21] that local boundary measurements uniquely determine an inclusion with C^2-conductivity in a known C^2-background and Druskin shows uniqueness for piecewise constant conductivities in a half space in [12]. For the case of measuring currents on S for voltages applied on some complementary part, Kenig, Sjöstrand and Uhlmann show unique identifiabilty of C^2-conductivities for $n \geq 3$ in [23], cf. also the preceding work of Bukhgeim and Uhlmann [8]. The unique identifiabilty of C^2-conductivities for local current-to-voltage measurements has been proven only recently by Isakov [22] for the special case of spherical or plane boundaries.

For measurements on the whole boundary, i.e. $S = \partial B$, the identifiability question was studied by Calderón in 1980 [9, 10]. Positive identifiabilty results were obtained under appropriate smoothness conditions in the seminal works of Sylvester and Uhlmann in 1987 [33] for three and higher space dimensions and of Nachmann in 1996 [31] for two space dimensions. Subsequent works reduced the smoothness assumptions and, recently, for two space dimensions the identifiabilty question was answered positively by Astala and Päivärinta [3] without any additional smoothness conditions, i.e., for L^∞-conductivities σ.

In this work we will present a new approach to construct localized potentials that enables us to weaken the regularity assumption on the conductivity σ down to a unique continuation principle (see Theorem 2.7). From this we deduce a simple, but new, identifiabilty result for the local current-to-voltage map. Roughly speaking, we show that we can distinguish between two conductivites when one is larger in some part of the body connected to S irrespective of smoothness (except for the unique continuation principle), or the geometry of the rest of the body, cf. Theorem 3.2 for the rigorous formulation of our identifiability result. Note that this covers the classical Kohn-Vogelius result and is not covered by the recent result of Isakov. On the other hand, using infinitely fast oscillating functions, one easily constructs C^2-conductivities for which our result does not apply. Note also, that our result is based on comparatively simple arguments that can easily be carried over to other real elliptic equations.

Our arguments are closely related to a class of non-iterative reconstruction algorithms known as sampling or factorization methods that have been developed for the special problem of detecting inclusions where a physical parameter differs from an otherwise known background value. The original factorization method was developed by Kirsch in 1998 [24] for inverse scattering problems and has been extended to electrical impedance tomography by Brühl and Hanke [7, 6, 17]. We show that the factorization method can also be interpreted in the sense of localized potentials and derive a new non-iterative reconstruction algorithm that might serve as a slightly less powerful but computationally cheaper alternative.

The outline of this paper is as follows. In Section 2 we proof the existence of localized potentials, and show how they can be constructed. In Section 3, we use these potentials to derive our new identifiability result for the Calderón problem with partial data, give a new interpretation of the factorization method and develop

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a similar non-iterative reconstruction method. Finally, we give some numerical examples for localized potentials and for the new reconstruction method in Section 4.

2. Existence and construction of localized potentials.

As stated in the introduction, the major tool for our new approach to construct localized potentials comes from a relatively new class of non-iterative methods for inverse scattering and diffusive tomography known as *linear sampling* or *factorization methods*. These methods are based on the observation that a subregion Ω of some domain B can be reconstructed from so-called *virtual measurements*, that correspond to applying currents on $\partial\Omega$ from the inside of Ω and measuring the resulting electric potential on the outer boundary of the domain. In this section we introduce a variant of these virtual measurements that have essentially the same properties but correspond to an electric source term on Ω rather than on its boundary $\partial\Omega$ (see also Kirsch [26], where such a variant is used for the factorization method). With the help of these operators we will give two independent proofs for the existence of localized potentials, the first is non-constructive but remarkably short, while the second one is slightly more technical, but constructive.

2.1. Virtual measurements.

We start with some notations. For a real Hilbert spaces H we use round brackets $(\cdot,\cdot)$ for the inner product and angle brackets $\langle\cdot,\cdot\rangle$ for the dual pairing on H and its dual space H'. For an operator $A \in \mathcal{L}(H_1, H_2)$ between real Hilbert spaces H_1 and H_2 we will we use the notation $A' \in \mathcal{L}(H_2', H_1')$ for the dual operator and $A^* \in \mathcal{L}(H_2, H_1)$ for the adjoint operator.

Throughout this work we fix $B \subset \mathbb{R}^n$, $n \geq 2$, to be a bounded domain with smooth boundary and $S \subseteq \partial B$ to be a relatively open subset of the boundary where we can apply electric currents and measure boundary potentials. The outer normal on ∂B is denoted by ν and, for an open subset $\Omega \subseteq B$, we denote by $H^1_\diamond(\Omega)'$ the space of all linear functionals $f \in H^1(\Omega)'$ that vanish on locally constant functions, i.e.,

$$\langle f, v\rangle = 0 \quad \text{for all } v \in H^1(\Omega) \text{ with } \nabla v = 0.$$

Its dual space $H^1_\diamond(\Omega)$ is the quotient space $H^1(\Omega)$ modulo the locally constant functions. If Ω is connected then we can identify $H^1_\diamond(\Omega)$ with the subspace of $H^1(\Omega)$-functions that have vanishing integral mean. Analogously, we define $H^1_\diamond(B)$, $L^2_\diamond(B)$ and $L^2_\diamond(S)$ and identify $L^2_\diamond(B)$ and $L^2_\diamond(S)$ with their duals.

In this work we will always assume without further notice that the conductivity is an isotropic real function $\sigma \in L^\infty_+(B)$, where the subscript $+$ denotes the subset of L^∞-functions with positive essential infima. For most of the results we will also need the following assumption. For an open connected subset $V \subseteq B$ with $S \subseteq \partial V$, we say that σ satisfies the *unique continuation property* (UCP) in V if constant functions are the only solutions of

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } V,$$

that are constant on an open subset of V and only the trivial solution possesses zero Cauchy data $u|_S = 0$ and $\sigma\partial_\nu u|_S = 0$. If σ is Lipschitz continuous then σ fulfills (UCP) in B, cf. e.g. Miranda [30], and by sequentially solving Cauchy problems this can be extended to piecewise Lipschitz continuous conductivities (see also Druskin [12]).

We define the operator of *virtual measurements* L_Ω by

$$L_\Omega :\ H^1_\diamond(\Omega)' \to L^2_\diamond(S), \qquad f \mapsto u|_S,$$

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where $u \in H^1_\diamond(B)$ solves

$$\int_B \sigma \nabla u \cdot \nabla w \, \mathrm{d}x = \langle f, w|_\Omega \rangle \quad \text{for all } w \in H^1_\diamond(B). \tag{1}$$

Note that when $\overline{\Omega} \subset B$ then we can consider $H^1_\diamond(\Omega)'$ as a subset of $H^{-1}(B)$ and (1) is then equivalent to

$$\nabla \cdot \sigma \nabla u = f \quad \text{and} \quad \sigma \partial_\nu u|_{\partial B} = 0.$$

On the other hand, if $\overline{\Omega}$ contains some part of ∂B then $H^1_\diamond(\Omega)'$ has to be considered as a subspace of $H^1(B)'$, so that in general (1) might also contain a non-homogeneous Neumann condition.

Lemma 2.1. *The range of L_Ω does not depend on $\sigma|_\Omega$.*

Proof. Let $\sigma_1, \sigma_2 \in L^\infty_+(B)$ with $\operatorname{supp}(\sigma_1 - \sigma_2) \subseteq \overline{\Omega}$ and denote by $L_\Omega^{(j)}$ the operator of virtual measurements for the conductivity $\sigma = \sigma_j$, $j = 1, 2$. If $\varphi \in \mathcal{R}(L_\Omega^{(1)})$ then there exists $f_1 \in H^1_\diamond(\Omega)'$ and $u_1 \in H^1_\diamond(B)$ such that $\varphi = u_1|_S$ and

$$\int_B \sigma_1 \nabla u_1 \cdot \nabla w \, \mathrm{d}x = \langle f_1, w|_\Omega \rangle \quad \text{for all } w \in H^1_\diamond(B).$$

We define $f_2 \in H^1(\Omega)'$ by setting

$$\langle f_2, w \rangle := \int_\Omega (\sigma_2 - \sigma_1) \nabla u_1 \nabla w \, \mathrm{d}x \quad \text{for all } w \in H^1_\diamond(\Omega),$$

then obviously $f_2 \in H^1_\diamond(\Omega)'$ and

$$\begin{aligned} \int_B \sigma_2 \nabla u_1 \cdot \nabla w \, \mathrm{d}x &= \int_B (\sigma_2 - \sigma_1) \nabla u_1 \cdot \nabla w \, \mathrm{d}x + \int_B \sigma_1 \nabla u_1 \cdot \nabla w \, \mathrm{d}x \\ &= \langle f_2, w|_\Omega \rangle + \langle f_1, w|_\Omega \rangle \end{aligned}$$

for all $w \in H^1_\diamond(B)$ which yields that $\varphi = u_1|_S = L_\Omega^{(2)}(f_1 + f_2)$.

Hence, $\mathcal{R}(L_\Omega^{(1)}) \subseteq \mathcal{R}(L_\Omega^{(2)})$ and the converse follows from interchanging $L_\Omega^{(1)}$ and $L_\Omega^{(2)}$. □

The dual operator of L_Ω

$$L'_\Omega : \; L^2_\diamond(S) \to H^1_\diamond(\Omega)$$

can be characterized as the composition of the solution operator and the restriction to Ω.

Lemma 2.2. *Let $\sigma \in L^\infty_+(B)$ and $\Omega \subset B$ be an open set . For $g \in L^2_\diamond(S)$ let $v \in H^1_\diamond(B)$ be the solution of the Neumann problem*

$$\nabla \cdot \sigma \nabla v = 0 \quad \text{and} \quad \sigma \partial_\nu v|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{on } \partial B \setminus S. \end{cases} \tag{2}$$

Then $L'_\Omega g = v|_\Omega$.

Proof. Let $f \in H^1_\diamond(\Omega)'$, $g \in L^2_\diamond(S)$ and $u, v \in H^1_\diamond(B)$ solve (1), resp., (2). Then

$$\langle f, L'_\Omega g \rangle = (g, L_\Omega f) = \int_B \sigma \nabla v \cdot \nabla u \, \mathrm{d}x = \langle f, v|_\Omega \rangle.$$

□

If σ satisfies (UCP) then L'_Ω is injective for every open subset $\Omega \subset B$ and thus $\mathcal{R}(L_\Omega)$ is dense in $L^2_\diamond(S)$. For two nested open subsets $\Omega_1 \subseteq \Omega_2 \subset B$ the ranges of the corresponding virtual measurements are obviously nested too, i.e., $\mathcal{R}(L_{\Omega_1}) \subseteq \mathcal{R}(L_{\Omega_2})$. The next lemma shows that under appropriate conditions also for two disjoint subsets of B, the ranges of the virtual measurements have no common (non-zero) elements.

Lemma 2.3. *Let $\Omega_1, \Omega_2 \subset B$ be two open sets with $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$. Futhermore let $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ be connected, $\overline{B} \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ contain S, and σ satisfy (UCP) on $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$. Then*

$$\mathcal{R}(L_{\Omega_1}) \cap \mathcal{R}(L_{\Omega_2}) = \{0\}.$$

Proof. The proof is a standard application of the unique continuation principle. If $\varphi \in L^2_\diamond(S)$ belongs to $\mathcal{R}(L_{\Omega_1}) \cap \mathcal{R}(L_{\Omega_2})$, then there exist $u_1, u_2 \in H^1_\diamond(B)$ such that $u_1|_S = u_2|_S = \varphi$ and

$$\int_B \sigma \nabla u_j \cdot \nabla w \, \mathrm{d}x = 0,$$

for all $w \in H^1_\diamond(B)$ with $\operatorname{supp} w \subseteq \overline{B} \setminus \overline{\Omega_j}$, $j = 1, 2$. In particular

$$\nabla \cdot \sigma \nabla u_1 = 0 \quad \text{in } B \setminus \overline{\Omega_1}, \qquad \nabla \cdot \sigma \nabla u_2 = 0 \quad \text{in } B \setminus \overline{\Omega_2}.$$

and $\sigma \partial_\nu u_1|_S = \sigma \partial_\nu u_2|_S = 0$. The unique continuation principle yields that $u_1 = u_2$ in $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$, so the function

$$u := \begin{cases} u_1 & \text{in } B \setminus \overline{\Omega_1}, \\ u_2 & \text{in } B \setminus \overline{\Omega_2}. \end{cases}$$

is well-defined. Since $u \in H^1_\diamond(B)$ satisfies

$$\int_B \sigma \nabla u \cdot \nabla w \, \mathrm{d}x = 0,$$

for all $w \in H^1_\diamond(B)$ with support in $\overline{B} \setminus \overline{\Omega_1}$ and also for all $w \in H^1_\diamond(B)$ with support in $\overline{B} \setminus \overline{\Omega_2}$, it follows that $u = 0$ and thus $\varphi = u|_S = 0$. □

2.2. **Existence of localized potentials.** Our short proof of the existence of localized potentials is based on the following functional analytic lemma that frequently appears in some form or another in applications of the factorization method. We state in the form as it is called the “14th important property of Banach spaces” in Bourbaki [5].

Lemma 2.4. *Let X,Y be two Banach spaces, let $A \in \mathcal{L}(X; Y)$ and $x' \in X'$. Then*

$$x' \in \mathcal{R}(A') \quad \text{if and only if} \quad \exists C > 0 \;:\; |\langle x', x\rangle| \leq C \|Ax\| \;\forall x \in X.$$

Proof. For convenience of the reader we copy our elementary proof from [14].

If $x' \in \mathcal{R}(A')$ then there exists $y' \in Y'$ such that $x' = A'y'$. Thus

$$|\langle x', x\rangle| = |\langle y', Ax\rangle| \leq \|y'\| \, \|Ax\| \;\forall x \in X,$$

and the assertion holds with $C = \|y'\|$.

Now let $x' \in X'$ be such that there exists $C > 0$ with $|\langle x', x\rangle| \leq C\|Ax\|$ for all $x \in X$. Define

$$f(z) := \langle x', x\rangle \text{ for every } z = Ax \in \mathcal{R}(A).$$

First published in Inverse Problems and Imaging in Vol. 2, No.2, 2008,
published by the American Institute of Mathematical Sciences

Then f is a well-defined, continuous linear functional, with $\|f(z)\| \leq C\|z\|$. Using the Hahn-Banach theorem there exists $y' \in Y'$ with $y'|_{\mathcal{R}(A)} = f$. For all $x \in X$ we have

$$\langle A'y', x\rangle = \langle y', Ax\rangle = f(A(x)) = \langle x', x\rangle$$

and thus $x' = A'y' \in \mathcal{R}(A')$. $\square$

From this lemma we deduce that the question whether one operator is bounded by another is related to the ranges of the corresponding dual operators.

Lemma 2.5. *Let X, Y_1 and Y_2 be three reflexive Banach spaces and let $A_i \in \mathcal{L}(Y_i, X)$, $i = 1, 2$. Then*

$$\mathcal{R}(A_1) \subseteq \mathcal{R}(A_2) \quad \textit{if and only if} \quad \exists C > 0 \,:\, \|A_1'x'\| \leq C\|A_2'x'\| \ \forall x' \in X'.$$

Proof. If there exists a $C > 0$ such that $\|A_1'x'\| \leq C\|A_2'x'\|$ for all $x' \in X'$ then $\mathcal{R}(A_1) \subseteq \mathcal{R}(A_2)$ immediately follows from Lemma 2.4.

To prove the converse, let $\mathcal{R}(A_1) \subseteq \mathcal{R}(A_2)$. The restriction of A_2 to an operator from the quotient space $Y_2/\mathcal{N}(A_2)$ to X is injective and has the same range as A_2. Therefore the mapping

$$B : Y_1 \to Y_2/\mathcal{N}(A_2), \quad By_1 := y_2 + \mathcal{N}(A_2), \quad \text{where } y_2 \text{ solves } A_2y_2 = A_1y_1,$$

is well-defined and obviously linear. To show that B is bounded we will apply the closed graph theorem. Let $(y_m^{(1)})_{m\in\mathbb{N}} \subset Y_1$ converge to some $y^{(1)} \in Y_1$ and $(By_m^{(1)})_{m\in\mathbb{N}}$ converge to some $y^{(2)} + \mathcal{N}(A_2) \in Y_2/\mathcal{N}(A_2)$. With the usual identification of $(Y_2/\mathcal{N}(A_2))'$ with $\mathcal{N}(A_2)^\perp = \overline{\mathcal{R}(A_2')}$ (cf., e.g., Rudin [32, Chp. 4]) we obtain for all $x' \in X'$

$$\begin{aligned}\langle A_2'x', y^{(2)} + \mathcal{N}(A_2)\rangle &= \lim_{m\to\infty} \langle A_2'x', By_m^{(1)}\rangle = \lim_{m\to\infty} \langle x', A_1y_m^{(1)}\rangle \\ &= \langle x', A_1y^{(1)}\rangle = \langle A_2'x, By^{(1)}\rangle.\end{aligned}$$

From the denseness of $\mathcal{R}(A_2')$ in $\mathcal{N}(A_2)^\perp$, it follows that $y^{(2)} + \mathcal{N}(A_2) = By^{(1)}$ and thus the continuity of B.

We thus obtain the existence of a continuous dual operator $B' : \overline{\mathcal{R}(A_2)} \to Y_1'$. Since

$$\langle B'A_2'x', y_1\rangle = \langle A_2'x', By_1\rangle = \langle x', A_1y_1\rangle = \langle A_1'x', y_1\rangle$$

holds for all $x' \in X'$, $y_1 \in Y_1$, we obtain that $B'A_2' = A_1'$ and thus

$$\|A_1'x'\| \leq \|B'\|\,\|A_2'x'\| \quad \text{for all } x' \in X'.$$

$\square$

An immediate consequence is the following corollary.

Corollary 2.6. *Let X, Y_1 and Y_2 be three reflexive Banach spaces and let $A_i \in \mathcal{L}(Y_i, X)$, $i = 1, 2$. If $\mathcal{R}(A_1)$ is not a subspace of $\mathcal{R}(A_2)$ then there exists a sequence $(x_m')_{m\in\mathbb{N}} \subset X'$ with*

$$\lim_{m\to\infty} \|A_1'x_m'\|^2 = \infty \quad \textit{and} \quad \lim_{m\to\infty} \|A_2'x_m'\|^2 = 0.$$

We now apply this result to the virtual measurements and obtain:

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published by the American Institute of Mathematical Sciences

Theorem 2.7. *Let $\Omega_1, \Omega_2 \subset B$ be two open sets with $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$. Futhermore let $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ be connected, $\overline{B} \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ contain S, and σ satisfy (UCP) on $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$.*

Then there exists a sequence of currents $(g_m)_{m\in\mathbb{N}} \subseteq L^2_\diamond(S)$ such that the electrical energy of the corresponding potentials $(u_m)_{m\in\mathbb{N}}$, i.e., the solutions of

$$\nabla \cdot \sigma \nabla u_m = 0 \quad \text{and} \quad \sigma \partial_\nu u_m|_{\partial B} = \begin{cases} g_m & \text{on } S, \\ 0 & \text{on } \partial B \setminus S \end{cases}$$

diverges on Ω_1 while tending to zero on Ω_2, i.e.,

$$\lim_{m\to\infty} \int_{\Omega_1} |\nabla u_m|^2 \, \mathrm{d}x = \infty \quad \text{and} \quad \lim_{m\to\infty} \int_{\Omega_2} |\nabla u_m|^2 \, \mathrm{d}x = 0.$$

Proof. Since $u \mapsto (\int_{\Omega_i} |\nabla u|^2 \, \mathrm{d}x)^{1/2}$ defines an equivalent norm on $H^1_\diamond(\Omega_i)$, $i = 1, 2$, the assertion follows from the combination of Lemma 2.3, Lemma 2.2 and Corollary 2.6. □

Theorem 2.7 guarantees that, by applying currents on S, we can generate an electric potential with arbitrarily high energy in any given part Ω_1 of the body B and arbitrarily low energy in some other given part Ω_2. The assumption on the complement $B \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ can be interpreted in the way that there must exist a connection from the boundary part S to the high energy part Ω_1 that does not intersect the low energy part Ω_2.

2.3. **Construction of localized potentials.** In this subsection we will show how to construct the localized potentials of Theorem 2.7. We start by proving a constructive version of Corollary 2.6 in Hilbert spaces.

Lemma 2.8. *Let X, Y_1 and Y_2 be three Hilbert spaces and let $A_i \in \mathcal{L}(Y_i, X)$, $i = 1, 2$. Furthermore we assume that A_2^* is injective.*

If $x \in \mathcal{R}(A_1)$ but $x \notin \mathcal{R}(A_2)$ and $x_\alpha \in X$, $\alpha > 0$, is defined by

$$x_\alpha := \frac{\xi_\alpha}{\|A_2^*\xi_\alpha\|^{\frac{3}{2}}}, \quad \text{where} \quad \xi_\alpha := (A_2A_2^* + \alpha I)^{-1}x,$$

then

$$\lim_{\alpha\to 0} \|A_1^* x_\alpha\| = \infty \quad \text{and} \quad \lim_{\alpha\to 0} \|A_2^* x_\alpha\| = 0.$$

Proof. Let $y \in Y_1$ be such that $x = A_1 y$. Then the scaled Tikhonov approximations $A_2^*\xi_\alpha$ satisfy

$$\begin{aligned} \|A_2^*\xi_\alpha\|^2 &= (A_2A_2^*\xi_\alpha, \xi_\alpha) = (x - \alpha\xi_\alpha, \xi_\alpha) = (x, \xi_\alpha) - \alpha\|\xi_\alpha\|^2 \\ &\le (x, \xi_\alpha) = (y, A_1^*\xi_\alpha) \le \|y\| \, \|A_1^*\xi_\alpha\|. \end{aligned}$$

Since $x \notin \mathcal{R}(A_2)$ in particular implies that $x \neq 0$, it follows that $\xi_\alpha \neq 0$ and $y \neq 0$. We now obtain

$$\|A_1^* x_\alpha\| = \frac{\|A_1^*\xi_\alpha\|}{\|A_2^*\xi_\alpha\|^{\frac{3}{2}}} \ge \frac{\|A_2^*\xi_\alpha\|^{\frac{1}{2}}}{\|y\|}. \tag{3}$$

Hence, the assertion follows if we can show that $\|A_2^*\xi_\alpha\| \to \infty$.

If $\|A_2^*\xi_\alpha\|$ had a bounded subsequence for $\alpha \to 0$ then there would be a bounded subsequence $(A_2^*\xi_{\alpha_m})_{m\in\mathbb{N}}$, $\alpha_m \to 0$, that weakly converges against some $y_2 \in Y_2$.

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But then we would have for all $z = A_2\eta_2 \in \mathcal{R}(A_2)$

$$\begin{aligned}(z, A_2 y_2) &= \lim_{m\to\infty} (z, A_2 A_2^* \xi_{\alpha_m}) = \lim_{m\to\infty} ((z, (A_2 A_2^* + \alpha_m I)\xi_{\alpha_m}) - \alpha_m (z, \xi_{\alpha_m})) \\ &= \lim_{m\to\infty} ((z, x) - \alpha_m(\eta_2, A_2^*\xi_{\alpha_m})) = (z, x)\end{aligned}$$

and since $\mathcal{R}(A_2)$ is dense in X it would follow that $x = A_2 y_2 \in \mathcal{R}(A_2)$ which contradicts our assumptions. Therefore $\|A_2^*\xi_\alpha\|$ cannot have a bounded subsequence which proves $\|A_2^*\xi_\alpha\| \to \infty$ and thus the assertion. □

To apply Lemma 2.8 on our problem of constructing localized potentials, we need a function that belongs to the range of the virtual measurements of some domain Ω_1 but not to that of another domain Ω_2. According to Lemma 2.3 we could take any non-zero function in the range of L_Ω for any subset $\Omega \subseteq \Omega_1$. If σ is constant in a neighbourhood of some point $z \in \Omega_1$ we can even shrink the subset to this point and take the boundary data of the electric potential of a dipole in z with arbitrary direction $d \in \mathbb{R}^3$, $|d| = 1$, i.e., $v_{z,d}|_S$, where $v_{z,d}$ solves

$$\nabla \cdot \sigma \nabla v_{z,d} = d \cdot \nabla \delta_z$$

with homogeneous Neumann boundary condition $\sigma \partial_\nu v_{z,d}|_{\partial B} = 0$. These dipole functions are also used by linear sampling or factorization methods to show that the range of the virtual measurements uniquely determines the corresponding domain (cf. Brühl and Hanke [7, 6, 17]). The shrinkage of a subset to a point is studied in the context of asymptotic factorization methods for the detection of small inclusions, cf., e.g., the recent work of Ammari, Griesmaier and Hanke [1] and the references therein.

Lemma 2.9. *Let σ satisfy (UCP) in B and be constant in a neighbourhood of some point $z \in B$. Furthermore let $\Omega \subseteq B$ be an open set, such that $z \notin \partial\Omega$, $B \setminus \overline{\Omega}$ is connected and $\overline{B} \setminus \overline{\Omega}$ contains S. Then*

$$v_{z,d}|_S \in \mathcal{R}(L_\Omega) \quad \textit{if and only if} \quad z \in \Omega.$$

Proof. For virtual measurement operators that are defined on the subregion's boundary $\partial\Omega$ and constant conductivity $\sigma = 1$ this was shown by Brühl in [6, Lemma 3.5], and in [26, Theorem 4.2] Kirsch shows a similar result for virtual measurement operators that are defined on a subregion rather that on its boundary. For the convenience of the reader we give a short independent proof here.

We first note that $v_{z,d}|_S \neq 0$, since otherwise our unique continuation assumption (UCP) would yield that $v_{z,d}$ vanishes on $B \setminus \{z\}$ which contradicts the fact that $v_{z,d}$ has the same singularity in the point z as the directional derivative of the fundamental solution of the Laplace equation, so that in particular $v_{z,d} \notin L^2(B \setminus \{z\})$.

From Lemma 2.3 we deduce that if $v_{z,d}|_S \in \mathcal{R}(L_\Omega)$ for every Ω that contains z, then it cannot be in the range of some $L_{\Omega'}$ where $z \notin \overline{\Omega'}$. So we only have to show that $v_{z,d}|_S \in \mathcal{R}(L_\Omega)$ for every Ω that contains z. To this end let $\epsilon > 0$ be small enough such that $\overline{B_\epsilon(z)} \subseteq \Omega$ and $u \in H^1(B_\epsilon(z))$ be a function whose trace on $\partial B_\epsilon(z)$ coincides with $v_{z,d}|_{\partial B_\epsilon(z)}$. We use u to replace the singular part of $v_{z,d}$, i.e., we define

$$\tilde{v} := \begin{cases} v_{z,d} & \text{in } B \setminus \overline{B_\epsilon(z)} \\ u & \text{in } B_\epsilon(z). \end{cases}$$

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Since $\tilde{v}$ is piecewise defined by H^1-functions that have the same traces on $\partial B_\epsilon(z)$, it is easily seen that $\tilde{v} \in H^1(B)$; cf., e.g., [15, Section 2.1]. From

$$\nabla \cdot \sigma \nabla \tilde{v} = \nabla \cdot \sigma \nabla v_{z,d} = 0 \quad \text{in } B \setminus \overline{B_\epsilon(z)} \tag{4}$$

we obtain that

$$\int_B \sigma \nabla \tilde{v} \cdot \nabla w \, \mathrm{d}x = \langle f, w|_\Omega \rangle \quad \text{for all } w \in H^1(B),$$

where $f \in H^1(\Omega)'$ is defined by

$$\langle f, w \rangle := \int_{B_\epsilon(z)} \nabla \tilde{v} \cdot \nabla w \, \mathrm{d}x - \langle \sigma \partial_\nu v_{z,d}|_{\partial B_\epsilon(z)}, w|_{\partial B_\epsilon(z)} \rangle \quad \text{for all } w \in H^1(\Omega).$$

From (4) and $\partial_\nu v_{z,d}|_S = 0$ it follows immediately that $\langle \sigma \partial_\nu v_{z,d}|_{\partial B_\epsilon(z)}, 1 \rangle = 0$. We conclude that $f \in H^1_\diamond(\Omega)$ and thus

$$v_{z,d}|_S = \tilde{v}|_S = L_\Omega f \in \mathcal{R}(L_\Omega).$$

□

Combining Lemma 2.8 with Lemma 2.9 we obtain our constructive version of Theorem 2.7.

Theorem 2.10. *Let σ be constant in a neighbourhood of some point $z \in B$ and $\Omega \subseteq B$ be an open set, such that $z \notin \overline{\Omega}$, $B \setminus \overline{\Omega}$ is connected, $\overline{B} \setminus \overline{\Omega}$ contains S and σ satisfies (UCP) in B.*

We define electric currents $(g_\alpha)_{\alpha>0} \subseteq L^2_\diamond(S)$ by setting

$$g_\alpha := \frac{1}{\|L_\Omega^* \gamma_\alpha\|^{3/2}} \gamma_\alpha, \quad \text{where} \quad \gamma_\alpha := (L_\Omega L_\Omega^* + \alpha I)^{-1} v_{z,d}|_S.$$

Then the electrical energy of the corresponding potentials $(u_\alpha)_{\alpha>0}$, i.e., the solutions of

$$\nabla \cdot \sigma \nabla u_\alpha = 0 \quad \text{and} \quad \sigma \partial_\nu u_\alpha|_{\partial B} = \begin{cases} g_\alpha & \text{on } S, \\ 0 & \text{on } \partial B \setminus S, \end{cases}$$

diverges on any neigbourhood of z while tending to zero on Ω, i.e., for any open U with $z \in U$,

$$\lim_{\alpha \to 0} \int_U |\nabla u_\alpha|^2 \, \mathrm{d}x = \infty \quad \text{and} \quad \lim_{\alpha \to 0} \int_\Omega |\nabla u_\alpha|^2 \, \mathrm{d}x = 0.$$

Proof. Lemma 2.9 yields that $v_{z,d}|_S \notin \mathcal{R}(L_\Omega)$ and $v_{z,d}|_S \in \mathcal{R}(L_U)$ for any open $U \subseteq B$ containing z. The assertion thus follows from Lemma 2.8. □

We finish this section with an interpretation of the operator $L_\Omega L_\Omega^*$.

Remark 2.11. The operator

$$L_\Omega L_\Omega^* : \ L^2_\diamond(S) \to L^2_\diamond(S)$$

maps an applied current $g \in L^2_\diamond(S)$ to the boundary values $u|_S$ of a solution $u \in H^1_\diamond(B)$ of

$$\int_B \sigma \nabla u \cdot \nabla w \, \mathrm{d}x = \int_\Omega \nabla v \cdot \nabla w \, \mathrm{d}x \quad \text{for all } w \in H^1_\diamond(B), \tag{5}$$

where $v \in H^1_\diamond(B)$ solves

$$\nabla \cdot \sigma \nabla v = 0 \quad \text{and} \quad \sigma \partial_\nu v|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{on } \partial B \setminus S. \end{cases}$$

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published by the American Institute of Mathematical Sciences

If $\overline{\Omega} \subset B$ then (5) is equivalent to

$$\nabla \cdot \sigma \nabla u = \nabla \cdot \chi_\Omega \nabla v \quad \text{and} \quad \sigma \partial_\nu u|_{\partial B} = 0.$$

The analogous equation

$$\nabla \cdot \sigma_1 \nabla (u_0 - u_1) = \nabla \cdot \chi_\Omega \nabla u_0 \quad \text{and} \quad \sigma \partial_\nu u|_{\partial B} = 0.$$

also determines the difference $u_0 - u_1$ of electric potentials $u_0, u_1 \in H^1_\diamond(B)$ that result from an applied current g in a body B with conductivity $\sigma_0 = \sigma$, resp., $\sigma_1 = \sigma + \chi_\Omega$. Such differences appear in the study of the factorization method, where the current-to-voltage-mapping of a homogenous body is compared to that of a body which contains an inclusion Ω; cf., e.g., [16, Sect. 3]. We will use this connection in Section 3.2 to develop a new interpretation of the factorization method and to derive a related new reconstruction algorithm.

3. Applications of localized potentials.

3.1. An identifiability result.

The existence of localized potentials has a consequence for the famous Calderón problem, that is, to the question whether the conductivity σ is uniquely determined by the Neumann-to-Dirichlet (or current-to-voltage) map

$$\Lambda_\sigma : \; L^2_\diamond(S) \to L^2_\diamond(S), \quad \Lambda_\sigma g := u|_S,$$

where $u \in H^1_\diamond(B)$ solves

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{and} \quad \sigma \partial_\nu u|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{on } \partial B \setminus S. \end{cases} \tag{6}$$

Note that in this work we consider *local* Neumann-to-Dirichlet operators; the measurements are taken on the (arbitrarily small) open part S of the boundary B. The Neumann-to-Dirichlet operators satisfy the following well-known monotonicity property; cf., e.g., [6, Lemma 2.1] for a similar result.

Lemma 3.1. *Let $\sigma_1, \sigma_2 \in L^\infty_+(B)$ and $\Lambda_1, \Lambda_2 : \; L^2_\diamond(S) \to L^2_\diamond(S)$ be the Neumann-to-Dirichlet operators corresponding to the conductivity $\sigma = \sigma_1$, resp., $\sigma = \sigma_2$. Then*

$$\int_B (\sigma_1 - \sigma_2) |\nabla u_2|^2 \, \mathrm{d}x \geq ((\Lambda_2 - \Lambda_1) g, g) \geq \int_B (\sigma_1 - \sigma_2) |\nabla u_1|^2 \, \mathrm{d}x, \tag{7}$$

for all $g \in L^2_\diamond(S)$, where u_1, resp., u_2 are the solutions of (6) with $\sigma = \sigma_1$, resp., $\sigma = \sigma_2$.

Proof. We proceed as in the work of Brühl [6, Lemma 2.1]; cf. also [15, Lemma 3.3] or [14, Lemma 3.3]. Let $g \in L^2_\diamond(S)$ and $u_1, u_2 \in H^1_\diamond(B)$ be the potentials corresponding to the conductivity $\sigma = \sigma_1$, resp., $\sigma = \sigma_2$. From the Lax-Milgram-Theorem (cf., e.g., Dautray and Lions [11, Chp. VII, §1, Remark 3]) it follows that u_1 minimizes the functional

$$w \mapsto \int_B \sigma_1 |\nabla w|^2 \, \mathrm{d}x - 2 \int_S g \, w|_S \, \mathrm{d}s$$

First published in Inverse Problems and Imaging in Vol. 2, No.2, 2008,
published by the American Institute of Mathematical Sciences

in $H^1_\diamond(B)$, so that

$$\begin{aligned}
-\int_B \sigma_1 |\nabla u_1|^2 \,\mathrm{d}x &= \int_B \sigma_1 |\nabla u_1|^2 \,\mathrm{d}x - 2\int_S g\, u_1|_S \,\mathrm{d}s \\
&\leq \int_B \sigma_1 |\nabla u_2|^2 \,\mathrm{d}x - 2\int_S g\, u_2|_S \,\mathrm{d}s \\
&= \int_B \sigma_1 |\nabla u_2|^2 \,\mathrm{d}x - 2\int_B \sigma_2 |\nabla u_2|^2 \,\mathrm{d}x,
\end{aligned}$$

and thus

$$\begin{aligned}
((\Lambda_2 - \Lambda_1)g, g) &= \int_B \sigma_2 |\nabla u_2|^2 \,\mathrm{d}x - \int_B \sigma_1 |\nabla u_1|^2 \,\mathrm{d}x \\
&\leq \int_B \sigma_2 |\nabla u_2|^2 \,\mathrm{d}x + \int_B \sigma_1 |\nabla u_2|^2 \,\mathrm{d}x - 2\int_B \sigma_2 |\nabla u_2|^2 \,\mathrm{d}x \\
&= \int_B (\sigma_1 - \sigma_2) |\nabla u_2|^2 \,\mathrm{d}x.
\end{aligned}$$

This yields the first inequality in (7). The second inequality then follows from interchanging Λ_1 and Λ_2. □

Before we rigorously state our identifiability result, let us give an intuitive derivation of it. Consider the second inequality in Lemma 3.1,

$$((\Lambda_2 - \Lambda_1)g, g) \geq \int_B (\sigma_1 - \sigma_2) |\nabla u_1|^2 \,\mathrm{d}x. \tag{8}$$

Using localized potentials we can make the energy $|\nabla u_1|^2$ arbitrarily large on any part of any neighbourhood of the measurement area S while staying arbitrarily small outside this neighbourhood. This means, that if σ_1 is larger than σ_2 in some part U of B and this part can be connected to S by a neighbourhood V where $\sigma_1 - \sigma_2$ does not change its sign, i.e., is not negative, then a potential u_1 with high energy in U and low energy outside V, will lead to a positive integral on the right hand side of (8), so that in particular $\Lambda_1 \neq \Lambda_2$. In other words, a region of higher conductivity that can be connected to the measurement area without crossing a region of lower conductivity can always be detected from the knowledge of the Neumann-to-Dirichlet-operator.

Theorem 3.2. *Let $\sigma_1, \sigma_2 \in L^\infty_+(B)$ and Λ_1, Λ_2 be the corresponding Neumann-to-Dirichlet maps. Assume that there exists an open, connected set V, with $S \subseteq \partial V$, on which $\sigma_2 \geq \sigma_1$ and σ_1 satisfies the unique continuation property (UCP) in V.*

If furthermore $\sigma_2|_U - \sigma_1|_U \in L^\infty_+(U)$ on an open subset $U \subseteq V$, then there exists a sequence $(g_m)_{m\in\mathbb{N}} \subset L^2_\diamond(S)$ such that

$$((\Lambda_2 - \Lambda_1)g_m, g_m) \to \infty.$$

In particular $\Lambda_1 \neq \Lambda_2$.

Proof. Without loss of generality we can assume that U is an open ball, with $\overline{U} \subset V$, so that $V \setminus \overline{U}$ is connected, its closure contains S and σ_1 satisfies the unique continuation property (UCP) in $V \setminus \overline{U}$. Now, we apply Theorem 2.7 with $\sigma := \sigma_1$, $\Omega_1 := U$ and $\Omega_2 := B \setminus \overline{V}$ to obtain a sequence of currents $(g_m)_{m\in\mathbb{N}} \subset L^2_\diamond(S)$ such that the corresponding solutions $u_m \in H^1_\diamond(B)$ of

$$\nabla \cdot \sigma_1 \nabla u_m = 0 \quad \text{and} \quad \sigma_1 \partial_\nu u_m|_{\partial B} = \begin{cases} g_m & \text{on } S, \\ 0 & \text{on } \partial B \setminus S. \end{cases}$$

satisfy

$$\lim_{m\to\infty}\int_U |\nabla u_m|^2\,\mathrm{d}x = \infty \quad \text{and} \quad \lim_{m\to\infty}\int_{B\setminus\overline{V}} |\nabla u_m|^2\,\mathrm{d}x = 0.$$

Then the assertion follows from Lemma 3.1. □

Note that Theorem 3.2 yields in particular that piecewise analytic conductivities are uniquely determined by the local Neumann-to-Dirichlet-map. We would also like to stress that our arguments solely rely on the unique continuation principle and the ellipticity of the equations of impedance tomography. Thus, analogous results can be achieved for similar real elliptic problems, e.g., for linear elasticity, electro- or magnetostatics.

3.2. **Detecting inclusions in EIT.** In Remark 2.11 we already observed a connection between localized potentials and the factorization method. We will now study this connection in more detail and derive a new interpretation of the factorization method as well as a related new reconstruction algorithm.

Consider the specific task of detecting an inclusion Ω inside a body B with an otherwise homogenous conductivity. We assume that the inclusion Ω is an open set with $\overline{\Omega}\subseteq B$ and that the conductivity in the body is given by $\sigma_1 = 1+\kappa(x)\chi_\Omega(x)$ with $\kappa\in L^\infty_+(\Omega)$. The factorization method determines Ω from comparing the corresponding Neumann-to-Dirichlet operator Λ_1 with that of a reference body with constant conductivity $\sigma_0 = 1$, denoted as Λ_0. Their difference $\Lambda := \Lambda_0 - \Lambda_1$ can be related to the virtual measurements L_Ω that we define as in Section 2.1 with respect to the homogenous conductivity $\sigma = \sigma_0$. (Note that the results of this section would also stay valid for inhomogenous σ_0 or negative perturbations κ).

Lemma 3.3. *There exist $c, C > 0$ such that*

$$c\|L_\Omega^* g\|^2 \leq (\Lambda g, g) \leq C\|L_\Omega^* g\|^2 \quad \textit{for all } g\in L^2_\diamond(S).$$

Proof. Denote by $u_0, u_1 \in H^1_\diamond(B)$ the solutions of (6) for the conductivity σ_0, resp., σ_1. From Lemma 3.1 we obtain with $c' := \inf\kappa$ and $C' := \sup\kappa$ that

$$c'\int_\Omega |\nabla u_1|^2\,\mathrm{d}x \leq (\Lambda g, g) \leq C'\int_\Omega |\nabla u_0|^2\,\mathrm{d}x.$$

Since Lemma 2.1 and Lemma 2.5 yield that $\int_\Omega |\nabla u_1|^2\,\mathrm{d}x$ and $\|L_\Omega^* g\|^2 = \int_\Omega |\nabla u_0|^2\,\mathrm{d}x$ are (up to multiplicative constants) bounded by each other, the assertion follows. □

Since Λ is positive and symmetric, there exists a positive and symmetric square root $\Lambda^{1/2}$ and Lemma 3.3 can be restated in the form that $\|\Lambda^{1/2} g\|$ and $\|L_\Omega^* g\|$ are bounded by each other, up to multiplicative constants. Thus $\mathcal{R}(\Lambda^{1/2}) = \mathcal{R}(L_\Omega)$ by Lemma 2.5, and we obtain, together with Lemma 2.9, the key result of the factorization method.

Corollary 3.4. *For every $z\notin\partial\Omega$ and arbitrary direction $d\in\mathbb{R}^n$*

$$v_{z,d}|_S \in \mathcal{R}(L_\Omega^*) = \mathcal{R}(\Lambda^{1/2}) \quad \textit{if and only if } z\in\Omega,$$

where $v_{z,d}$ is the electric dipole potential introduced in Section 2.3 for $\sigma=\sigma_0$, i.e., the solution of

$$\Delta v_{z,d} = d\cdot\nabla\delta_z, \qquad \partial_\nu v_{z,d}|_{\partial B} = 0.$$

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Such a characterization was originally proven for inverse scattering problems by Kirsch in [24]. For the case of impedance tomography that we consider here it is proven in Brühl [6], cf. also the works of Brühl and Hanke [7, 17], the related work on the halfspace case by Hanke and Schappel [18], on electrode models by Hyvönen [19], Hyvönen, Hakula and Pursiainen [20], and Lechleiter, Hyvönen and Hakula [29]. Generalizations to more general elliptic equations are developed in Kirsch [25] and [15], irregularly bounded inclusions and smooth conductivity deviations are treated in [16], and Azzouz, Hanke, Oesterlein and Schilcher apply the method to real EIT data in [4].

Corollary 3.4 provides a binary criterion to decide whether a point belongs to the inclusion or not from knowledge of the measurements Λ. It is usually implemented by constructing regularized approximations ψ_α to the solution of

$$\Lambda^{1/2}\psi = v_{z,d}|_S \tag{9}$$

and checking whether the norm of ψ_α stays bounded for $\alpha \to 0$ (implying that (9) has a solution, i.e., that $v_{z,d}|_S \in \mathcal{R}(\Lambda^{1/2})$), or not (implying that $v_{z,d}|_S \notin \mathcal{R}(\Lambda^{1/2})$).

The square root operator $\Lambda^{1/2}$ seems to have no natural physical interpretation; one would rather work with regularized approximations g_α to the equation $\Lambda g = v_{z,d}|_S$. If we use Tikhonov regularization and define

$$g^\alpha_{z,d} := (\Lambda^*\Lambda + \alpha I)^{-1}\Lambda^* v_{z,d}|_S, \tag{10}$$

then we can interpret $g^\alpha_{z,d}$ as (regularized approximations to) an applied current for which the difference of the corresponding potentials u_0 and u_1 looks like that of a dipole in z. Using these currents we obtain the following characterization result.

Lemma 3.5. *For every $z \notin \partial\Omega$ and arbitrary direction $d \in \mathbb{R}^n$, $\|\Lambda^{1/2} g^\alpha_{z,d}\|$ stays bounded if and only if $z \in \Omega$.*

Proof. Note that Λ is a compact, symmetric and injective operator. Thus, it has dense range and also $\Lambda^{1/2}$ is injective.

For $z \notin \partial\Omega$, Corollary 3.4 yields that $z \in \Omega$ is equivalent to $v_{z,d}|_S \in \mathcal{R}(\Lambda^{1/2})$. From the injectivity of $\Lambda^{1/2}$, it follows that this is equivalent to $\Lambda^{1/2} v_{z,d}|_S \in \mathcal{R}(\Lambda)$.

Since $\Lambda^{1/2}$ commutes with $(\Lambda^*\Lambda + \alpha I)^{-1}\Lambda^*$, the functions $\Lambda^{1/2} g^\alpha_{z,d}$ are simply the Tikhonov regularized preimages of $\Lambda^{1/2} v_{z,d}$ with respect to the operator Λ. Thus we obtain from classical theory on Tikhonov regularization (cf., e.g., Engl, Hanke and Neubauer [13, Sect. 5.1]) that $\Lambda^{1/2} g^\alpha_{z,d}$ converges against $\Lambda^{-1}\Lambda^{1/2} v_{z,d}$ for $\Lambda^{1/2} v_{z,d} \in \mathcal{R}(\Lambda)$ and (cf., e.g., [13, Prop. 3.6]) that $\|\Lambda^{1/2} g^\alpha_{z,d}\| \to \infty$ for $\Lambda^{1/2} v_{z,d} \notin \mathcal{R}(\Lambda)$. □

Since, by Lemma 3.3, the boundedness of $\|\Lambda^{1/2} g\|$ is equivalent to that of $\|L^*_\Omega g\|$, the criterion in Lemma 3.5 allows the interpretation that (the energy of) the homogeneous potentials $u^\alpha_{z,d}$ created by the currents $g^\alpha_{z,d}$ diverges on the unknown inclusion if $z \notin \Omega$, while staying bounded for $z \in \Omega$. The next theorem shows that it even suffices to examine the energy in the point z and that $u^\alpha_{z,d}$ can be interpreted as localized potentials.

Theorem 3.6. *Let $d \in \mathbb{R}^n$ be an arbitrary direction, for every $z \in \Omega$ we define $g^\alpha_{z,d}$ by (10) and let $u^\alpha_{z,d} \in H^1_\diamond(B)$ be the corresponding homogeneous potentials, i.e., the solutions of*

$$\Delta u^\alpha_{z,d} = 0 \quad \text{and} \quad \partial_\nu u^\alpha_{z,d}|_{\partial B} = \begin{cases} g^\alpha_{z,d} & \text{on } S, \\ 0 & \text{on } \partial B \setminus S. \end{cases}$$

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(a) *For every* $z \notin \partial\Omega$,

$$|\nabla u^\alpha_{z,d}(z)| \to \infty \quad \textit{if and only if} \quad z \notin \Omega.$$

(b) *For every* $z \notin \overline{\Omega}$ *the energy of the normalized potential*

$$x \mapsto \frac{u^\alpha_{z,d}(x)}{(\Lambda g^\alpha_{z,d}, g^\alpha_{z,d})^{3/2}}$$

diverges in the point z *while tending to zero on* Ω. *More precisely,*

$$\frac{|\nabla u^\alpha_{z,d}(z)|}{(\Lambda g^\alpha_{z,d}, g^\alpha_{z,d})^{3/2}} \to \infty \quad \textit{and} \quad \int_\Omega \frac{|\nabla u^\alpha_{z,d}(x)|^2}{\left(\Lambda g^\alpha_{z,d}, g^\alpha_{z,d}\right)^{3/2}}\, \mathrm{d}x \to 0 \quad \textit{for } \alpha \to 0.$$

Proof. From standard regularity theory for the Laplace equation it follows that $u^\alpha_{z,d}$ is smooth in a neighbourhood of z and, if $z \in \Omega$, the evaluation $\nabla u^\alpha_{z,d}(z)$ depends continuously on $\nabla u^\alpha_{z,d}|_\Omega \in L^2(\Omega)^n$ and thus on $L^*_\Omega g^\alpha_{z,d}$. In particular, $|\nabla u^\alpha_{z,d}(z)|$ stays bounded if $z \in \Omega$.

To prove the converse we apply Lemma 3.3 and obtain a $C > 0$ such that

$$\begin{aligned}\frac{1}{C}\|L^*_\Omega g^\alpha_{z,d}\|^2 &\leq (\Lambda g^\alpha_{z,d}, g^\alpha_{z,d}) = \left(\Lambda\Lambda^*(\Lambda\Lambda^* + \alpha I)^{-1} v_{z,d}|_S, g^\alpha_{z,d}\right)\\ &= \left(v_{z,d}|_S, g^\alpha_{z,d}\right) - \alpha((\Lambda\Lambda^* + \alpha I)^{-1} v_{z,d}|_S, \Lambda^*(\Lambda\Lambda^* + \alpha I)^{-1} v_{z,d}|_S)\\ &\leq \left(v_{z,d}|_S, g^\alpha_{z,d}\right).\end{aligned}$$

For the last term one can show that

$$\left(v_{z,d}|_S, g^\alpha_{z,d}\right) = d \cdot \nabla u^\alpha_{z,d}(z).$$

Formally, this is just Green's formula; rigorously, it follows from approximating $v_{z,d}$ by a regularized sequence and using that $u^\alpha_{z,d}$ is smooth in a neighbourhood of z. Thus, we obtain

$$|\nabla u^\alpha_{z,d}(z)| \geq \|L^*_\Omega g^\alpha_{z,d}\|^2, \tag{11}$$

which yields that $|\nabla u^\alpha_{z,d}(z)| \to \infty$ for $z \notin \overline{\Omega}$.

Assertion b) follows from (11) and Lemma 3.3. □

For the inverse problem of detecting Ω from the boundary measurements Λ, Theorem 3.6 contains two different reconstruction strategies. The first one consists in calculating $u^\alpha_{z,d}$ *for every* z in B, and checking for each z whether the energy of the corresponding homogeneous potential $|\nabla u^\alpha_{z,d}(z)|$, *evaluated in the same point* z, tends to infinity or not. This is essentially the factorization method, however, the term $|\nabla u^\alpha_{z,d}(z)|$ is physically interpretable and does not utilize the square root operator $\Lambda^{1/2}$. For inverse scattering problems an analogous criterion was already proposed by Arens in [2].

The new second method is to calculate $u^\alpha_{z,d}$ *for a single* z in B, that is known to lie outside the inclusion, and to check for every $x \in B$ whether (after the normalization in Theorem 3.6(b)) $|\nabla u^\alpha_{z,d}(x)|$ tends to infinity or not. The advantage of this method is that the dipole potential has to be calculated only once and (10) has to be solved only for a single z. The disadvantage is that Theorem 3.6(b) does not guarantee that (after normalization) $|\nabla u^\alpha_{z,d}(x)|$ diverges in every point x outside the inclusion, so that the second method may find a larger set than Ω. However, our numerical experiments in the next section suggest that this is still enough to obtain a quick rough estimate of the inclusion's location.

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published by the American Institute of Mathematical Sciences

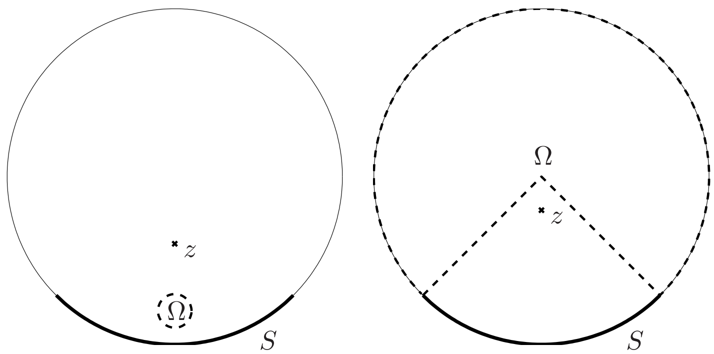

Figure 1. Geometry of the two examples.

4. **Numerical examples.** In this section we will consider some numerical examples to demonstrate the construction of localized potentials in Theorem 2.10 and for the new reconstruction strategy from Theorem 3.6. We will restrict ourselves to simple cases that will serve as a *proof of concept*. Detailed studies of the numerical construction of localized potentials and of the performance of the new reconstruction strategy are beyond the scope of this article and will be the subject of further work. In all cases B is the two-dimensional unit circle and $S \subset \partial B$ is the subset of all points on the boundary with angle $\varphi \in (\frac{5}{4}\pi, \frac{7}{4}\pi)$.

4.1. **Construction of localized potentials.** Figure 1 shows two examples of a set Ω and a point $z \in B$ for which we try to find an electric potential in a homogenous body ($\sigma = 1$) that is large in z but stays small in Ω. According to Theorem 2.10 such a potential is created by the input current

$$g_{z,d}^{\alpha} := \frac{1}{\|L_\Omega^* \gamma_\alpha\|^{3/2}} \gamma_\alpha, \quad \text{where} \quad \gamma_\alpha := (L_\Omega L_\Omega^* + \alpha I)^{-1} v_{z,d}|_S, \tag{12}$$

for small $\alpha > 0$. The numerical implementation of $L_\Omega L_\Omega^*$ is done similar as in [16]. According to Remark 2.11, $L_\Omega L_\Omega^*$ maps an input current $g \in L^2_\diamond(S)$ to the boundary values $u|_S$ of the solution $u \in H^1_\diamond(B)$ of

$$\int_B \nabla u \cdot \nabla w \, \mathrm{d}x = \int_\Omega \nabla v \cdot \nabla w \, \mathrm{d}x \quad \text{for all } w \in H^1_\diamond(B), \tag{13}$$

where $v \in H^1_\diamond(B)$ solves

$$\Delta v = 0 \quad \text{and} \quad \partial_\nu v|_{\partial B} = \begin{cases} g & \text{on } S, \\ 0 & \text{on } \partial B \setminus S. \end{cases} \tag{14}$$

As input currents we apply on ∂B the $L^2_\diamond(S)$-orthonormal functions

$$\left\{ \frac{2}{\sqrt{\pi}} \cos(4k\varphi - 5\pi)\chi_S, \frac{2}{\sqrt{\pi}} \sin(4k\varphi - 5\pi)\chi_S \;\middle|\; k = 1, \ldots, 128 \right\}, \tag{15}$$

where here and in the following (r, φ) denotes the polar coordinates. These currents are expanded into $L^2_\diamond(B)$-orthonormal basis functions

$$\left\{ \frac{1}{\sqrt{\pi}} \cos(m\varphi), \frac{1}{\sqrt{\pi}} \sin(m\varphi) \;\middle|\; m = 1, \ldots, 512 \right\},$$

First published in Inverse Problems and Imaging in Vol. 2, No.2, 2008,
published by the American Institute of Mathematical Sciences

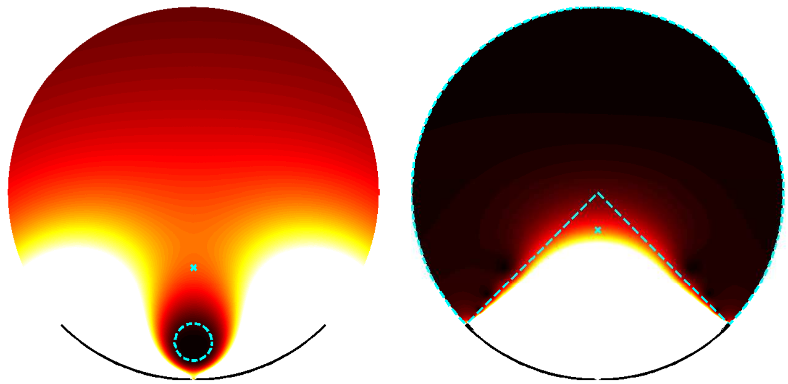

FIGURE 2. Localized potentials for the two examples.

for which the solution v of (14) is explicitly known,

$$v = \frac{1}{m\sqrt{\pi}} \cos(m\varphi) r^m, \quad \text{resp.,} \quad v = \frac{1}{m\sqrt{\pi}} \sin(m\varphi) r^m.$$

Using these exact expressions for v we compute u by solving (13) with the commercial finite element software Comsol and expand $u|_S$ into the orthonormal functions given in (15). The boundary data of the dipole function $v_{z,d}|_S$ can be written as (cf., e.g., Brühl and Hanke [6])

$$v_{z,d}(x) = \frac{1}{\pi} \frac{(z-x) \cdot d}{|z-x|^2}, \quad \text{for all } x \in \partial B.$$

$v_{z,d}|_S$ is then also expanded into the orthonormal functions in (15). Thus we obtain a discrete approximation $M \in \mathbb{R}^{256,256}$ of $L_\Omega L_\Omega^*$ and a discrete approximation $V_{z,d} \in \mathbb{R}^{256}$ of $v_{z,d}|_S$. Applying (12) we now calculate a discrete approximation $G_{z,d}^\alpha \in \mathbb{R}^{256}$ of $g_{z,d}^\alpha$:

$$G_{z,d}^\alpha := \frac{1}{(\Gamma_\alpha^* M \Gamma_\alpha)^{3/4}} \Gamma_\alpha, \quad \text{where} \quad \Gamma_\alpha := (M + \alpha I)^{-1} V_{z,d}.$$

If there was no discretization error then the potentials $u_{z,d}^\alpha$ created by the input currents corresponding to $G_{z,d}^\alpha$ would diverge in z while tending to zero on Ω for $\alpha \to 0$. However, since L_Ω is a compact operator, a smaller α also leads to a stronger amplification of discretization errors, so that the result will worsen when α is chosen too small. For our examples we choose α by hand and set $d := (1,0)^T$. Figure 2 shows the energy $|\nabla u_{z,d}^\alpha|$ of the resulting potentials for the two examples. The color axes are truncated in order to obtain a good color contrast in the interesting regions of the plots. More precisely, every energy value that is more than twice as high as the energy in the point z is plotted with the brightest (white) color. The point z, resp., the set Ω are plotted with a cyan cross, resp., a cyan dashed line. The potentials clearly show the theoretically expected behaviour and attain large energy values around the point z while staying small in the set Ω.

4.2. **Reconstruction of inclusions.** We now turn to the new reconstruction strategy from Theorem 3.6(b). Consider the set Ω that consists of the two inclusions plotted in the top left position of Figure 3. We assume that the conductivity equals one outside Ω and equals two inside Ω. As in Section 3.2 we denote by Λ_1 the