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# A sampling method for detecting buried objects using electromagnetic scattering

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**Abstract**

We consider a simple (but fully three-dimensional) mathematical model for the electromagnetic exploration of buried, perfect electrically conducting objects within the soil underground. Moving an electric device parallel to the ground at constant height in order to generate a magnetic field, we measure the induced magnetic field within the device, and factor the underlying mathematics into a product of three operations which correspond to the primary excitation, some kind of reflection on the surface of the buried object(s) and the corresponding secondary excitation, respectively. Using this factorization we are able to give a justification of the so-called sampling method from inverse scattering theory for this particular set-up.

(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

The exploration of the ground's subsurface to detect and identify buried objects is an important task in various applications, for example, the removal of buried land mines. In this context, a standard technique consists of moving hand-held metal detectors based on electromagnetic sensors above a certain area of interest. It is the aim of this paper to enhance the viability of this approach using advanced imaging techniques based on methods from inverse scattering and mathematical tomography.

The algorithms that we have in mind belong to a class of comparatively new noniterative methods, now known as *factorization* or *sampling methods*. Basically, these methods make use of some sort of symmetric or, even better, self-adjoint factorization

$$M = LFL^T \tag{1.1}$$

of some (measurement) operator $M$ which takes the primary excitations on the surface of what we shall call the *device* further on, and maps them onto corresponding measurements of some secondary fields measured with the same device. The operators $L^T$ and $L$ can be considered to propagate the fields from the device to the scatterer and vice versa, whereas $F$ represents some kind of refraction of the field at the boundary of the scatterer.

The operators $L$ and $L^T$ are dual or even adjoint to each other, depending on the particular application (the so-called *reciprocity laws* are the physical reason). They encode—in a sense to be made precise below—the distance between the device and the scatterer. If it is possible to characterize the range of $L$ numerically using the given range of $M$, then one has ideal prerequisites to reconstruct the domain occupied by the scattering object: for which one can generate synthetic measurements corresponding to an infinitely strong point scatterer at some point $z$, and these data belong to the range of $L$, if and only if $z$ is located within the scatterer or at most on its boundary. This is the basic argument employed by the so-called *factorization method* developed in [13], which has since been applied to a variety of different applications, cf, e.g. the papers [10, 11, 14, 15, 18], and the many references therein.

A somewhat more immediate consequence of the factorization (1.1) is the obvious property that the range of $M$ is always contained in the range of $L$. While this does not suffice to numerically characterize the range of $L$, it nevertheless allows us to conclude that some point $z$ belongs to the domain of the scatterer if the aforementioned synthetic measurement data belong to the range of $M$. However, no conclusion is possible when the latter fails to hold. Methods using this approach originated with the paper [5] and are now called *sampling methods*, cf, e.g., [2–4, 7] for more recent contributions.

The particular sampling method that we are going to analyse goes back to a method suggested by Coyle [7]: its main features are

- a *two-layered background* medium containing the scatterer, and
- the use of *local near-field data* rather than the far-field operator as in other works.

In contrast to Coyle, however, we do not restrict our attention to a two-dimensional approximation (TM-mode) of this problem, but consider the full three-dimensional (time-harmonic) Maxwell equations. We have to stress, though, that our interpretation of sampling methods, and the formal justification that we have outlined above, differ from the usual presentations.

In this work we restrict ourselves to the case, where the buried object is a perfect electrical conductor. We are, however, convinced that our results can be extended to penetrable objects. This will be the subject of a forthcoming publication.

The outline of this paper is as follows. In section 2, we set up the model that we are going to use and we introduce the corresponding (near-field) measurement operator $M$. Basic properties of $M$, i.e. symmetry and injectivity, are investigated in section 3. We then continue and derive a factorization (1.1) with certain operators $L$ and $L^T$ to be defined in section 4. The factorization itself is the main result of section 5. Section 6 recalls Coyle's sampling method and gives a formal justification of this method for our particular setting. Preliminary numerical results (for a homogeneous space without layers) follow in section 7. The paper concludes with an outlook on future work.

## 2. The mathematical setting

We decompose the space $\mathbb{R}^3 = \mathbb{R}^3_+ \cup \Sigma_0 \cup \mathbb{R}^3_-$ in a hyperplane $\Sigma_0$ of $\mathbb{R}^3$ corresponding to the surface of the ground, and the two half spaces $\mathbb{R}^3_\pm$ above and below $\Sigma_0$ representing air and ground, respectively. We assume that both half spaces are filled with homogeneous materials with dielectricity $\varepsilon$ and permeability $\mu$ given by

$$\varepsilon(x) = \begin{cases} \varepsilon_+, & x \in \mathbb{R}^3_+, \\ \varepsilon_-, & x \in \mathbb{R}^3_-, \end{cases} \qquad \mu(x) = \begin{cases} \mu_+, & x \in \mathbb{R}^3_+, \\ \mu_-, & x \in \mathbb{R}^3_-. \end{cases}$$

We require that $\varepsilon_+$ as well as $\mu_\pm$ are positive numbers, whereas $\varepsilon_-$ may be complex with positive real and nonnegative imaginary parts to allow for soil materials that are conducting.

Throughout the paper, we investigate radiating solutions of the time-harmonic Maxwell system

$$\operatorname{curl} H = -\mathrm{i}\omega\varepsilon E, \qquad \operatorname{curl} E = \mathrm{i}\omega\mu H \tag{2.1}$$

in the exterior of some compact set $B \subset \mathbb{R}^3$. By this we understand (cf, e.g., Cutzach and Hazard [8] or Monk [19]) solutions $E, H \in H_{\mathrm{loc}}(\mathrm{curl}; \mathbb{R}^3 \setminus \overline{B})$[3] which obey the radiation condition

$$\int_{\partial B_r} \left| \frac{x}{r} \times H + (\varepsilon/\mu)^{1/2} E \right|^2 \mathrm{d}\sigma = o(1) \qquad \text{as } r \to \infty, \tag{2.2}$$

where $B_r$ denotes the ball of radius $r > 0$ around the origin, and

$$\kappa = \omega\sqrt{\varepsilon\mu} = \begin{cases} \kappa_+, & x \in \mathbb{R}^3_+, \\ \kappa_-, & x \in \mathbb{R}^3_-, \end{cases}$$

is the associated (discontinuous) wave number. If $\varepsilon_- \notin \mathbb{R}$ then $\kappa$ is taken to have positive imaginary part. We mention that $E, H \in H_{\mathrm{loc}}(\mathrm{curl}; \mathbb{R}^3 \setminus \overline{B})$ implies that their tangential traces do not jump across the surface $\Sigma_0 \cap (\mathbb{R}^3 \setminus B)$ and the radiation condition (2.2) implies that

$$\int_{\partial B_r} (|H|^2 + |E|^2)\, \mathrm{d}\sigma = O(1) \tag{2.3}$$

as $r \to \infty$, compare [8].

Denote by $\{e_1, e_2, e_3\}$ the usual Cartesian basis in $\mathbb{R}^3$, such that $e_3$ is the normal vector on $\Sigma_0$ pointing into $\mathbb{R}^3_+$, and

$$\Sigma_d = \left\{ x \in \mathbb{R}^3_+ : x \cdot e_3 = d \right\} \subset \mathbb{R}^3_+$$

is the hyperplane parallel to the surface of the ground at height $d > 0$. We assume that measurements and excitations are restricted to an open bounded sheet $\mathcal{M} \subset \Sigma_d$ supporting the device. A time-harmonic excitation given by a tangential magnetic dipole density $\varphi$ on $\mathcal{M}$ leads to a primary electromagnetic field $(E^i, H^i)$ satisfying (2.1) in $\mathbb{R}^3 \setminus \mathcal{M}$, where the magnetic field has the form

$$H^i(x) = -\kappa_+^2 \int_{\mathcal{M}} \mathbb{G}(x; x_0)\varphi(x_0)\, \mathrm{d}x_0. \tag{2.4}$$

Here, $\mathbb{G}(x; x_0)$ is the magnetic dyadic Green's function for the Maxwell system (2.1) with wave number $\kappa$, and

$$\varphi \in \mathcal{X} := \mathcal{L}^2_\tau(\mathcal{M}) = \{\varphi \in \mathcal{L}^2(\mathcal{M}; \mathbb{C}^3) : \varphi \cdot e_3 = 0 \text{ a.e. on } \mathcal{M}\}. \tag{2.5}$$

Let $\Omega \Subset \mathbb{R}^3_-$ be an open bounded set with smooth boundary and connected complement $\mathbb{R}^3 \setminus \Omega$ (compare figure 1 for a sketch of the geometry). Throughout, we denote by $\nu$ the outer

[3] Given an unbounded open set $G$, we define the space $H_{\mathrm{loc}}(\mathrm{curl}; G)$ as in [19]: a field $H$ belongs to $H_{\mathrm{loc}}(\mathrm{curl}; G)$, if and only if $H \in H(\mathrm{curl}; B_r \cap G)$ for every ball $B_r$ of radius $r > 0$ around the origin.

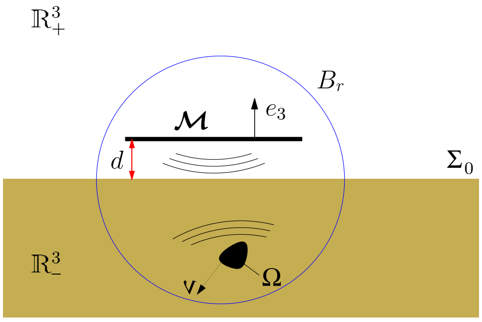


**Figure 1.** Sketch of the geometrical set-up.

normal vector on $\partial\Omega$. A perfect conductor sitting in $\Omega$ induces a secondary field $(E^s, H^s)$ which is a radiating solution of (2.1) in $\mathbb{R}^3\backslash\overline{\Omega}$ subject to the boundary conditions

$$\nu \times E^s = -\nu \times E^i \qquad \text{on } \partial\Omega. \tag{2.6}$$

We emphasize that we do not assume that $\Omega$ itself is connected (as in figure 1); in fact, a particularly appealing feature of our approach is that we are in a position to reconstruct several perfect conductors using our method without knowing their number *a priori*. Formally, $\Omega$ may even be the empty set (an important aspect for real applications). In this case there is no scattering, i.e. the secondary field vanishes.

Now we are ready to define the operator $M$ which maps the given excitation $\varphi$ onto the measured tangential component

$$H^s_\tau := (e_3 \times H^s) \times e_3 \tag{2.7}$$

of the secondary magnetic field on $\mathcal{M}$, i.e.

$$M : \begin{cases} \mathcal{X} \to \mathcal{X}, \\ \varphi \mapsto H^s_\tau|_{\mathcal{M}}. \end{cases} \tag{2.8}$$

Note that, since $H^s$ is analytic, cf, e.g., Colton and Kress [6], its (complex) tangential component is a well-defined member of $\mathcal{X}$.

**Theorem 2.1.** *The operator $M : \mathcal{X} \to \mathcal{X}$ is compact.*

**Proof.** By superposition, we can represent $M$ as an integral operator

$$M\varphi(x) = \int_{\mathcal{M}} H^s_\tau(x; x_0)\varphi(x_0)\,\mathrm{d}x_0,$$

where the $j$th column of $H^s_\tau(x; x_0) \in \mathbb{C}^{3\times3}$ is the tangential component in $\mathcal{M}$ of the secondary magnetic field induced by a magnetic dipole in $x_0 \in \mathcal{M}$ with polarization $e_j$. Every entry of $H^s_\tau$ is a square-integrable function of the two variables $x$ and $x_0$, and hence $M$ is compact, cf, e.g., Kress [16]. □

## 3. Symmetry and injectivity of $M$

For our setting the following fundamental *reciprocity law* holds true (see, e.g., Kress [17, section 4.2.7] for a corresponding result in a homogeneous medium):

Let $p_1, p_2 \in \mathbb{R}^3$ be two unit vectors and consider the two secondary fields $H_j^s$, $j = 1, 2$, corresponding to magnetic dipole sources at $x_j \in \mathcal{M}$ with polarizations $p_j$. Then there holds $p_2 \cdot H_1^s(x_2) = p_1 \cdot H_2^s(x_1)$.

This observation is the essence of the following result for which we introduce the standard bilinear form

$$\langle \varphi_1, \varphi_2 \rangle_{\mathcal{M}} = \int_{\mathcal{M}} \varphi_1 \cdot \varphi_2 \, \mathrm{d}\sigma \qquad \text{for } \varphi_1, \varphi_2 \in \mathcal{X}. \tag{3.1}$$

In fact, our proof of theorem 3.1 implies the reciprocity principle only for tangential vectors $p_j$ in $\mathcal{M}$; however, the proof of the general case is exactly the same.

**Theorem 3.1.** *The operator $M$ is symmetric, i.e.*

$$\langle \varphi_1, M\varphi_2 \rangle_{\mathcal{M}} = \langle M\varphi_1, \varphi_2 \rangle_{\mathcal{M}} \qquad \textit{for } \varphi_1, \varphi_2 \in \mathcal{X}.$$

**Proof.** Without loss of generality, we can assume that $\Omega \neq \emptyset$. We denote by $\left(E_j^s, H_j^s\right)$ the secondary electromagnetic field corresponding to the excitation density $\varphi_j$, where $j = 1, 2$ and by $(E_j, H_j)$ the associated total field, i.e. the superposition of primary and secondary fields. The total field is a radiating solution of the inhomogeneous Maxwell system

$$\operatorname{curl} H_j = -\mathrm{i}\omega\varepsilon E_j, \qquad \operatorname{curl} E_j = \mathrm{i}\omega\mu(H_j - \phi_j) \qquad \text{in } \mathbb{R}^3 \backslash \overline{\Omega}, \tag{3.2}$$

where $\phi_j$ is the surface distribution associated with the density $\varphi_j$ on $\mathcal{M}$. The second part of (3.2) is equivalent (cf [1, section 3.1.3]) to

$$\operatorname{curl} E_j = \mathrm{i}\omega\mu H_j \qquad \text{in } \mathbb{R}^3 \backslash (\bar{\Omega} \cup \bar{\mathcal{M}}), \qquad [e_3 \times E_j]_{\mathcal{M}} = \mathrm{i}\omega\mu_+\varphi_j,$$

where $[e_3 \times E_j]_{\mathcal{M}} = e_3 \times E_j|_{\mathcal{M}^-} - e_3 \times E_j|_{\mathcal{M}^+}$ is the jump of the trace taken from below and from above $\mathcal{M}$. Furthermore, by virtue of (2.6), $E_j$ satisfies the homogeneous boundary condition

$$\nu \times E_j = 0 \qquad \text{on } \partial\Omega. \tag{3.3}$$

Let $r > 0$ be such that the ball $B_r$ of radius $r$ around the origin contains both $\mathcal{M}$ and $\Omega$, compare figure 1. Then partial integrations above and below $\mathcal{M}$ yield

$$\begin{aligned}
\mathrm{i}\omega\mu_+\langle \varphi_1, M\varphi_2 \rangle_{\mathcal{M}} &= \int_{\mathcal{M}} \mathrm{i}\omega\mu_+\varphi_1 \cdot H_2^s \, \mathrm{d}\sigma \\
&= \int_{B_r\backslash\overline{\Omega}} \left(\operatorname{curl} E_1 \cdot H_2^s - E_1 \cdot \operatorname{curl} H_2^s\right) \mathrm{d}x \\
&\quad + \int_{\partial\Omega} \nu \times E_1 \cdot H_2^s \, \mathrm{d}\sigma - \int_{\partial B_r} \frac{x}{r} \times E_1 \cdot H_2^s \, \mathrm{d}\sigma \\
&= -\int_{\partial B_r} \frac{x}{r} \times E_1 \cdot H_2^s \, \mathrm{d}\sigma + \int_{B_r\backslash\overline{\Omega}} \left(\mathrm{i}\omega\mu H_1 \cdot H_2^s - E_2^s \cdot \operatorname{curl} H_1\right) \mathrm{d}x,
\end{aligned}$$

where we have also used the boundary condition (3.3) for $E_1$ and the Maxwell identities for $E_1$ and $\operatorname{curl} H_2^s$. Note that the normal components of the electromagnetic fields jump across the interface $\Sigma_0$, however, the fact that all fields belong to $H_{\mathrm{loc}}(\operatorname{curl}; G)$ in a neighbourhood $G$ of $\Sigma_0$ legitimates the above transformation.

In a second step, we split the total magnetic field $H_1$ in $B_r \setminus \overline{\Omega}$ into its primary and secondary components. With another partial integration, we thus obtain

$$\begin{aligned}
\mathrm{i}\omega\mu_+\langle\varphi_1, M\varphi_2\rangle_{\mathcal{M}} &= -\int_{\partial B_r} \frac{x}{r} \times E_1 \cdot H_2^s \,\mathrm{d}\sigma + \int_{B_r\setminus\overline{\Omega}} \left(\mathrm{i}\omega\mu H_1^s \cdot H_2^s - E_2^s \cdot \mathrm{curl}\, H_1^s\right) \mathrm{d}x \\
&\quad + \int_{\partial B_r} \frac{x}{r} \times E_2^s \cdot H_1^i \,\mathrm{d}\sigma - \int_{\partial\Omega} \nu \times E_2^s \cdot H_1^i \,\mathrm{d}\sigma \\
&\quad + \int_{B_r\setminus\overline{\Omega}} H_1^i \cdot \left(\mathrm{i}\omega\mu H_2^s - \mathrm{curl}\, E_2^s\right) \mathrm{d}x \\
&= \int_{\partial B_r} \left(\frac{x}{r} \times E_2^s \cdot H_1^i - \frac{x}{r} \times E_1 \cdot H_2^s\right) \mathrm{d}\sigma + \int_{\partial\Omega} \nu \times E_2^i \cdot H_1^i \,\mathrm{d}\sigma \\
&\quad + \int_{B_r\setminus\overline{\Omega}} \left(\mathrm{i}\omega\mu H_1^s \cdot H_2^s + \mathrm{i}\omega\varepsilon E_2^s \cdot E_1^s\right) \mathrm{d}x,
\end{aligned}$$

using the boundary condition (2.6) for $E_2^s$ and the Maxwell identities for $H_2^s$ and curl $H_1^s$. We rewrite our result as

$$\mathrm{i}\omega\mu_+\langle\varphi_1, M\varphi_2\rangle_{\mathcal{M}} = I_{\partial B_r} + I_{\partial\Omega} + I_{B_r\setminus\overline{\Omega}}, \tag{3.4}$$

where the three terms $I_{\dots}$ on the right-hand side refer to the integrals over the boundaries or the domain indicated by the respective indices.

The two boundary integrals are now treated separately. The integral $I_{\partial\Omega}$ over $\partial\Omega$ can be transformed by partial integration over $\Omega$ and by using the Maxwell identities as follows:

$$\begin{aligned}
I_{\partial\Omega} = \int_{\partial\Omega} \nu \times E_2^i \cdot H_1^i \,\mathrm{d}\sigma &= \int_{\Omega} \left(\mathrm{curl}\, E_2^i \cdot H_1^i - E_2^i \cdot \mathrm{curl}\, H_1^i\right) \mathrm{d}x \\
&= \int_{\Omega} \left(\mathrm{i}\omega\mu H_2^i \cdot H_1^i + \mathrm{i}\omega\varepsilon E_2^i \cdot E_1^i\right) \mathrm{d}x.
\end{aligned} \tag{3.5}$$

For the integral $I_{\partial B_r}$ over the boundary of $B_r$, we use the radiation condition (2.2) and (2.3) to obtain

$$\begin{aligned}
I_{\partial B_r} &= \int_{\partial B_r} \left(\frac{x}{r} \times E_2^s \cdot H_1^i - \frac{x}{r} \times E_1 \cdot H_2^s\right) \mathrm{d}\sigma = \int_{\partial B_r} \left(\frac{x}{r} \times H_2^s \cdot E_1 - \frac{x}{r} \times H_1^i \cdot E_2^s\right) \mathrm{d}\sigma \\
&= \int_{\partial B_r} \left(\left(\frac{x}{r} \times H_2^s + (\varepsilon/\mu)^{1/2} E_2^s\right) \cdot E_1 - \left(\frac{x}{r} \times H_1^i + (\varepsilon/\mu)^{1/2} E_1^i\right) \cdot E_2^s\right) \mathrm{d}\sigma \\
&\quad + (\varepsilon/\mu)^{1/2} \int_{\partial B_r} \left(E_1^i \cdot E_2^s - E_2^s \cdot E_1\right) \mathrm{d}\sigma \\
&= -(\varepsilon/\mu)^{1/2} \int_{\partial B_r} E_1^s \cdot E_2^s \,\mathrm{d}\sigma + o(1)
\end{aligned}$$

as $r \to \infty$. Inserting this and (3.5) into (3.4), we thus have

$$\begin{aligned}
\mathrm{i}\omega\mu_+\langle\varphi_1, M\varphi_2\rangle_{\mathcal{M}} &= \mathrm{i}\omega \int_{B_r\setminus\overline{\Omega}} \left(\mu H_1^s \cdot H_2^s + \varepsilon E_1^s \cdot E_2^s\right) \mathrm{d}x \\
&\quad + \mathrm{i}\omega \int_{\Omega} \left(\mu H_1^i \cdot H_2^i + \varepsilon E_1^i \cdot E_2^i\right) \mathrm{d}x - (\varepsilon/\mu)^{1/2} \int_{\partial B_r} E_1^s \cdot E_2^s \,\mathrm{d}\sigma + o(1)
\end{aligned}$$

as $r \to \infty$, and the symmetry of this expression implies the symmetry of $M$. □

If we replace $\phi_j$ in (3.2) by the delta distribution $\delta(x_j)p_j$ with $x_j \in \mathcal{M}$ and unit vector $p_j$ tangential to $\mathcal{M}$, the above proof yields the aforementioned reciprocity law.

We emphasize that $\mathcal{X}$ is a complex vector space, so that $\langle\cdot,\cdot\rangle_{\mathcal{M}}$ is *not* the associated scalar product in $\mathcal{X}$, and hence $M : \mathcal{X} \to \mathcal{X}$ is *not* self-adjoint. Now we turn to the injectivity of $M$.

**Theorem 3.2.** *Let $\Omega \neq \emptyset$ and assume that $\kappa_-^2$ is no resonance of $\Omega$, i.e. eigenvalue of the* curlcurl *operator in $\Omega$ with natural boundary condition. Then the operator $M$ is injective with dense range $\mathcal{R}(M)$ in $\mathcal{X}$. In particular, this assumption is fulfilled if $\kappa_-^2 \notin \mathbb{R}$.*

**Proof.** Let $\varphi \in \mathcal{X}$ belong to the null space of $M$, i.e. $H_\tau^s = 0$ on $\mathcal{M}$. Since $H^s$ is an analytic function on $\mathbb{R}_+^3$, it can locally be represented by a converging Taylor series of the three components of the space variable $x$. Freezing the vertical component to be $x \cdot e_3 = d > 0$ and considering only the two horizontal components of the field, it follows immediately that these tangential components on $\Sigma_d$ are real analytic functions of the two horizontal variables. Since they vanish on $\mathcal{M}$ they must vanish everywhere in $\Sigma_d$, i.e. $(E^s, H^s)$ is a radiating solution of the homogeneous Maxwell system in the half space

$$\left\{x \in \mathbb{R}_+^3 : x \cdot e_3 > d\right\}$$

with boundary condition

$$H_\tau^s = 0 \qquad \text{on } \Sigma_d. \tag{3.6}$$

At this point, we employ the reflection principle, and extend $H^s$ by

$$\hat{H}^s(x) = \begin{cases} H^s(x), & x \cdot e_3 > d, \\ -(H^s(x'))', & x \cdot e_3 < d, \end{cases}$$

to all of $\mathbb{R}^3$, where $x' = x - 2(x \cdot e_3 - d)e_3$ denotes the reflection of $x$ at $\Sigma_d$, and

$$(H^s)' = \left(H_1^s, H_2^s, -H_3^s\right), \qquad \text{given } H^s = \left(H_1^s, H_2^s, H_3^s\right).$$

Because of (3.6) $\hat{H}^s$ belongs to $H_{\text{loc}}(\text{curl}, \mathbb{R}^3)$ and agrees with $H^s$ in $\mathbb{R}_+^3$. Moreover, as the tangential component of curl $\hat{H}^s$ is also continuous across $\Sigma_0$, $\hat{H}^s$ is a radiating solution of a homogeneous Maxwell equation with constant coefficient,

$$\operatorname{curl}\operatorname{curl} \hat{H}^s - \kappa_+^2 \hat{H}^s = 0 \qquad \text{in } \mathbb{R}^3.$$

As a consequence, cf, e.g., Cessenat [1, section 3.1.2.2], $\hat{H}^s$ vanishes everywhere in $\mathbb{R}^3$, and therefore, $H^s$ as well as $E^s$ vanish in $\mathbb{R}_+^3$.

In particular, $E^s$ and $H^s$ have vanishing tangential components on $\Sigma_0$. Since these tangential components are continuous across the interface $\Sigma_0$ (cf, e.g., [19]), it now follows from Holmgren's theorem ([17, theorem 4.1.2.4]) that the field $(E^s, H^s)$ is also zero in a neighbourhood of $\Sigma_0$ in $\mathbb{R}_-^3$, since it is a solution of a homogeneous Maxwell system with constant coefficients $\varepsilon_-$ and $\mu_-$.

Accordingly, $E^s = 0$ everywhere in $\mathbb{R}_-^3 \backslash \overline{\Omega}$ because of its analyticity, and hence $\nu \times E^i = 0$ on $\partial\Omega$ because of (2.6). Since $(E^i, H^i)$ solve the homogeneous Maxwell system in $\Omega$, and $\kappa_-^2$ is no resonance of $\Omega$ by assumption, it follows that the field vanishes identically in $\Omega$ and, because of its analyticity, everywhere in $\mathbb{R}_-^3$. As in the first part of this proof, this implies that $H^i$ is zero everywhere in $\mathbb{R}^3 \backslash \mathcal{M}$, and hence we necessarily have $\varphi = 0$.

Having established that $\mathcal{N}(M)$ is trivial, it follows immediately from the symmetry of $M$ that

$$\overline{\mathcal{R}(M)} = \mathcal{N}(M)^\perp = \mathcal{X},$$

where $\mathcal{N}(M)^\perp$ refers to orthogonality with respect to the bilinear form (3.1), see [16].

Since all eigenvalues of the curlcurl operator with natural boundary conditions are real, resonances can only occur when $\kappa_-^2 \in \mathbb{R}$. □

## 4. The operators $L$ and $L^T$

Assume now that $\Omega \neq \emptyset$ and consider an arbitrary (complex) tangential vector field

$$\psi \in \mathcal{Y} := H^{-1/2}(\mathrm{div}; \partial\Omega). \tag{4.1}$$

We denote by $(E^\psi, H^\psi)$ the associated radiating solution of the Maxwell system (2.1) in $\mathbb{R}^3\backslash\overline{\Omega}$ subject to the boundary condition

$$\nu \times E^\psi = \psi \qquad \text{on } \partial\Omega; \tag{4.2}$$

see, for example, [19] for existence and uniqueness of the solution $(E^\psi, H^\psi)$ of this problem. Given $\psi$ we define the operator

$$L : \begin{cases} \mathcal{Y} \to \mathcal{X}, \\ \psi \mapsto H_\tau^\psi, \end{cases} \tag{4.3}$$

where $H_\tau^\psi = (e_3 \times H^\psi) \times e_3$ is the tangential component of this field on $\mathcal{M}$.

In particular, we mention that if $E^i$ and $H^s$ are the primary electric and secondary magnetic fields introduced in section 2, respectively, then $\psi = -\nu \times E^i|_{\partial\Omega}$ belongs to $\mathcal{Y}$, and this choice of $\psi$ yields $H^\psi = H^s$. This means that we have

$$L : -\nu \times E^i|_{\partial\Omega} \mapsto H_\tau^s \tag{4.4}$$

with $H_\tau^s$ from (2.7).

As usual, we will identify the dual space $\mathcal{Y}'$ of $\mathcal{Y}$ with the space $H^{-1/2}(\mathrm{curl}; \partial\Omega)$ using the bilinear form

$$\langle \psi, \chi \rangle_{\partial\Omega} = \int_{\partial\Omega} \psi \cdot \chi \, \mathrm{d}\sigma \qquad \text{for } \psi \in \mathcal{Y}, \chi \in \mathcal{Y}'. \tag{4.5}$$

The corresponding transpose $L^T : \mathcal{X} \to \mathcal{Y}'$ of $L$ is defined by the identity

$$\langle L\psi, \varphi \rangle_{\mathcal{M}} = \langle \psi, L^T\varphi \rangle_{\partial\Omega} \qquad \text{for all } \varphi \in \mathcal{X}, \quad \psi \in \mathcal{Y}.$$

**Theorem 4.1.** *Let $\Omega \neq \emptyset$ and $\varphi \in \mathcal{X}$ be given. Denote by $H^i$ and $H^s$ the associated primary and secondary magnetic fields introduced in section 2 (see (2.4)). Then there holds $L^T\varphi = -H_\tau/(\mathrm{i}\omega\mu_+)$, where*

$$H_\tau = (\nu \times H) \times \nu \qquad \text{on } \partial\Omega$$

*is the tangential component of the total field $H = H^i + H^s$ on the boundary of the scatterer(s).*

**Proof.** Given $\psi \in \mathcal{Y}$, let $(E^\psi, H^\psi)$ be defined as above. Furthermore, let $E = E^i + E^s$ be the total electric field corresponding to the excitation $\varphi$. We proceed as in the first step of the proof of theorem 3.1 to obtain

$$\begin{aligned} \mathrm{i}\omega\mu_+\langle \varphi, L\psi \rangle_{\mathcal{M}} &= \int_{\mathcal{M}} \mathrm{i}\omega\mu_+\varphi \cdot H^\psi \, \mathrm{d}\sigma \\ &= -\int_{\partial B_r} \frac{x}{r} \times E \cdot H^\psi \, \mathrm{d}\sigma + \int_{B_r\backslash\overline{\Omega}} (\mathrm{i}\omega\mu H \cdot H^\psi - E^\psi \cdot \mathrm{curl}\, H) \, \mathrm{d}x. \end{aligned}$$

Then, another partial integration yields

$$\begin{aligned} \mathrm{i}\omega\mu_+\langle \varphi, L\psi \rangle_{\mathcal{M}} = &\int_{\partial B_r} \left( \frac{x}{r} \times E^\psi \cdot H - \frac{x}{r} \times E \cdot H^\psi \right) \mathrm{d}\sigma - \int_{\partial\Omega} \nu \times E^\psi \cdot H \, \mathrm{d}\sigma \\ &+ \int_{B_r\backslash\overline{\Omega}} H \cdot (\mathrm{i}\omega\mu H^\psi - \mathrm{curl}\, E^\psi) \, \mathrm{d}x. \end{aligned}$$

Here, the integral over $B_r\backslash\overline{\Omega}$ disappears because of the Maxwell equations, and the boundary condition (4.2) for $E^\psi$ can be inserted into the boundary integral over $\partial\Omega$: thus, we have

$$\mathrm{i}\omega\mu_+\langle\varphi, L\psi\rangle_{\mathcal{M}} = \int_{\partial B_r}\left(\frac{x}{r}\times E^\psi\cdot H - \frac{x}{r}\times E\cdot H^\psi\right)\mathrm{d}\sigma - \int_{\partial\Omega}\psi\cdot H_\tau\,\mathrm{d}\sigma. \tag{4.6}$$

Rewriting the integral over the surface of the sphere $B_r$ as

$$\int_{\partial B_r}\left(\frac{x}{r}\times E^\psi\cdot H - \frac{x}{r}\times E\cdot H^\psi\right)\mathrm{d}\sigma = \int_{\partial B_r}\left(\frac{x}{r}\times H^\psi\cdot E - \frac{x}{r}\times H\cdot E^\psi\right)\mathrm{d}\sigma$$

$$= \int_{\partial B_r}\left(\left(\frac{x}{r}\times H^\psi + (\varepsilon/\mu)^{1/2}E^\psi\right)\cdot E - \left(\frac{x}{r}\times H + (\varepsilon/\mu)^{1/2}E\right)\cdot E^\psi\right)\mathrm{d}\sigma$$

we conclude that this integral vanishes as $r\to\infty$ because of the radiation condition (2.2), (2.3). Accordingly, as $r\to\infty$, (4.6) yields

$$\langle L\psi,\varphi\rangle_{\mathcal{M}} = -\frac{1}{\mathrm{i}\omega\mu_+}\langle\psi, H_\tau\rangle_{\partial\Omega}.$$

Since $\varphi\in\mathcal{X}$ and $\psi\in\mathcal{Y}$ can be chosen arbitrarily this proves the assertion. □

## 5. The factorization of *M*

The next ingredient on our way to the factorization (1.1) is the diffraction problem

$$\operatorname{curl} H^d = -\mathrm{i}\omega\varepsilon E^d, \qquad \operatorname{curl} E^d = \mathrm{i}\omega\mu\, H^d \qquad \text{in } \mathbb{R}^3\backslash\partial\Omega \tag{5.1}$$

with the jump conditions

$$[H_\tau^d]_{\partial\Omega} = \chi \qquad \text{and} \qquad [\nu\times E^d]_{\partial\Omega} = 0 \tag{5.2}$$

on the boundary of $\Omega$. Here, we assume that $\Omega\neq\emptyset$, and $\chi\in H^{-1/2}(\mathrm{curl};\partial\Omega)$ is a given tangential field on the boundary of $\Omega$. The square brackets denote the differences between the respective traces from outside and inside. We are looking for the radiating solution of this problem, the existence and uniqueness of which follow as in [1, section 3.1.3] by replacing Green's function for the homogeneous Helmholtz equation by the corresponding one for our layered background medium.

Given the solution $(E^d, H^d)$ of this problem, we define

$$F:\begin{cases}\mathcal{Y}'\to\mathcal{Y},\\ \chi\mapsto\nu\times E^d|_{\partial\Omega},\end{cases} \tag{5.3}$$

and note that $\nu\times E^d|_{\partial\Omega}\in H^{-1/2}(\mathrm{div};\partial\Omega)$ is a well-defined quantity because of the second condition in (5.2).

For the special case when $\chi = H_\tau$, i.e. the tangential component of the total magnetic field corresponding to some excitation $\varphi\in\mathcal{X}$ as described in section 2, the solution of the diffraction problem (5.1), (5.2) can be constructed from the corresponding primary and the secondary fields, namely

$$E^d = \begin{cases}E^s, & x\in\mathbb{R}^3\backslash\overline{\Omega},\\ -E^i, & x\in\Omega,\end{cases} \qquad H^d = \begin{cases}H^s, & x\in\mathbb{R}^3\backslash\overline{\Omega},\\ -H^i, & x\in\Omega.\end{cases}$$

Obviously, $(E^d, H^d)$ is a radiating solution of the Maxwell system (5.1), and satisfies the appropriate jump conditions (5.2) on the boundary of $\Omega$:

$$[H_\tau^d]_{\partial\Omega} = H_\tau^s + H_\tau^i = H_\tau = \chi$$

and, cf (2.6),

$$[\nu \times E^d]_{\partial\Omega} = \nu \times E^s + \nu \times E^i = 0.$$

Consequently, we have

$$F : H_\tau \mapsto \nu \times E^s|_{\partial\Omega} = -\nu \times E^i|_{\partial\Omega}, \tag{5.4}$$

and hence, the mapping sequence

$$\varphi \overset{L^T}{\mapsto} -\frac{1}{\mathrm{i}\omega\mu_+}H_\tau \overset{F}{\mapsto} \frac{1}{\mathrm{i}\omega\mu_+}\nu \times E^i|_{\partial\Omega} \overset{L}{\mapsto} -\frac{1}{\mathrm{i}\omega\mu_+}H_\tau^s.$$

Thus, we have achieved the following goal:

**Theorem 5.1.** *Assume that* $\Omega \neq \emptyset$*. Given L of (4.3) and F of (5.3) the measurement operator M of (2.8) admits the factorization*

$$M = -\mathrm{i}\omega\mu_+ LFL^T. \tag{5.5}$$

## 6. A sampling method for the inverse problem

Finally, we consider the inverse problem of locating the scatterer given the measurement operator $M$. To this end, we take up a variant of the sampling method introduced first by Coyle [7] for a TM-mode approximation of our problem. For our method, we require a test whether the tangential component

$$H_\tau(\cdot; z, p) = (e_3 \times \mathbb{G}(\cdot; z)p) \times e_3 \qquad \text{on } \mathcal{M} \tag{6.1}$$

of a magnetic dipole in some point $z \notin \mathcal{M}$ with polarization $p \in \mathbb{R}^3$ belongs to the range $\mathcal{R}(M)$ of $M$. Accordingly, we define

$$\Omega_1 := \{z \in \mathbb{R}^3_- : H_\tau(\cdot; z, p) \in \mathcal{R}(M) \text{ for some } p \in \mathbb{R}^3\} \tag{6.2}$$

to be the set of all points for which this test is positive for some polarization vector $p$.

**Theorem 6.1.** *Let* $\Omega \neq \emptyset$ *and p be a unit polarization vector in* $\mathbb{R}^3$*. A point* $z \in \mathbb{R}^3_-$ *belongs to* $\Omega$*, if and only if the function* $H_\tau(\cdot; z, p)$ *of (6.1) belongs to the range of L.*

**Proof.** First, assume that $z \in \Omega \subset \mathbb{R}^3_-$ and denote by $E(\cdot; z, p)$ the electric field of the magnetic dipole $\mathbb{G}(\cdot; z)p$. As $z$ lies in the interior of $\Omega$, this electromagnetic dipole field is obviously a radiating solution of the Maxwell system (2.1) subject to the boundary condition (4.2), where

$$\psi := \nu \times E(\cdot; z, p)|_{\partial\Omega}.$$

Therefore, $L\psi = H_\tau(\cdot; z, p)$ on $\mathcal{M}$, i.e. the latter belongs to the range of $L$.

Vice versa, assume that $H_\tau(\cdot; z, p) = L\psi$ for some $z \in \mathbb{R}^3_-$ and $\psi \in \mathcal{Y}$. As in section 4, we denote by $(E^\psi, H^\psi)$ the radiating solution of (2.1) associated with the boundary condition (4.2). By assumption, the tangential components of $\mathbb{G}(\cdot; z)p$ and $H^\psi$ on $\mathcal{M}$ coincide, i.e. the magnetic field

$$H = \mathbb{G}(\cdot; z)p - H^\psi,$$

together with the associated electric field $E$, is a radiating solution of (2.1) in $\mathbb{R}^3 \setminus (\{z\} \cup \overline{\Omega})$ that satisfies

$$H_\tau = 0 \qquad \text{on } \mathcal{M}.$$

Now we follow the proof of theorem 3.2 and conclude that $(E, H)$ vanishes near the interface $\Sigma_0$, and hence everywhere in $\mathbb{R}^3\setminus(\{z\} \cup \overline{\Omega})$. Assuming that $z \notin \overline{\Omega}$, it follows that $H$ can be extended continuously (by zero) into the point $z$. However, since $H^\psi$ is analytic near $z$ this is a contradiction to the fact that $\mathbb{G}(\cdot; z)p$ has a singularity there.

A refinement of this argument applies when $z \in \partial\Omega$: in this case, we obtain that $H = 0$ outside of $\overline{\Omega}$ and $H^\psi$ belongs to $H_{\text{loc}}(\text{curl}, \mathbb{R}^3\setminus\overline{\Omega})$. However, the strength of the singularity of $\mathbb{G}(\cdot; z)p$ in $z$ prevents that $\mathbb{G}(\cdot; z)p$ belongs to $H_{\text{loc}}(\text{curl}, \mathbb{R}^3\setminus\overline{\Omega})$.

Thus, in either case the assumption that $z \notin \Omega$ has led to a contradiction, which was to be shown. □

As a corollary, we obtain the following result:

**Corollary 6.2.** *For $\Omega_1$ of (6.2), we always have $\Omega_1 \subset \Omega$, i.e. if $H_\tau(\cdot; z, p)$ of (6.1) belongs to the range of $M$ for some $z \in \mathbb{R}^3_-$ and $p \in \mathbb{R}^3$ then $z \in \Omega$.*

**Proof.** We consider first the degenerate case $\Omega = \emptyset$. In this case, we have $M = 0$ and the magnetic dipole field $H_\tau(\cdot; z, p)$ belongs to the range of $M$ if it vanishes everywhere on $\mathcal{M}$. Again, we can follow the proof of theorem 3.2 to conclude that the electromagnetic field of this dipole vanishes entirely in $\mathbb{R}^3\setminus\{z\}$, which is a contradiction to its singularity at $z$. Therefore, in this case we have $\Omega_1 = \Omega = \emptyset$.

When $\Omega \neq \emptyset$, the factorization (5.5) implies that $H_\tau(\cdot; z, p)$ belongs to the range of $L$ whenever it belongs to the range of $M$, and hence in this case we have $z \in \Omega$ by theorem 6.1. □

Corollary 6.2 motivates the following sampling method: a certain region of interest $\mathcal{R}$ within the lower half space $\mathbb{R}^3_-$ is sampled, and for each sampling point $z \in \mathcal{R}$ we test whether it belongs to the set $\Omega_1$ of (6.2). This test is cheap and comparatively easy to implement (see the following section), although the numerics are somewhat subtle in order to achieve maximal robustness of the test.

Note that for the results of this section (and, in particular, for the aforementioned sampling method) it has been irrelevant whether $\kappa_-^2$ is a resonance of $\Omega$, or not. However, it has to be emphasized that if $\kappa_-^2$ is no resonance of $\Omega$ then the test function $H_\tau(\cdot; z, p)$ of (6.1) *almost* belongs to $\mathcal{R}(M)$ for every $z \in \mathbb{R}^3_-$ and polarization $p \in \mathbb{R}^3$, i.e. for every $\delta > 0$ there is some $\varphi = \varphi^\delta_{z,p}$, such that

$$\left\| H_\tau(\cdot; z, p) - M\varphi^\delta_{z,p} \right\|_{\mathcal{X}} \leqslant \delta,$$

but only for $z \in \Omega_1$ this inequality also holds true for $\delta = 0$. This follows from theorem 3.2.

Yet in other words, in this situation we have

$$\mathcal{R}(M) \subset \mathcal{R}(L) \subset \mathcal{X},$$

and all inclusions are dense. However, while this implies that $\Omega_1 \subset \Omega$—as stated in corollary 6.2—we currently have no tool to describe the size of the remainder set $\Omega\setminus\Omega_1$.

## 7. Numerical results

In this section, we show some preliminary numerical results for the following benchmark problem: a perfectly conducting ball of radius 4 cm buried in the ground is to be found from low-frequent electromagnetic measurements ($\omega = 1$ kHz), where the device operates on a square $\mathcal{M}$ of size $2 \times 2$ m located $d = 5$ cm above the ground. The centre of the ball is 15 cm below the centre of the measurement area, i.e. 10 cm below the surface of the ground. These figures are meant to represent a realistic test case for the detection of buried land mines using commercial off-the-shelf metal detectors.

The numerical results are preliminary in that so far we have only access to simulated data (kindly provided by Klaus Erhard from Göttingen, using a boundary element method) corresponding to a homogeneous background, i.e. vacuum

$$\varepsilon_+ = \varepsilon_- = \varepsilon_0 = 8.854 \times 10^{-12}\,\mathrm{A\,s\,V^{-1}\,m^{-1}}$$
$$\mu_+ = \mu_- = \mu_0 = 4\pi \times 10^{-7}\,\mathrm{V\,s\,A^{-1}\,m^{-1}}$$

Accordingly, $\kappa \approx 2.1\times10^{-5}\,\mathrm{m^{-1}}$. We also show a result for a homogeneous lossy medium, using a complex $\kappa$.

To impose primary fields, we use a square $6 \times 6$ equidistant grid $\mathcal{M}_h \subset \mathcal{M}$, and apply (tangential) magnetic dipoles at these grid points as excitation. Given these 36-grid points and two independent polarization vectors $p$ as well as two measured components of the secondary field $H_\tau^s$ per grid point, we are talking about a $(72 \times 72)$-dimensional matrix $A$ approximating the continuous operator $M$ of (2.8). With an appropriate ordering of the rows and columns of $A$, this is a complex symmetric matrix due to the reciprocity principle.

As region of interest, we choose a box $\mathcal{R} \subset \mathbb{R}^3_-$, aligned with the three coordinate axes and centred around the scatterer, such that the upper face, which is $80 \times 80$ cm large, belongs to $\Sigma_0$. Given a point $z \in \mathcal{R}$, we have to check whether the function $H_\tau(\cdot; z, p)$ of (6.1) belongs to the range of $M$; throughout, we restrict our attention to $p = e_3$. This criterion can be rewritten in terms of the so-called *Picard criterion* [9, 12]: let

$$Mv_j = \sigma_j u_j, \qquad M^* u_j = \sigma_j v_j, \qquad j = 1, 2, 3, \ldots, \tag{7.1}$$

be the singular value decomposition of $M$, with orthonormal bases $\{u_j\}, \{v_j\} \subset \mathcal{X}$, a nonincreasing sequence $\{\sigma_j\}$ of positive (or nonnegative) numbers and $M^*$ the complex Hilbert space adjoint of $M$. Then the function $H_\tau(\cdot; z, e_3)$ belongs to $\mathcal{R}(M)$, if and only if the series

$$f(z) = \frac{1}{\|H_\tau(\cdot; z, e_3)\|^2_{\mathcal{X}}} \sum_{j=1}^{\infty} \frac{|\langle H_\tau(\cdot; z, e_3), \overline{u_j}\rangle_{\mathcal{M}}|^2}{\sigma_j^2} \tag{7.2}$$

converges. Note that $\sigma_j \to 0$ as $j \to \infty$, since $M$ is compact.

In accordance with the discrete matrix $A$, we can sample any two functions $u, v \in \mathcal{X}$ at the mesh points of $\mathcal{M}_h$ and store the corresponding function values in 72-dimensional vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively, using the same ordering as for the rows of $A$. Then, if $D$ is the diagonal matrix whose entries are the square roots of the weights of the tensor trapezoidal quadrature rule for the integral over $\mathcal{M}$ given the function values on $\mathcal{M}_h$, we obtain the approximation

$$\langle v, Mu\rangle_{\mathcal{M}} \approx (D\boldsymbol{v})^T (DAD)(D\boldsymbol{u}).$$

As a consequence, we can approximate the singular value decomposition (7.1) of $M$ by computing the singular value decomposition of the matrix $DAD$,

$$DAD\boldsymbol{v}_j = \sigma_{h,j}\boldsymbol{u}_j, \qquad DA^* D\boldsymbol{u}_j = \sigma_{h,j}\boldsymbol{v}_j, \qquad j = 1, 2, 3, \ldots, n, \tag{7.3}$$

and considering $D^{-1}\boldsymbol{u}_j$ (respectively $D^{-1}\boldsymbol{v}_j$) as approximate values of $u_j$ (respectively $v_j$) on $\mathcal{M}_h$. Again, we take $\sigma_{h,j}$ to be in nonincreasing order and denote by $n$ the number of reasonable approximations $\sigma_{h,j} \approx \sigma_j$. This number depends on the quality of the data. If there are good reasons to believe that $DAD$ is known up to a perturbation of size $\delta > 0$ (with respect to the spectral norm), then we can only trust in those $n$ singular values which are larger

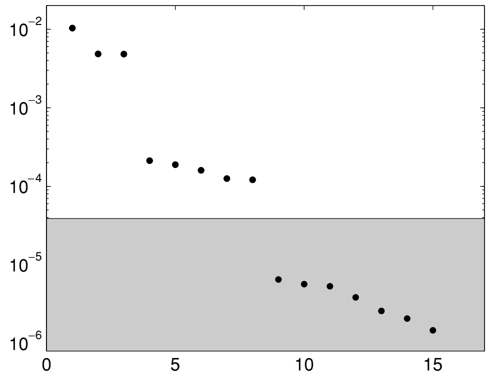


**Figure 2.** Singular values of $DAD$.

than $\delta$. Having specified $n$ accordingly, the function $f$ of (5.3) can be approximated by

$$f_h(z) = \sum_{j=1}^{n} \frac{|\boldsymbol{h}_z^* D\boldsymbol{u}_j|^2}{\sigma_{h,j}^2} \Bigg/ \sum_{j=1}^{n} |\boldsymbol{h}_z^* D\boldsymbol{u}_j|^2, \tag{7.4}$$

where $\boldsymbol{h}_z$ is the vector with the values of $H_\tau(\cdot; z, e_3)$ on $\mathcal{M}_h$.

Even though we use simulated data, these are not exact because of all sorts of approximation and discretization errors in the forward solver. To estimate reasonable parameters $\delta$ and $n$, we use the magnitude of the nonsymmetric part of $DAD$, i.e. we take

$$\delta = \|D(A - A^T)D\|, \tag{7.5}$$

since this quantity should be zero for exact data and be of the order of the data error, otherwise. Figure 2 shows the first few singular values $\sigma_{h,j}$ of (7.3); the grey shaded area indicates the threshold $\sigma_{h,j} \leqslant \delta = \|D(A - A^T)D\|$. According to this graph, we can only trust in the first $n = 8$ singular values of $DAD$.

By virtue of corollary 6.2, the set $\Omega_1$ of (6.2) is a subset of the scatterer and this set $\Omega_1$ contains all points $z \in \mathcal{R}$ for which the function $f(z)$ of (7.2) attains the value $+\infty$. In our code, we approximate this function by $f_h$ of (7.4), i.e. a finite sum whose values are always finite. Accordingly, we need another threshold value $C_\infty > 0$ to distinguish points $z \in \mathcal{R}$ with 'large' values $f_h(z) \geqslant C_\infty$ from test points $z$ with 'smaller' values of $f_h$. Given this threshold, we consider

$$\Omega_{h,1} := \{z \in \mathcal{R} : f_h(z) < C_\infty\} \tag{7.6}$$

as approximation of $\Omega_1$.

Figure 3 illustrates this procedure using a colour coded plot (left) of log $f_h$ in a horizontal layer of $\mathcal{R}$ at height $d = -10$ cm, which cuts the centre of the scatterer. The boundary of the scatterer is indicated by a solid line; units in figure 3 are in cm. Even though we have only used $n = 8$ terms of the series, the function $f_h$ is much larger outside the scatterer. The subplot on the right compares the true boundary of the scatterer (solid line) with some level contour of $f_h(z)$ (dashed line). The corresponding value of $f_h$ has been used as $C_\infty$ for the three-dimensional reconstructions. This reconstruction, i.e. the set $\Omega_{h,1}$ of (7.6) using this particular value of $C_\infty$, is shown in figure 4. The left-hand subplot indicates the reconstruction within the geometrical set-up, including the ground level (the green surface) and the mesh $\mathcal{M}_h$ of the operating device; units are again in cm. The right-hand subplot zooms in on the reconstruction as compared to the true scatterer.

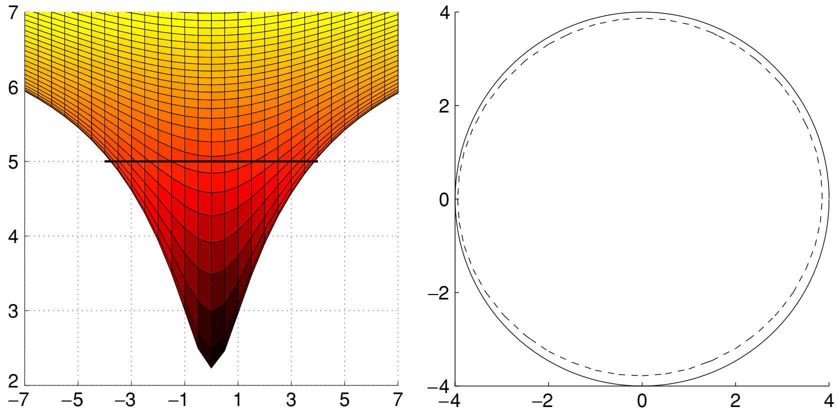


**Figure 3.** Choice of threshold $C_\infty$.

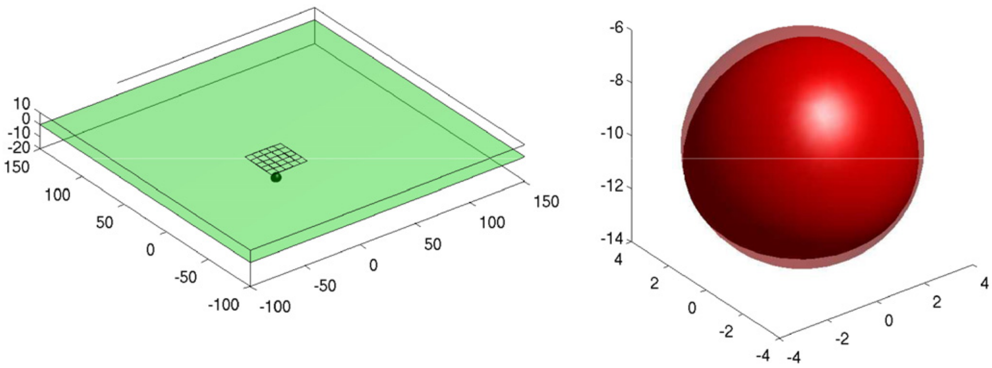


**Figure 4.** Numerical reconstructions (vacuum).

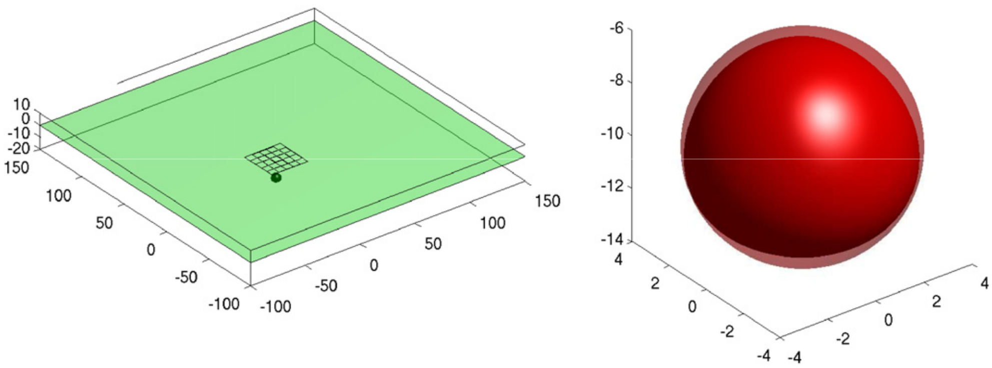


**Figure 5.** Numerical reconstructions (lossy medium).

Figure 5 shows that our method works equally well in a lossy medium. For this example, we have chosen a relative dielectricity of 10 and a conductivity of $10^{-2}$A V$^{-1}$ m$^{-1}$.

Accordingly,

$$\varepsilon_+ = \varepsilon_- = \left(8.854 \times 10^{-11} + \frac{\mathrm{i}}{2\pi} \times 10^{-5}\right) \mathrm{A\,s\,V^{-1}\,m^{-1}},$$

thus $\kappa \approx (6.283 + 6.283\mathrm{i}) \times 10^{-3}\mathrm{m}^{-1}$.

## 8. Conclusion

We have analysed a sampling method for the detection of buried objects using inverse electromagnetic scattering within a two-layered medium in three space dimensions. Numerical results for simulated data corresponding to a homogeneous background and small frequencies indicate that this method may be appropriate to detect land mines using standard hand-held detector technology. To verify this claim, and to calibrate parameters like $C_\infty$, more numerical tests will be necessary, with simulated data for a two-layered background and nonconvex or multiple obstacles, and eventually with field data of real objects.

Our theoretical investigations are based on a reciprocity principle, and a corresponding symmetric factorization of the operator $M$ which is constructed from the measured data. Such kind of factorization may also form the basis to develop some variant of the so-called factorization methods, which may achieve significant improvements in the numerical reconstructions; compare e.g., [15].

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