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Unique shape detection in transient eddy current problems‡

L. Arnold[1], B. Harrach[2] §

[1] Department of Mathematics, University of Stuttgart, Germany

[2] Department of Mathematics, University of Stuttgart, Germany

E-mail: `lilian.arnold@mathematik.uni-stuttgart.de`

Abstract. Transient excitation currents generate electromagnetic fields which in turn induce electric currents in proximal conductors. For slowly varying fields, this can be described by the eddy current equations, which are obtained by neglecting the dielectric displacement currents in Maxwell's equations. The eddy current equations are of parabolic-elliptic type: In insulating regions, the field instantaneously adapts to the excitation (elliptic behavior), while in conducting regions, this adaptation takes some time due to the induced eddy currents (parabolic behavior).

The subject of this work is to locate the conductor(s) surrounded by a non-conducting medium from electromagnetic measurements, i.e., from knowledge of the excitation currents and measurements of the corresponding electromagnetic fields. We show, that the conductors are uniquely determined by the measurements, and give an explicit criterion to decide whether a given point is inside or outside the conducting domain. This criterion serves as a base for rigorously justified non-iterative numerical reconstruction strategies.

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§ Present address: Department of Mathematics, University of Stuttgart, Germany

1. Introduction

Transient excitation currents $J(x,t)$ through excitation coils generate electromagnetic fields $E(x,t)$ and $H(x,t)$. These fields induce electric currents inside proximal conductors which in turn affect the fields. The resulting fields can be measured by sensing coils. The aim in several practical applications is to obtain information about the electromagnetic properties from such measurements.

The electromagnetic fields can be described by Maxwell's equations

$$\begin{aligned}\operatorname{curl} H &= \epsilon \partial_t E + \sigma E + J,\\ \operatorname{curl} E &= -\mu \partial_t H,\end{aligned}$$

where the curl-operator acts on the three spatial coordinates, ∂_t denotes the time-derivative, and (under the assumption of linear and isotropic time-independent material laws) $\sigma(x)$, $\epsilon(x)$ and $\mu(x)$ are the conductivity, permittivity and permeability of the considered domain. For slowly varying fields, the displacement currents $\epsilon \frac{\partial E}{\partial t}$ can be neglected. Then, elimination of H leads to the *transient eddy current equation*

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu} \operatorname{curl} E\right) = I \qquad (1)$$

with the source term $I := -\partial_t J$. A rigorous mathematical justification for the eddy current model has been derived by Alonso [2], Pepperl [29] and Ammari, Buffa and Nédélec [3] for the (low-frequency) time-harmonic case. This also justifies the transient model when the excitation is composed of low-frequency components; cf. [3, Section 8].

Inferring information about the electromagnetic properties from knowledge of the excitation currents and the corresponding measured fields corresponds to the inverse problem of reconstructing the coefficients σ and μ in (1) from knowledge of the excitations I and a part of the solutions E of (1).

Various applications of inverse eddy current problems have been studied in the engineering literature. Reconstruction of electromagnetic properties in time harmonic problems is the subject of magnetic induction tomography (MIT) which is used for medical and industrial imaging (see for example [22, 30] and the references therein). An overview about non-destructive evaluation is given in [9], see also [27, 32]. Inverse problems in transient eddy current problems are considered, for instance, in [19, 13]. In the mathematical literature, a more detailed analysis on inverse time harmonic eddy current problems is provided, for instance, in [4, 33, 31].

In this paper the main focus is on locating the conductors surrounded by a non-conducting medium. Mathematically this corresponds to detecting the support of the conductivity coefficient σ in (1). The measurements are modeled in the following way (cf. [20, 21]): Transient excitation currents through an idealized measurement instrument given by a two-dimensional sheet S (representing infinitely many infinitesimal excitation coils and measurement coils) are used to generate the fields. Then, the induced voltages in the sensing coils are detected on S, again. Mathematically, this is encoded in a measurement operator Λ, that maps I (the negative time-derivative of the transient excitation current J) on the electric field E, the solution of (1), restricted to S:

$$\Lambda : I \mapsto E|_S.$$

A proper definition of Λ is given in section 4.

The aim of this work is to show, that the conducting domains are uniquely determined by Λ. Moreover, we propose a strategy for the reconstruction of the shape of the conductor. To this end we consider (1) on both, conducting regions ($\sigma(x) > 0$) and non-conducting regions ($\sigma(x) = 0$). The consequence is that equation (1) is of parabolic-elliptic type. Several well-posed variational formulations have been proposed for the transient eddy current equation, cf., e.g., [10, 28, 1, 26], but these approaches concentrate on solving the equation with a fixed conducting region. Accordingly, the variational formulations, with their underlying solution spaces, depend on the support of the conductivity. Our main tool to treat the inverse problem is a variational formulation for (1) derived by the authors in [8], that is unified with respect to σ.

A well-established method for shape reconstruction in several inverse problems is the *factorization method* invented by Kirsch [24]. Here, an explicit criterion is developed, which determines whether a given point is inside or outside the domain of interest. In [25], Kirsch applies this method to an inverse problem involving the time harmonic Maxwell system. In the context of land mine detection, the magnetostatic limit of Maxwell's equations is treated in [21]. Results on a scalar parabolic-elliptic problem can be found in [18]. Another approach are *linear sampling methods*, originated by Colton and Kirsch in [14]. Like the factorization method, a sufficient (but not necessary) condition on a point to be inside the domain of interest is produced.

In this paper we apply the linear sampling method for shape detection in transient eddy current problems. Beyond that, considering diamagnetic materials, we show that the conducting domain is uniquely determined by the measurement operator and state an explicit criterion to determine whether a given point is inside or outside the domain. Finally we show that this criterion is equivalent to the one used in the factorization method.

This paper is organized as follows. In section 2 we introduce the necessary notations and assumptions. Section 3 summarizes our variational solution theory from [8] for the direct problem. The setting for the inverse problem and the definition of the measurement operator is provided in section 4. In section 5, we show that the linear sampling method can be applied to detect a subset of the conducting domain. Our main result is presented in section 6: In case of diamagnetic materials, the conductor is uniquely determined by the measurement operator. We present an explicit criterion for detecting the conducting domain and show its equivalence to the factorization method. Finally, section 7 contains the proof of our main result. A conclusion can be found in section 8.

2. Notations and assumptions

Let $\Omega \subset \mathbb{R}^3$ denote the conductor, i.e., let the closure of Ω be the support of the conductivity $\sigma \in L^\infty(\mathbb{R}^3)$. We assume, that

$$\exists\, s \in \mathbb{N}: \ \Omega = \cup_{i=1}^s \Omega_i, \text{ where } \Omega_i, \ i = 1, \ldots, s, \text{ are smoothly bounded}$$
$$\text{domains such that } \overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset, \ i \neq j, \text{ and } \mathbb{R}^3 \setminus \overline{\Omega} \text{ is connected,}$$

and that $\sigma|_\Omega \in L^\infty_+(\Omega)$, where we denote by $L^\infty_+(\Omega)$ the space of $L^\infty(\Omega)$-functions with positive (essential) infima. Let Γ denote the union of the boundaries of Ω_i and ν the outer unit normal on Γ. The permeability $\mu \in L^\infty_+(\mathbb{R}^3)$ is assumed to be constant outside of Ω, for simplicity we assume

$$\mu|_{\mathbb{R}^3 \setminus \overline{\Omega}} \equiv 1.$$

Let $T > 0$. $\mathcal{D}(\mathbb{R}^3)$, respectively, $\mathcal{D}(\mathbb{R}^3 \times]0, T[)$ denotes the space of C^∞-functions which are compactly supported in $\mathbb{R}^3$, respectively, $\mathbb{R}^3 \times]0, T[$. We will also use the notation $\mathcal{D}(\mathbb{R}^3 \times [0, T[)$ for the space of restrictions of functions from $\mathcal{D}(\mathbb{R}^3 \times]{-\infty}, T[)$ to $\mathbb{R}^3 \times]0, T[$ and analogously for any closed subset of $\mathbb{R}^3$. $\mathcal{D}'(\mathbb{R}^3)$ denotes the space of distributions, i.e., continuous linear mappings from $\mathcal{D}(\mathbb{R}^3)$ to $\mathbb{R}$. $\mathcal{D}'(\mathbb{R}^3 \times]0, T[)^3$ is defined likewise. We will also use the spaces

$$\begin{aligned}
&L^2_\rho(\mathbb{R}^3) := \{E \in \mathcal{D}'(\mathbb{R}^3) \,|\, (1 + |x|^2)^{-\frac{1}{2}} E \in L^2(\mathbb{R}^3)\},\\
&W(\mathrm{curl}, \mathbb{R}^3) := \{E \in L^2_\rho(\mathbb{R}^3)^3 \mid \mathrm{curl}\, E \in L^2(\mathbb{R}^3)^3\},\\
&W^1(\mathbb{R}^3) := \{E \in L^2_\rho(\mathbb{R}^3) \,|\, \nabla E \in L^2(\mathbb{R}^3)^3\},\\
&W^1(\mathbb{R}^3)^3 := \{E \in L^2_\rho(\mathbb{R}^3)^3 \,|\, \nabla E \in L^2(\mathbb{R}^3)^{3\times 3}\}.
\end{aligned}$$

They are Hilbert spaces with respect to the norms

$$\begin{aligned}
&\|\cdot\|^2_{L^2_\rho(\mathbb{R}^3)} = \|(1 + |x|^2)^{-\frac{1}{2}} \cdot \|^2_{L^2(\mathbb{R}^3)}, && \|\cdot\|^2_{W(\mathrm{curl},\mathbb{R}^3)} = \|\cdot\|^2_{L^2_\rho(\mathbb{R}^3)^3} + \|\,\mathrm{curl}\,\cdot\|^2_{L^2(\mathbb{R}^3)^3},\\
&\|\cdot\|_{W^1(\mathbb{R}^3)} = \|\nabla \cdot\|_{L^2(\mathbb{R}^3)^3}, && \|\cdot\|_{W^1(\mathbb{R}^3)^3} = \|\nabla \cdot\|_{L^2(\mathbb{R}^3)^{3\times 3}}.
\end{aligned}$$

The spaces $W^1(\mathbb{R}^3 \setminus \overline{\Omega})$, $W(\mathrm{curl}, \mathbb{R}^3 \setminus \overline{\Omega})$, $W(\mathrm{curl}, \mathbb{R}^3 \setminus \Gamma)$ and $H(\mathrm{curl}, \Omega)$ are defined likewise. We frequently use the closed subspace $W^1_\diamond \subset W^1(\mathbb{R}^3)^3$ of functions with vanishing divergence. Beyond that, we make use of the space $TH^{-1/2}(\mathrm{curl}_\Gamma)$ and its dual space $TH^{-1/2}(\mathrm{div}_\Gamma)$, cf., e.g., [12, Chp. 2], and the surjective trace mappings

$$\begin{aligned}
&H(\mathrm{curl}, \Omega) \to TH^{-1/2}(\mathrm{curl}_\Gamma), \quad E \mapsto \gamma_\Gamma E := (\nu \times E|_\Gamma) \times \nu,\\
&H(\mathrm{curl}, \Omega) \to TH^{-1/2}(\mathrm{div}_\Gamma), \quad E \mapsto \nu \times E|_\Gamma.
\end{aligned}$$

For a Banach space X, $C(0, T, X)$ and $L^2(0, T, X)$ denote the space of vector-valued functions

$$E : [0, T] \to X$$

which are continuous, respectively, square integrable; cf. e.g., [16, XVIII, §1]. $H^1(0, T, X)$ is defined likewise.

The inner product on a real Hilbert space H is denoted by $(\,\cdot\,,\,\cdot\,)_H$ and the dual pairing on $H' \times H$ by $\langle\,\cdot\,,\,\cdot\,\rangle_H$. They are related by the isometry $\iota_H : H \to H'$, that "identifies H with its dual":

$$\langle \iota_H u,\,\cdot\,\rangle_H := (u,\,\cdot\,)_H \quad \text{for all } u \in H.$$

We denote the dual operator of an operator $A \in \mathcal{L}(H_1, H_2)$ between real Hilbert spaces H_1, H_2 by A'. For all $h_2' \in H_2'$, A' is defined by

$$\langle A'h_2', h_1\rangle_{H_1} := \langle h_2', Ah_1\rangle_{H_2} \quad \text{for all } h_1 \in H_1.$$

Throughout this work we rigorously distinguish between the dual and the adjoint operator (denoted by A^*). They satisfy the identity $A^* = \iota_{H_1}^{-1} A' \iota_{H_2}$.

We usually omit the arguments x and t and only use them where we expect them to improve readability.

3. The direct problem

This section summarizes the results of [8] on the solution theory of the direct problem. All proofs and details can be found there.

We assume that we are given an arbitrary right hand side $l \in L^2(0, T, W(\mathrm{curl}, \mathbb{R}^3)')$ that obeys $\operatorname{div} l = 0$ and initial values σE^0 with $E^0 \in L^2(\mathbb{R}^3)^3$ that fulfill $\operatorname{div}(\sigma E^0) = 0$.

Theorem 3.1. *([8, Thm. 2.1]) Let $E \in L^2(0, T, W(\mathrm{curl}, \mathbb{R}^3))$. The eddy current problem reads*

$$\partial_t(\sigma(x)E(x,t)) + \mathrm{curl}\left(\frac{1}{\mu(x)}\,\mathrm{curl}\,E(x,t)\right) = l(x,t) \qquad \text{in } \mathbb{R}^3\times]0,T[, \tag{2}$$

$$\sqrt{\sigma(x)}E(x,0) = \sqrt{\sigma(x)}E^0(x) \qquad \text{in } \mathbb{R}^3. \tag{3}$$

The following holds:

a) For every solution $E \in L^2(0, T, W(\mathrm{curl}, \mathbb{R}^3))$ of (2) it holds, that $\sqrt{\sigma}E \in C(0, T, L^2(\mathbb{R}^3)^3)$.

b) $E \in L^2(0, T, W(\mathrm{curl}, \mathbb{R}^3))$ solves (2)–(3) if and only if E solves

$$-\int_0^T \int_{\mathbb{R}^3} \sigma E \cdot \partial_t \Phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu}\,\mathrm{curl}\,E \cdot \mathrm{curl}\,\Phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle l, \Phi\rangle_{W(\mathrm{curl},\mathbb{R}^3)}\,\mathrm{d}t + \int_{\mathbb{R}^3} \sigma E^0 \cdot \Phi(0)\,\mathrm{d}x \tag{4}$$

for all $\Phi \in \mathcal{D}(\mathbb{R}^3 \times [0,T[)^3$.

c) Equations (2)–(3) uniquely determine $\mathrm{curl}\,E$ and $\sqrt{\sigma}E$. Moreover, if $E \in L^2(0, T, W(\mathrm{curl}, \mathbb{R}^3))$ solves (2)–(3), then every $F \in L^2(0, T, W(\mathrm{curl}, \mathbb{R}^3))$ with $\mathrm{curl}\,F = \mathrm{curl}\,E$ and $\sqrt{\sigma}F = \sqrt{\sigma}E$ also solves (2)–(3).

Here, the time-derivative of σE is to be understood in the following way: Every solution $E \in L^2(0, T, W(\mathrm{curl}, \mathbb{R}^3))$ of (2) is an element of the space

$$\mathcal{W}_\sigma := \left\{E \in L^2(0, T, W(\mathrm{curl}, \mathbb{R}^3)) \,|\, (\sigma E)^{\cdot} \in L^2(0, T, W(\mathrm{curl}, \mathbb{R}^3)')\right\},$$

cf. [8, Lemma 2.3], where $(\sigma E)^{\cdot}$ denotes the time-derivative of $\sigma E \in L^2(0,T,L^2(\mathbb{R}^3)^3)$ in the sense of vector-valued distributions with respect to the canonical injection $L^2(\mathbb{R}^3)^3 \hookrightarrow W(\mathrm{curl},\mathbb{R}^3)'$. For $E,F \in \mathcal{W}_\sigma$ the following integration by parts formula holds:

$$\begin{aligned}&\int_0^T \left[\langle(\sigma E)^{\cdot}(t),F(t)\rangle_{W(\mathrm{curl},\mathbb{R}^3)} + \langle(\sigma F)^{\cdot}(t),E(t)\rangle_{W(\mathrm{curl},\mathbb{R}^3)}\right]\,\mathrm{d}t\\ &= \int_{\mathbb{R}^3} \left[\sigma E(T)\cdot F(T) - \sigma E(0)\cdot F(0)\right]\mathrm{d}x,\end{aligned} \tag{5}$$

cf., e.g., [16, XVIII, §1, Thms. 1,2] and [8, Lemma 2.2]. Note, that it is equivalent to find $E \in L^2(0,T,W(\mathrm{curl},\mathbb{R}^3))$ that solves (4), or to find $E \in \mathcal{W}_\sigma$ that solves (3) and

$$\int_0^T \langle(\sigma E)^{\cdot},F\rangle_{W(\mathrm{curl},\mathbb{R}^3)}\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^3}\frac{1}{\mu}\,\mathrm{curl}\,E\cdot\mathrm{curl}\,F\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\langle l,F\rangle_{W(\mathrm{curl},\mathbb{R}^3)}\,\mathrm{d}t \tag{6}$$

for all $F \in L^2(0,T,W(\mathrm{curl},\mathbb{R}^3))$, see [8, Thm. 2.4].

Lemma 3.2. *([8, Lemma 3.1]) There is a continuous linear map*

$$L^2_\rho(\mathbb{R}^3)^3 \to H(\mathrm{curl}\,0,\mathbb{R}^3) := \{E \in L^2(\mathbb{R}^3)^3 \mid \mathrm{curl}\,E = 0\},\quad E \mapsto \nabla u_E,$$

with $\mathrm{div}(\sigma(E+\nabla u_E)) = 0$ *in* $\mathbb{R}^3$.

We define the bilinear form $a_\sigma : L^2(0,T,W^1(\mathbb{R}^3)^3)\times H^1(0,T,W^1(\mathbb{R}^3)^3) \to \mathbb{R}$ by

$$a_\sigma(E,\Phi) := -\int_0^T\int_{\mathbb{R}^3}\sigma(E+\nabla u_E)\cdot\dot{\Phi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^3}\frac{1}{\mu}\,\mathrm{curl}\,E\cdot\mathrm{curl}\,\Phi\,\mathrm{d}x\,\mathrm{d}t.$$

Theorem 3.3. (Well-posedness of the eddy current problem, [8, Thm. 3.2])

a) If $E \in L^2(0,T,W^1_\diamondsuit)$ *solves*

$$a_\sigma(E,\Phi) = \int_0^T\langle l,\Phi\rangle_{W(\mathrm{curl},\mathbb{R}^3)}\,\mathrm{d}t + \int_{\mathbb{R}^3}\sigma E^0\cdot\Phi(0)\,\mathrm{d}x \quad \text{for all } \Phi \in H^1_{T0}(0,T,W^1_\diamondsuit), \tag{7}$$

then $E + \nabla u_E \in L^2(0,T,W(\mathrm{curl},\mathbb{R}^3))$ *solves (2)–(3), where*

$$H^1_{T0}(0,T,W^1_\diamondsuit) := \{\Psi \in H^1(0,T,W^1_\diamondsuit) \mid \Psi(T) = 0\}.$$

b) There is a unique solution $E \in L^2(0,T,W^1_\diamondsuit)$ *of (7).* E *depends continuously on* l *and* $\sqrt{\sigma}E^0$. $E + \nabla u_E$ *solves the eddy current equations (2)–(3) and any other solution* $F \in L^2(0,T,W(\mathrm{curl},\mathbb{R}^3))$ *of (2)–(3) fulfills*

$$\mathrm{curl}\,F = \mathrm{curl}\,E,\quad \sqrt{\sigma}F = \sqrt{\sigma}(E+\nabla u_E).$$

$\mathrm{curl}\,F$ *and* $\sqrt{\sigma}F$ *depend continuously on* l *and* $\sqrt{\sigma}E^0$.

We will also consider the case $\sigma \equiv 0$ and $\mu \equiv 1$, that we will call the *reference problem*. This case corresponds to the eddy current problem without any conducting medium. Then, theorem 3.1 and theorem 3.3 reduce to

Theorem 3.4. (Well-posedness of the reference problem, cf. [8, Thm. 3.3])
Let $E \in L^2(0,T,W(\mathrm{curl},\mathbb{R}^3))$.

a) The reference problem reads

$$\operatorname{curl}\operatorname{curl} E(x,t) = l(x,t) \qquad \text{in } \mathbb{R}^3\times]0,T[. \tag{8}$$

b) E solves (8) if and only if E solves

$$\begin{aligned} a_0(E,\Phi) &:= \int_0^T \int_{\mathbb{R}^3} \operatorname{curl} E \cdot \operatorname{curl}\Phi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \langle l, \Phi\rangle_{W(\operatorname{curl},\mathbb{R}^3)}\,\mathrm{d}t \qquad \text{for all } \Phi \in L^2(0,T,W^1_\diamondsuit), \end{aligned} \tag{9}$$

where $a_0 : L^2(0,T,W(\operatorname{curl},\mathbb{R}^3))^2 \to \mathbb{R}$.

c) There exists a unique solution $E \in L^2(0,T,W^1_\diamondsuit)$ of (9) and this solution depends continuously on l. Any other solution $F \in L^2(0,T,W(\operatorname{curl},\mathbb{R}^3))$ fulfills

$$\operatorname{curl} F = \operatorname{curl} E$$

and $\operatorname{curl} F$ *depends continuously on l.*

4. Electromagnetic measurements

We now turn to the description of our idealized measurement instrument. As in, e.g., [20, 21], we assume that the electric field E is generated by transient surface currents on a two-dimensional sheet S. In this way we assume that we can apply every divergence-free tangential function I supported in S as excitation on the right hand side of (2). Our idealized measurement instrument also measures the tangential component of the electric field on S.

Mathematically, the setting is as follows. We assume that

$$S \subset \mathbb{R}^3_0 := \{(x_1,x_2,0)^T \in \mathbb{R}^3\}$$

is (as a subset of $\mathbb{R}^2$) a bounded Lipschitz domain. Let n be the outer unit normal on S, i.e., $n = (0,0,1)^T$. We assume that Ω is placed below S and that $\overline{\Omega}\cap\overline{S} = \emptyset$, i.e.,

$$\overline{\Omega} \subset \{(x_1,x_2,x_3)^T \in \mathbb{R}^3 \,|\, x_3 < 0\}.$$

We consider the excitation I as an element of the space $L^2(0,T,TL^2_\diamondsuit(S))$. Here, the space $TL^2_\diamondsuit(S)$ denotes the subspace of the space $TL^2(S)$ of elements with vanishing divergence, where

$$TL^2(S) := \{u \in L^2(S)^3 \,|\, n\cdot u = 0\}$$

is the space of tangential functions. Using the continuous extension of the identification of an element $I \in TL^2(S)$ with the distribution

$$\Phi \mapsto \int_S I\cdot\Phi\,\mathrm{d}S = \int_S I\cdot((n\times\Phi|_S)\times n)\,\mathrm{d}S \quad \text{for all } \Phi \in \mathcal{D}(\mathbb{R}^3)^3$$

to $W(\operatorname{curl},\mathbb{R}^3)$, we consider the spaces $TL^2(S)$ and $TL^2_\diamondsuit(S)$ as subspaces of $W(\operatorname{curl},\mathbb{R}^3)'$. Both, $TL^2(S)$ and $TL^2_\diamondsuit(S)$ are Hilbert spaces equipped with the usual $L^2(S)^3$-inner product. Hence, every $I \in L^2(0,T,TL^2_\diamondsuit(S))$ defines an element of the space

$L^2(0,T,W(\mathrm{curl},\mathbb{R}^3)')$ that satisfies $\operatorname{div} I = 0$. In this sense we can consider the surface current $I \in L^2(0,T,TL^2_\diamondsuit(S))$ as a source term for the eddy current equation (2), respectively, the reference problem (8). In the following we don't distinguish between $I \in L^2(0,T,TL^2_\diamondsuit(S))$ and the corresponding element of $L^2(0,T,W(\mathrm{curl},\mathbb{R}^3)')$ and still write the dual pairing as a $L^2(S)^3$-product.

To define the measurement operator we first remark, that the mapping

$$W^1(\mathbb{R}^3)^3 \to TL^2(S),\ E \mapsto \gamma_S E := (n \times E|_S) \times n$$

is linear and continuous. Moreover, let

$$N_S := \overline{\mathcal{R}\,(\gamma_S \nabla D(\mathbb{R}^3))} \subset TL^2(S).$$

It can easily be verified, that $N_S \oplus^\perp TL^2_\diamondsuit(S) = TL^2(S)$ and

$$TL^2(S)/N_S \cong TL^2_\diamondsuit(S)'. \tag{10}$$

Together with the identification of $TL^2_\diamondsuit(S)$ with its dual we consider the measurements as elements of $L^2(0,T,TL^2_\diamondsuit(S))$. This can be interpreted as measuring the electric field, so that it is adequately gauged to be divergence-free on S. Now, theorems 3.3 and 3.4 yield the following linear continuous operators.

Definition 4.1. (Measurement operator)
We define the measurement operator

$$\Lambda := \Lambda_0 - \Lambda_\sigma : L^2(0,T,TL^2_\diamondsuit(S)) \to L^2(0,T,TL^2_\diamondsuit(S)).$$

Here, Λ_0 *and* Λ_σ *are the mappings*

$$\begin{aligned}\Lambda_0, \Lambda_\sigma : L^2(0,T,TL^2_\diamondsuit(S)) &\to L^2(0,T,TL^2_\diamondsuit(S))\\ I &\mapsto \gamma_S E_0, \quad \text{respectively,} \quad \gamma_S E_\sigma,\end{aligned} \tag{11}$$

where $E_0, E_\sigma \in L^2(0,T,W^1_\diamondsuit)$ *are the unique solutions of*

$$a_0(E_0,F) = \int_0^T (\gamma_S F, I)_{L^2(S)^3}\,\mathrm{d}t \qquad \text{for all } F \in L^2(0,T,W^1_\diamondsuit), \tag{12}$$

$$a_\sigma(E_\sigma,F) = \int_0^T (\gamma_S F, I)_{L^2(S)^3}\,\mathrm{d}t \qquad \text{for all } F \in H^1_{T0}(0,T,W^1_\diamondsuit). \tag{13}$$

Note, that if E_0 and E_σ solve (12) and (13), then they are the unique solutions of (9) and (7) with right hand side I. This means that $E_\sigma + \nabla u_{E_\sigma} \in L^2(0,T,W(\mathrm{curl},\mathbb{R}^3))$ solves (2) with right hand side I and zero initial condition, cf. theorem 3.3 b). Especially, the just defined operators do not match the tangential value of the "real" electric field but just the tangential value of its divergence-free part.

Let us stress, that even if (2)–(3) does not determine the solution uniquely, in the measurement space, measurements of different solutions coincide. This is up to (10) and the fact, that in a neighborhood of S all solutions $E \in L^2(0,T,W(\mathrm{curl},\mathbb{R}^3))$ of (2)–(3) equal up to gradient fields. Hence, the evaluation of $\gamma_S E$ in $L^2(0,T,TL^2_\diamondsuit(S))$ is also well-defined, linear and continuous and defines the same element as $\gamma_S E_\sigma$. Therefore, we understand Λ as a gauged measurement operator, where $\gamma_S E_0$, $\gamma_S E_\sigma$ actually represent equivalence classes, cf. (10).

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Before we start with the inverse problem, we introduce the time-integral operator

$$\Xi : L^2(0,T,TL^2_\diamondsuit(S)) \to TL^2_\diamondsuit(S), \qquad h \mapsto \int_0^T h(t)\,\mathrm{d}t.$$

Its adjoint operator maps a time-independent function $I \in TL^2_\diamondsuit(S)$ on its counterpart in $L^2(0,T,TL^2_\diamondsuit(S))$ that is constant in time, i.e.

$$(\Xi^* I)(t) = I,\ t \in (0,T).$$

To maintain lucidity, we will usually omit Ξ^*.

5. Linear sampling method

In this section we show, that a subset of Ω is determined by the measurements. Therefore, we factorize the measurement operator into

$$\Lambda = LN,$$

where N maps an excitation on S to its effect on the conductor, and L measures then the induced electric field on S. In linear sampling or factorization method context, L is often called the *virtual measurement operator*. Its range contains information to detect Ω.

We start with this operator. Let $H(\mathrm{curl},\Omega)'_\diamondsuit$ denote the subspace of $H(\mathrm{curl},\Omega)'$ of elements with vanishing divergence,

$$H(\mathrm{curl},\Omega)'_\diamondsuit := \{g \in H(\mathrm{curl},\Omega)' \,|\, \langle g, \nabla\phi\rangle_{H(\mathrm{curl},\Omega)} = 0 \ \text{ for all } \phi \in \mathcal{D}(\overline{\Omega})\}.$$

Then $H(\mathrm{curl},\Omega)'_\diamondsuit$ is a Hilbert space and the following operator is linear and continuous:

$$L : L^2(0,T,H(\mathrm{curl},\Omega)'_\diamondsuit) \to L^2(0,T,TL^2_\diamondsuit(S)),\ B \mapsto \gamma_S H,$$

where $H \in L^2(0,T,W^1_\diamondsuit)$ solves

$$a_0(H,F) = \int_0^T \langle B, F|_\Omega\rangle_{H(\mathrm{curl},\Omega)}\,\mathrm{d}t \qquad \text{for all } F \in L^2(0,T,W^1_\diamondsuit). \tag{14}$$

We show the following relation between L and Λ:

Lemma 5.1. *It holds that* $\mathcal{R}(\Lambda) \subset \mathcal{R}(L)$.

Proof. We show that $\Lambda = LN$ with an appropriate operator N.

The assumption $\overline{\Omega} \cap \overline{S} = \emptyset$ ensures, that for solutions $E \in L^2(0,T,W(\mathrm{curl},\mathbb{R}^3))$ of (2) the evaluation $\nu \times \mathrm{curl}\, E|_\Gamma^+ \in L^2(0,T,TH^{-\frac{1}{2}}(\mathrm{div}_\Gamma))$ is linear and continuous, where we denote by the $+$-sign the value from the outside of Ω. Moreover, for $t \in (0,T)$ a.e. we have, that

$$F \mapsto \langle \nu \times \mathrm{curl}\, E(t)|_\Gamma^+, \gamma_\Gamma F\rangle_{TH^{-\frac{1}{2}}(\mathrm{curl}_T)} \qquad \text{for all } F \in H(\mathrm{curl},\Omega)$$

defines an element of $H(\mathrm{curl},\Omega)'_\diamondsuit$. Hence, the following operator is linear and continuous:

$$N :\ L^2(0,T,TL^2_\diamondsuit(S)) \to L^2(0,T,H(\mathrm{curl},\Omega)'_\diamondsuit), \quad I \mapsto h,$$

with $\quad h : F \mapsto -\int_0^T \langle \nu \times \mathrm{curl}\, E_\sigma|_\Gamma^+, \gamma_\Gamma F\rangle_{TH^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)}\,\mathrm{d}t + \int_0^T \int_\Omega \mathrm{curl}\, E_\sigma \cdot \mathrm{curl}\, F\,\mathrm{d}x\,\mathrm{d}t$

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for all $F \in L^2(0,T,H(\mathrm{curl},\Omega))$, and where E_σ solves (13) with source I,

To show that $\Lambda = LN$, let $I \in L^2(0,T,TL^2_\diamondsuit(S))$ and E_0 and E_σ denote the solutions of (12) and (13) with source I. For $t \in (0,T)$ a.e. a short computation using (6) shows, that for every $\Phi \in \mathcal{D}(\mathbb{R}^3)^3$

$$\begin{aligned}&\langle(\sigma(E_\sigma + \nabla u_{E_\sigma}))^{\cdot}(t), \Phi\rangle_{W(\mathrm{curl},\mathbb{R}^3)}\\ &= -\int_\Omega \frac{1}{\mu}\,\mathrm{curl}\,E_\sigma(t)\cdot \mathrm{curl}\,\Phi\,\mathrm{d}x - \langle \nu\times \mathrm{curl}\,E_\sigma(t)|_\Gamma^+, \gamma_\Gamma\Phi\rangle_{TH^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)}.\end{aligned}$$

The right hand side depends continuously on $\Phi|_\Omega \in \mathcal{D}(\overline{\Omega})^3 \subset H(\mathrm{curl},\Omega)$, thus denseness implies that it defines an element of $H(\mathrm{curl},\Omega)'$. Using this, (6) and integration by parts (5), we obtain for every $\Phi \in \mathcal{D}(\mathbb{R}^3\times]0,T[)^3$, that

$$\begin{aligned}&a_0(E_0 - E_\sigma, \Phi) = a_\sigma(E_\sigma,\Phi) - a_0(E_\sigma,\Phi)\\ &= \int_0^T \langle(\sigma(E_\sigma + \nabla u_{E_\sigma}))^{\cdot}, \Phi\rangle_{W(\mathrm{curl},\mathbb{R}^3)}\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^3}\frac{1}{\mu}\,\mathrm{curl}\,E_\sigma\cdot\mathrm{curl}\,\Phi\,\mathrm{d}x\,\mathrm{d}t - a_0(E_\sigma,\Phi)\\ &= -\int_0^T \langle \nu\times\mathrm{curl}\,E_\sigma|_\Gamma^+, \gamma_\Gamma\phi\rangle_{TH^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)}\,\mathrm{d}t - \int_0^T\int_\Omega \mathrm{curl}\,E_\sigma\cdot\mathrm{curl}\,\Phi\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T \langle NI, \Phi|_\Omega\rangle_{H(\mathrm{curl},\Omega)}\,\mathrm{d}t.\end{aligned}$$

On the other hand, let $H \in L^2(0,T,W^1_\diamondsuit)$ be the solution of (14) with $B = NI$. Then, again denseness implies

$$a_0(E_0 - E_\sigma, \Phi) = a_0(H,\Phi) \quad \text{for all } \Phi \in L^2(0,T,W^1_\diamondsuit),$$

and then uniqueness implies $H = E_0 - E_\sigma$, cf. theorem 3.4 c). It follows
$\Lambda I = \gamma_S(E_0 - E_\sigma) = \gamma_S H = LNI.$ □

To characterize the conductor, we introduce for an arbitrary direction $d \in \mathbb{R}^3$, $|d| = 1$ the functions

$$G_{z,d} : \mathbb{R}^3\setminus\{z\} \to \mathbb{R}^3, \quad x \mapsto \mathrm{curl}\,\frac{d}{|x-z|},$$

that have a dipole in $z \in \mathbb{R}^3$. In $\mathbb{R}^3\setminus\{z\}$, every component of $G_{z,d}$ solves the homogeneous Laplace equation. Therefore, $G_{z,d}$ is analytic in $\mathbb{R}^3\setminus\{z\}$.

The following theorem shows, that a subset of Ω is determined by Λ.

Theorem 5.2. (Linear sampling method)
For every direction $d \in \mathbb{R}^3$, $|d| = 1$, and every point $z \in \mathbb{R}^3$, z below S, $z \notin \Gamma$,

$$\gamma_S G_{z,d} \in \mathcal{R}(\Xi\Lambda) \quad \textit{implies} \quad z \in \Omega.$$

Proof. Let $\gamma_S G_{z,d} \in \mathcal{R}(\Xi\Lambda)$. Lemma 5.1 yields $\mathcal{R}(\Lambda) \subset \mathcal{R}(L)$, hence there is a preimage $B \in L^2(0,T,H(\mathrm{curl},\Omega)'_\diamondsuit)$ and some $H \in L^2(0,T,W^1_\diamondsuit)$ that solves (14) and that fulfills

$$\Xi\gamma_S H = \gamma_S G_{z,d}.$$

We consider $E := \int_0^T H(t)\,\mathrm{d}t \in W^1_\diamondsuit$ and obtain $\gamma_S E = \gamma_S G_{z,d}$, i.e., $\gamma_S(E - G_{z,d}) \in N_S$, and

$$\operatorname{curl}\operatorname{curl} E = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \qquad \operatorname{div} E = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}.$$

Thus E is analytic in $\mathbb{R}^3 \setminus \overline{\Omega}$. Moreover, $G_{z,d}$ is analytic in $\mathbb{R}^3 \setminus \{z\}$, and it follows that $\operatorname{curl}(E - G_{z,d})$ is analytic in $\mathbb{R}^3 \setminus \left(\overline{\Omega} \cup \{z\}\right)$. Now, following [21], we obtain by unique continuation of analytic functions, that

$$\operatorname{curl} E = \operatorname{curl} G_{z,d} \quad \text{in } \mathbb{R}^3 \setminus \{z\}.$$

The fact, that $\operatorname{curl} E \in L^2(\mathbb{R}^3 \setminus \overline{\Omega})$ but $\operatorname{curl} G_{z,d} \in L^2(\mathbb{R}^3 \setminus \overline{\Omega})$ only if $z \in \Omega$ yields the assertion. □

Further results on unique characterization can be obtained if we assume some additional feature on the permeability μ. This is done in the following sections.

6. Unique shape identification

For the rest of this paper we assume in addition, that the permeability is smaller on the conductor than on the background:

$$1 - \mu|_\Omega \in L^\infty_+(\Omega).$$

This is the case, for instance, for diamagnetic materials.

We moreover assume, that the connected components of Ω are simply connected. This is only due to technical reasons, we expect our theory also to hold for multiply connected domains, that fulfill [15, IX, Part A, §3, (1.45)], for instance, if Ω has the form of a torus.

Now we formulate our main result. The proof is postponed to section 7.

Theorem 6.1. (Unique shape identification)
It holds for every direction $d \in \mathbb{R}^3$, $|d| = 1$, and every point $z \in \mathbb{R}^3$, z below S, $z \notin \Gamma$, that

$$\begin{aligned} & z \in \Omega \qquad \textit{if and only if} \\ \exists C > 0: \quad & (G_{z,d}, I)^2_{L^2(S)^3} \leq C \int_0^T (\Lambda I, I)_{L^2(S)^3}\,\mathrm{d}t \qquad \text{for all } I \in TL^2_\diamondsuit(S), \end{aligned} \tag{15}$$

where

$$G_{z,d}(x) = \operatorname{curl} \frac{d}{|x - z|}.$$

In particular, Λ uniquely determines Ω. Let us stress, that therefore only time-independent I are needed. This means, that the applied source currents J on S (recall, that I denotes the time-derivative of J) only depend linearly on time.

To formulate an equivalent formulation of theorem 6.1, we make the following observation. Let $I \in L^2(0, T, TL^2_\diamondsuit(S))$ and E_0 and E_σ be the solutions of (12) and

(13) with source I. Then, integrating E_σ by parts in time (5), and using the fact, that E_0 minimizes the functional

$$L^2(0,T,W^1_\diamondsuit) \to \mathbb{R}, \quad E \mapsto \frac{1}{2}a_0(E,E) - \int_0^T (\gamma_S E, I)_{L^2(S)^3}\,\mathrm{d}t,$$

leads to

$$\int_0^T (\Lambda I, I)_{L^2(S)^3}\,\mathrm{d}t \geq \int_0^T (\gamma_S E_\sigma, I)_{L^2(S)^3}\,\mathrm{d}t - a_0(E_\sigma, E_\sigma)$$
$$\geq \frac{1}{2}\|\sqrt{\sigma}(E_\sigma + \nabla u_{E_\sigma})(T)\|^2_{L^2(\Omega)^3} + \inf_\Omega \left[\frac{1}{\mu} - 1\right] \|\operatorname{curl} E_\sigma\|^2_{L^2(0,T,L^2(\Omega)^3)} \geq 0. \tag{16}$$

An immediate consequence is the following. The linear continuous and (by construction) self adjoint operator

$$\tilde{\Lambda} := \Xi(\Lambda + \Lambda^*)\Xi^* : TL^2_\diamondsuit(S) \to TL^2_\diamondsuit(S)$$

is positive, as for every $I \in TL^2_\diamondsuit(S)$ it holds

$$(\tilde{\Lambda} I, I)_{L^2(S)^3} = (\Xi(\Lambda + \Lambda^*)\Xi^* I, I)_{L^2(S)^3} = \int_0^T ((\Lambda + \Lambda^*)\Xi^* I, \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t$$
$$= 2\int_0^T (\Lambda \Xi^* I, \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t \geq 0.$$

Hence, the square root $\tilde{\Lambda}^{\frac{1}{2}}$ exists.

Corollary 6.2. (Factorization method)
It holds for every direction $d \in \mathbb{R}^3$, $|d| = 1$, and every point $z \in \mathbb{R}^3$, z below S, $z \notin \Gamma$, that

$$z \in \Omega \quad \textit{if and only if} \quad \gamma_S G_{z,d} \in \mathcal{R}\left(\tilde{\Lambda}^{1/2}\right).$$

Proof. Theorem 6.1 yields that $z \in \Omega$ if and only if

$$\exists C > 0 : \quad (\gamma_S G_{z,d}, I)^2_{L^2(S)^3} \leq C \int_0^T (\Lambda \Xi^* I, \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t \quad \text{for all } I \in TL^2_\diamondsuit(S). \tag{17}$$

For every $I \in TL^2_\diamondsuit(S)$, (17) equals

$$(\gamma_S G_{z,d}, I)^2_{L^2(S)^3} \leq C \int_0^T (\Lambda \Xi^* I, \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t = \frac{C}{2}(\tilde{\Lambda} I, I)_{L^2(S)^3} = \frac{C}{2}\|\tilde{\Lambda}^{1/2} I\|^2_{L^2(S)^3}.$$

A standard result on the relation between the norm of an operator and the range of its dual, cf., e.g., the “14th important property of Banach spaces“ in Bourbaki [11] or [18, Lemma 3.4] in case of Hilbert spaces, yields, that this is equivalent to

$$\gamma_S G_{z,d} \in \mathcal{R}\left(\tilde{\Lambda}^{1/2}\right).$$

$\square$

7. Constraining operators for Λ

The key of the proof of theorem 6.1 is to find adequate operators that control the measurement operator from below and from above, cf. [23]. To be more precise, we are looking for operators R_1 and R_2 mapping into particular Hilbert spaces, that fulfill

$$c\|R_1 I\|^2 \leq \int_0^T (\Lambda I, I)_{L^2(S)^3}\,\mathrm{d}t \leq c'\|R_2 I\|^2$$

for all $I \in L^2(0,T,TL^2_\diamondsuit(S))$ with some positive constants c, c'. These Hilbert spaces will depend on Ω, so that the operators can be used to determine Ω uniquely.

In this section we introduce the operators R_1 and R_2 and show, how they can be used to characterize Ω. At the end of this section we give a proof of theorem 6.1.

7.1. Lower bound

For the lower bound, an appropriate candidate for R_1 can be found easily. Therefore, let $I \in L^2(0,T,TL^2_\diamondsuit(S))$ and E_0 and E_σ be the solutions of (12) and (13) with source I. Then, (16) yields

$$\begin{aligned}\int_0^T (\Lambda I, I)_{L^2(S)^3}\,\mathrm{d}t &\geq \frac{1}{2}\|\sqrt{\sigma}(E_\sigma + \nabla u_{E_\sigma})(T)\|^2_{L^2(\Omega)^3} + \inf_\Omega \left[\frac{1}{\mu} - 1\right] \|\operatorname{curl} E_\sigma\|^2_{L^2(0,T,L^2(\Omega)^3)} \\ &\geq c\left[\|\sigma(E_\sigma + \nabla u_{E_\sigma})(T)\|^2_{L^2(\Omega)^3} + \|\operatorname{curl} E_\sigma\|^2_{L^2(0,T,L^2(\Omega)^3)}\right] \\ &=: c\,\|R_1 I\|^2 \qquad (18)\end{aligned}$$

with the constant

$$c = \min\left\{\frac{1}{2\|\sigma\|_\infty}, \inf_\Omega\left(\frac{1}{\mu} - 1\right)\right\}.$$

To define R_1 rigorously, let us first introduce the following factor space.

$$X := H(\operatorname{curl}, \Omega)/\mathcal{N}, \qquad \text{where } \mathcal{N} := \ker \operatorname{curl} = \nabla H^1(\Omega),$$

cf. [15, IX, Part A, §1, Prop. 2 & Rem. 6], is a Hilbert space with respect to the induced norm

$$\|u + \mathcal{N}\|_X := \inf_{m\in\mathcal{N}} \|u - m\|_{H(\operatorname{curl},\Omega)}.$$

Lemma 7.1. *An equivalent norm on X is given by*

$$u + \mathcal{N} \mapsto \|\operatorname{curl} u\|_{L^2(\Omega)}.$$

Proof. We consider $u + \mathcal{N} \in X$. Then we have

$$\|u + \mathcal{N}\|^2_X = \inf_{m\in\mathcal{N}} \|u - m\|^2_{H(\operatorname{curl},\Omega)} \geq \|\operatorname{curl} u\|^2_{L^2(\Omega)}.$$

Moreover, [15, IX, Part A, §1, Cor. 5 & Rem. 6] yields that every u has a unique orthogonal decomposition

$$u = \nabla p + \operatorname{curl} w,$$

where $p \in H^1(\Omega)$ and $w \in H^1(\Omega)^3$ with $\nu \cdot \operatorname{curl} w|_\Gamma = 0$ (w must not be unique, but $\operatorname{curl} w$ is). A short computation shows

$$\|u + \mathcal{N}\|_X^2 = \|\operatorname{curl} w\|_{L^2(\Omega)}^2 + \|\operatorname{curl} u\|_{L^2(\Omega)}^2.$$

Now, [15, IX, Part A, §1, Rems. 4 & 6] yields that

$$\operatorname{curl} : \{a \in H^1(\Omega)^3 \mid \operatorname{div} a = 0, \nu \cdot a|_\Gamma = 0\} \to \operatorname{curl} H^1(\Omega)^3$$

is an isomorphism and therefore has a continuous linear inverse. As $\operatorname{curl} w$ is an element of that space, it follows

$$\begin{aligned}\|u + \mathcal{N}\|_X^2 &= \|\operatorname{curl} w\|_{L^2(\Omega)}^2 + \|\operatorname{curl} u\|_{L^2(\Omega)}^2 \\ &\leq c''\|\operatorname{curl}\operatorname{curl} w\|_{L^2(\Omega)}^2 + \|\operatorname{curl} u\|_{L^2(\Omega)}^2 = (c''+1)\|\operatorname{curl} u\|_{L^2(\Omega)}^2\end{aligned}$$

with a constant c'' independent of u (or its decomposition). □

Let $L^2(\Omega)^3_\diamondsuit$ be the space of $L^2(\Omega)^3$-functions with vanishing divergence. Obviously, $L^2(\Omega)^3_\diamondsuit$ is a Hilbert space.

Corollary 7.2. *The following mapping is linear and continuous:*

$$\begin{aligned}R_1 : L^2(0,T,TL^2_\diamondsuit(S)) &\to L^2(\Omega)^3_\diamondsuit \times L^2(0,T,X), \\ I &\mapsto (\sigma(E_\sigma + \nabla u_{E_\sigma})(T)|_\Omega, E_\sigma|_\Omega + \mathcal{N}),\end{aligned}$$

where E_σ solves (13) with source I. Its dual mapping is given by

$$R_1' : (L^2(\Omega)^3_\diamondsuit)' \times L^2(0,T,X') \to L^2(0,T,TL^2_\diamondsuit(S)), \quad (v,w) \mapsto h,$$

where h obeys for every $I \in L^2(0,T,TL^2_\diamondsuit(S))$

$$\begin{aligned}\int_0^T (h,I)_{L^2(S)^3}\,\mathrm{d}t &= \int_0^T (R_1'(v,w),I)_{L^2(S)^3}\,\mathrm{d}t \\ &= \langle v, \sigma(E_\sigma + \nabla u_{E_\sigma})(T)|_\Omega\rangle_{L^2(\Omega)^3_\diamondsuit} + \int_0^T \langle w, E_\sigma|_\Omega + \mathcal{N}\rangle_X\,\mathrm{d}t,\end{aligned}$$

where E_σ denotes the solution of (13) with source I, again.

Now, the inequality (18) reads: There is a positive constant c such that

$$c\,\|R_1 I\|^2_{L^2(\Omega)^3_\diamondsuit \times L^2(0,T,X)} \leq \int_0^T (\Lambda I, I)_{L^2(S)^3}\,\mathrm{d}t \quad \text{for all } I \in L^2(0,T,TL^2_\diamondsuit(S)). \tag{19}$$

The following lemma shows, that the range of R_1' determines a superset of Ω: Whenever a point z is inside Ω, then $\gamma_S G_{z,d}$ is contained in the range of the dual operator of R_1.

Lemma 7.3. *Let $z \in \Omega$. For every direction $d \in \mathbb{R}^3$, $|d| = 1$, there is a preimage $(v,w) \in (L^2(\Omega)^3_\diamondsuit)' \times L^2(0,T,X')$ of $\Xi R_1'$ with*

$$\gamma_S G_{z,d} = \Xi R_1'(v,w).$$

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Proof. For every $z \in \Omega$ there is an $\varepsilon > 0$ such that for the open ball $B_\varepsilon(z)$ it holds $\overline{B_\varepsilon(z)} \subset \Omega$. Now we choose a smooth cutoff function $\varphi \in C^\infty(\mathbb{R}^3)$ with $\varphi \equiv 1$ outside of $B_\varepsilon(z)$ and $\varphi \equiv 0$ in $B_{\frac{\varepsilon}{2}}(z)$. We obtain

$$\tilde{G}_{z,d}(x) := \operatorname{curl}\left(\frac{\varphi(x)d}{|x-z|}\right) \in H(\operatorname{curl}, \mathbb{R}^3) \quad \text{and we have} \quad \tilde{G}_{z,d} = G_{z,d} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}.$$

Let $\tilde{G}_{z,d}(t) := \tilde{G}_{z,d}$. Then, we have $\tilde{G}_{z,d} \in L^2(0,T,W^1_\diamondsuit)$, $\operatorname{curl} \tilde{G}_{z,d} \in L^2(0,T,H(\operatorname{curl},\mathbb{R}^3))$ and $\operatorname{curl}\operatorname{curl} \tilde{G}_{z,d} = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$.

We define $v \in (L^2(\Omega)^3_\diamondsuit)'$ and $w \in L^2(0,T,X')$ by

$$v : H \mapsto \int_\Omega H \cdot \tilde{G}_{z,d} \,\mathrm{d}x,$$

$$w : F + \mathcal{N} \mapsto \int_0^T \int_\Omega \left[\operatorname{curl}\operatorname{curl} \tilde{G}_{z,d} \cdot F + \left(\frac{1}{\mu} - 1\right) \operatorname{curl} \tilde{G}_{z,d} \cdot \operatorname{curl} F\right] \mathrm{d}x \,\mathrm{d}t.$$

We use the fact, that for all $F \in L^2(0,T,W^1_\diamondsuit)$ it holds

$$\int_0^T \int_{\mathbb{R}^3 \setminus \overline{\Omega}} \operatorname{curl} \tilde{G}_{z,d} \cdot \operatorname{curl} F \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \left[\operatorname{curl}\operatorname{curl} \tilde{G}_{z,d} \cdot F - \operatorname{curl} \tilde{G}_{z,d} \cdot \operatorname{curl} F\right] \mathrm{d}x\,\mathrm{d}t,$$

the identity (6) and the integration by parts formula (5) and obtain, that for every $I \in TL^2_\diamondsuit(S)$ it holds

$$\begin{aligned}
&(\Xi R_1'(v,w), I)_{L^2(S)^3} = \int_0^T (R_1'(v,w), \Xi^* I)_{L^2(S)^3} \,\mathrm{d}t \\
&= \int_\Omega \sigma(E_\sigma + \nabla u_{E_\sigma})(T) \cdot \tilde{G}_{z,d} \,\mathrm{d}x \\
&\qquad + \int_0^T \int_\Omega \left[\operatorname{curl}\operatorname{curl} \tilde{G}_{z,d} \cdot E_\sigma + \left(\frac{1}{\mu} - 1\right) \operatorname{curl} \tilde{G}_{z,d} \cdot \operatorname{curl} E_\sigma\right] \mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}^3} \langle (\sigma(E_\sigma + \nabla u_{E_\sigma}))^\cdot, \tilde{G}_{z,d} \rangle_{W(\operatorname{curl},\mathbb{R}^3)} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} \tilde{G}_{z,d} \cdot \operatorname{curl} E_\sigma \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T (\gamma_S \tilde{G}_{z,d}, \Xi^* I)_{L^2(S)^3} \,\mathrm{d}t = (\gamma_S G_{z,d}, I)_{L^2(S)^3},
\end{aligned}$$

where E_σ denotes the solution of (13) with source $\Xi^* I$. □

7.2. Upper bound

To define R_2, we consider the subspace of elements with vanishing divergence of $TH^{-1/2}(\operatorname{div}_\Gamma)$,

$$TH^{-1/2}_\diamondsuit(\Gamma) := \{g \in TH^{-1/2}(\operatorname{div}_\Gamma) \mid \operatorname{div} g = 0\},$$

where we understand $TH^{-1/2}(\operatorname{div}_\Gamma)$ as a subspace of $W(\operatorname{curl},\mathbb{R}^3)'$ by

$$E \mapsto \langle g, \gamma_\Gamma E \rangle_{TH^{-1/2}(\operatorname{curl}_\Gamma)} \quad \text{for all } E \in W(\operatorname{curl},\mathbb{R}^3).$$

Clearly, $TH_{\diamondsuit}^{-1/2}(\Gamma)$ is a Hilbert space with respect to $\|\cdot\|_{TH^{-1/2}(\mathrm{div}_\Gamma)}$. As the tangential components of elements of $W(\mathrm{curl},\mathbb{R}^3)$ are in $TH^{-1/2}(\mathrm{curl}_\Gamma)$, every $E\in W(\mathrm{curl},\mathbb{R}^3)$ defines an element of $TH_{\diamondsuit}^{-1/2}(\Gamma)'$ by

$$g\mapsto \langle g,\gamma_\Gamma E\rangle_{TH^{-1/2}(\mathrm{curl}_\Gamma)}\quad \text{for all } g\in TH_{\diamondsuit}^{-1/2}(\Gamma).$$

Now, theorems 3.3 and 3.4 yield the following corollary.

Corollary 7.4. *For $i=0,\sigma$, linear continuous mappings are given by*

$$K_i: L^2(0,T,TL^2_{\diamondsuit}(S))\to L^2(0,T,TH_{\diamondsuit}^{-1/2}(\Gamma)'),\quad I\mapsto d_i,$$
$$\text{with}\quad d_i: g\mapsto \int_0^T \langle g,\gamma_\Gamma E_i\rangle_{TH^{-1/2}(\mathrm{curl}_\Gamma)}\,\mathrm{d}t,$$

and where $E_0,E_\sigma\in L^2(0,T,W^1_{\diamondsuit})$ are the solutions of (12) and (13) with source I.

Their dual operators are given by

$$K_i': L^2(0,T,TH_{\diamondsuit}^{-1/2}(\Gamma))\to L^2(0,T,TL^2_{\diamondsuit}(S)),\ \ g\mapsto \gamma_S H_i,$$

where $H_0\in L^2(0,T,W^1_{\diamondsuit})$ solves the variational problem

$$a_0(H_0,\Phi)=\int_0^T\langle g,\gamma_\Gamma\Phi\rangle_{TH^{-1/2}(\mathrm{curl}_\Gamma)}\,\mathrm{d}t$$

for all $\Phi\in L^2(0,T,W^1_{\diamondsuit})$, and $H_\sigma\in L^2(0,T,W^1_{\diamondsuit})$ solves

$$a_\sigma(H_\sigma,\Phi)=\int_0^T\langle g,\gamma_\Gamma\Phi\rangle_{TH^{-1/2}(\mathrm{curl}_\Gamma)}\,\mathrm{d}t$$

for all $\Phi\in H^1(0,T,W^1_{\diamondsuit})$ with $\Phi(0)=0$.

We need two more operators and their duals:

Lemma 7.5. *For $i=0,\sigma$, linear continuous mappings are given by*

$$M_i: L^2(0,T,TL^2_{\diamondsuit}(S))\to L^2(0,T,TH^{-1/2}(\mathrm{div}_\Gamma)),\ \ I\mapsto \nu\times\mathrm{curl}\,E_i|_\Gamma^+,$$

where $E_0,E_\sigma\in L^2(0,T,W^1_{\diamondsuit})$ are the solutions of (12) and (13) with source I.

Their dual operators obey

$$M_i': L^2(0,T,TH^{-1/2}(\mathrm{curl}_\Gamma))\to L^2(0,T,TL^2_{\diamondsuit}(S)),\ \ f\mapsto -\gamma_S G_i$$

for some $G_i\in L^2(0,T,W(\mathrm{curl},\mathbb{R}^3\setminus\Gamma))$ that fulfill

$$\begin{aligned}\gamma_\Gamma G_i^+-\gamma_\Gamma G_i^- &= f &&\text{in } \Gamma\times(0,T),\\ \mathrm{curl}\,\mathrm{curl}\,G_i &= 0 &&\text{in } \mathbb{R}^3\setminus\overline{\Omega}\times(0,T).\end{aligned}$$

Proof. Again, the first assertion follows from theorem 3.3, theorem 3.4, and the fact, that the evaluation of $\nu\times\mathrm{curl}\,E|_\Gamma^+$ for solutions of (12) or (13) in $TH^{-1/2}(\mathrm{div}_\Gamma)$ is linear and continuous.

For the second assertion, let γ_Γ^{-1} be a linear, continuous right inverse of

$$\gamma_\Gamma: W(\mathrm{curl},\mathbb{R}^3\setminus\overline{\Omega})\to TH^{-1/2}(\mathrm{curl}_\Gamma).$$

For $f \in L^2(0,T,TH^{-1/2}(\mathrm{curl}_\Gamma))$ we denote $U^f := \gamma_\Gamma^{-1} f \in L^2(0,T,W(\mathrm{curl},\mathbb{R}^3 \setminus \overline{\Omega}))$. Let $U_0 \in L^2(0,T,W^1_\diamondsuit)$ be the solution of

$$\int_0^T \int_{\mathbb{R}^3} \mathrm{curl}\, U_0 \cdot \mathrm{curl}\, F \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_{\mathbb{R}^3\setminus\overline{\Omega}} \mathrm{curl}\, U^f \cdot \mathrm{curl}\, F \,\mathrm{d}x\,\mathrm{d}t$$

for all $F \in L^2(0,T,W^1_\diamondsuit)$. Then, for every $I \in L^2(0,T,TL^2_\diamondsuit(S))$ we obtain

$$\begin{aligned}
\int_0^T \langle M_0' f, I\rangle_{L^2(S)^3}\,\mathrm{d}t &= \int_0^T \langle M_0 I, f\rangle_{TH^{-1/2}(\mathrm{curl}_\Gamma)}\,\mathrm{d}t \\
&= \int_0^T \langle \nu \times \mathrm{curl}\, E_0|_\Gamma^+, \gamma_\Gamma U^f\rangle_{TH^{-1/2}(\mathrm{curl}_\Gamma)}\,\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}^3\setminus\overline{\Omega}} \mathrm{curl}\, E_0 \cdot \mathrm{curl}\, U^f \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle \gamma_S U^f + N_S, I\rangle_{L^2(S)^3}\,\mathrm{d}t \\
&= -\int_0^T \int_{\mathbb{R}^3} \mathrm{curl}\, E_0 \cdot \mathrm{curl}\, U_0 \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle \gamma_S U^f + N_S, I\rangle_{L^2(S)^3}\,\mathrm{d}t \\
&= -\int_0^T (\gamma_S(U_0 + U^f), I)_{L^2(S)^3}\,\mathrm{d}t,
\end{aligned}$$

where $E_0 \in L^2(0,T,W^1_\diamondsuit)$ is the solution of (12) with source I. The assertion for M_0' follows now by the choice

$$G_0 := \begin{cases} U_0 + U^f & \mathbb{R}^3 \setminus \overline{\Omega} \times (0,T) \\ U_0 & \Omega \times (0,T). \end{cases}$$

The assertion for M_σ' follows similarly by replacing U_0 with the solution $U \in L^2(0,T,W^1_\diamondsuit)$ of

$$\int_0^T \int_{\mathbb{R}^3} \left[\sigma(U + \nabla u_U)\cdot \dot{F} + \frac{1}{\mu}\mathrm{curl}\, U \cdot \mathrm{curl}\, F\right] \mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_{\mathbb{R}^3\setminus\overline{\Omega}} \mathrm{curl}\, F \cdot \mathrm{curl}\, U^f \,\mathrm{d}x\,\mathrm{d}t$$

for all $F \in H^1(0,T,W^1_\diamondsuit)$ with $F(0) = 0$. □

Now we are prepared to define the operator R_2:

$$\begin{aligned}
R_2 : L^2(0,T,TL^2_\diamondsuit(S)) &\to L^2(0,T,TH^{-\frac{1}{2}}(\mathrm{div}_\Gamma))^2 \times L^2(0,T,TH_\diamondsuit^{-1/2}(\Gamma)')^2, \\
I &\mapsto (M_0 I, M_\sigma I, K_0 I, K_\sigma I).
\end{aligned}$$

Obviously, its dual is given by

$$\begin{aligned}
&R_2' : L^2(0,T,TH^{-\frac{1}{2}}(\mathrm{curl}_\Gamma))^2 \times L^2(0,T,TH_\diamondsuit^{-1/2}(\Gamma))^2 \to L^2(0,T,TL^2_\diamondsuit(S)) \\
&(e,f,g,h) \mapsto M_0' e + M_\sigma' f + K_0' g + K_\sigma' h.
\end{aligned}$$

A reformulation of the measurement operator in terms of $M_0, M_\sigma, K_0, K_\sigma$ yields the estimation

$$\begin{aligned}
\int_0^T (\Lambda I, I)_{L^2(S)^3}\,\mathrm{d}t &= \left|\int_0^T \left[\langle M_0 I, K_\sigma I\rangle_{TH^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)} - \langle M_\sigma I, K_0 I\rangle_{TH^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)}\right]\mathrm{d}t\right| \\
&\le \frac{1}{2}\|R_2 I\|^2_{L^2(0,T,TH^{-\frac{1}{2}}(\mathrm{div}_\Gamma))^2 \times L^2(0,T,TH_\diamondsuit^{-1/2}(\Gamma)')^2}. \qquad (20)
\end{aligned}$$

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In the following lemma we show likewise to theorem 5, that the dual of R_2 determines a subset of Ω.

Lemma 7.6. *For every direction $d \in \mathbb{R}^3$, $|d| = 1$, and every point $z \in \mathbb{R}^3$, z below S, $z \notin \Gamma$,*

$$\gamma_S G_{z,d} \in \mathcal{R}\left(\Xi R_2'\right) \quad \text{implies} \quad z \in \Omega.$$

Proof. Assume $\gamma_S G_{z,d} \in \mathcal{R}\left(\Xi R_2'\right)$. Then, there are $g_0, g_\sigma \in L^2(0,T,TH^{-\frac{1}{2}}(\mathrm{curl}_\Gamma))$ and $f_0, f_\sigma \in L^2(0,T,TH_\diamondsuit^{-1/2}(\Gamma))$ such that

$$\gamma_S G_{z,d} = \Xi(M_0' g_0 + M_\sigma' g_\sigma + K_0' f_0 + K_\sigma' f_\sigma) = \Xi\left(\gamma_S H_0 + \gamma_S H_\sigma + \gamma_S G_0 + \gamma_S G_\sigma\right)$$

with $H_0, H_\sigma \in L^2(0,T,W_\diamondsuit^1)$ such as in corollary 7.4 and $G_0, G_\sigma \in L^2(0,T,W(\mathrm{curl},\mathbb{R}^3 \backslash \Gamma)$ such as in lemma 7.5. Let $V_i = \int_0^T H_i(t)\,\mathrm{d}t \in W_\diamondsuit^1$ and $P_i = \int_0^T G_i(t)\,\mathrm{d}t \in W(\mathrm{curl},\mathbb{R}^3 \backslash \Gamma)$ for $i = 0, \sigma$ and consider

$$E := (V_0 + V_\sigma + P_0 + P_\sigma)|_{\mathbb{R}^3 \backslash \overline{\Omega}}.$$

Then, we have $E \in W(\mathrm{curl}, \mathbb{R}^3 \setminus \overline{\Omega})$ and $\mathrm{curl}\,\mathrm{curl}\,E = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$, moreover it holds $\gamma_S E = \gamma_S G_{z,d}$ and especially $\gamma_S(E - G_{z,d}) \in N_S$.

Now we study the function

$$Z := \mathrm{curl}(E - G_{z,d}).$$

As a start, Z is analytic in $\mathbb{R}^3 \setminus \left(\overline{\Omega} \cup \{z\}\right)$, as $\mathrm{curl}\,G_{z,d}$ is analytic in $\mathbb{R}^3 \setminus \{z\}$ and $\mathrm{curl}\,E$ is analytic in $\mathbb{R}^3 \setminus \overline{\Omega}$. Further, the third component of Z (denoted by Z_3) vanishes on $\mathbb{R}^3_0$. To see this we add a gradient field ∇a that fulfills $\mathrm{div}(E + \nabla a) = 0$ in a neighborhood of S and we obtain that $E + \nabla a - G_{z,d}$ is analytic in this neighborhood. Beyond that,

$$\gamma_S(E + \nabla a - G_{z,d}) \in N_S$$

implies that there is a sequence $(\varphi_n) \in \mathcal{D}(\mathbb{R}^3)$ with

$$\gamma_S \nabla\varphi_n \to \gamma_S(E + \nabla a - G_{z,d}) \quad \text{in } TL^2(S)$$

and hence, as $\gamma_S F = n \times (F|_S \times n) = (F_1|_S, F_2|_S, 0)^T$ for every $F \in W(\mathrm{curl},\mathbb{R}^3)$, we have

$$(\nabla\varphi_n)_1|_S \to (E + \nabla a - G_{z,d})_1|_S, \quad (\nabla\varphi_n)_2|_S \to (E + \nabla a - G_{z,d})_2|_S \quad \text{in } L^2(S).$$

Because of $\partial_2(\nabla\varphi_n)_1 = \partial_1(\nabla\varphi_n)_2$ it follows in a distributional sense, that

$$\partial_2(E + \nabla a - G_{z,d})_1 - \partial_1(E + \nabla a - G_{z,d})_2 = 0 \quad \text{on } S.$$

Moreover, as $E + \nabla a$ and $G_{z,d}$ are analytic on S, the classical derivatives exist and are equal to the distributional ones. It follows that

$$\mathrm{curl}(E + \nabla a - G_{z,d})_3 = \partial_1(E + \nabla a - G_{z,d})_2 - \partial_2(E + \nabla a - G_{z,d})_1 = 0 \quad \text{on } S$$

and hence, that $Z_3 = \mathrm{curl}(E - G_{z,d})_3 = \mathrm{curl}(E + \nabla a - G_{z,d})_3 = 0$ on S. As Z_3 is analytic in $\mathbb{R}^3_0$ and vanishes on S, unique continuation implies that

$$Z_3 = 0 \quad \text{in } \mathbb{R}^3_0.$$

The next step is to conclude, that Z vanishes in $\mathbb{R}^3_{x_3>0} := \{(x_1,x_2,x_3)^T \in \mathbb{R}^3 \,|\, x_3 > 0\}$. By choosing a transformation $\alpha : \mathbb{R}^3 \to \mathbb{R}^3,\ x \mapsto x - 2x_3(0,0,1)^T$ and analyzing the function

$$\tilde{Z}(x) := \begin{cases} Z(x) & x_3 \geq 0 \\ \alpha(Z(\alpha(x))) & x_3 < 0 \end{cases},$$

one ends up with

$$\tilde{Z} \in L^2(\mathbb{R}^3)^3 \qquad \text{and} \qquad \operatorname{div} \tilde{Z} = \operatorname{curl} \tilde{Z} = 0 \quad \text{in } \mathbb{R}^3.$$

Hence, there is some $U \in W^1_\diamondsuit$ with $\operatorname{curl} U = \tilde{Z}$. This U also solves

$$\operatorname{curl}\operatorname{curl} U = 0 \quad \text{in } \mathbb{R}^3.$$

It follows $U = 0$ and thus $Z|_{\mathbb{R}^3_{x_3>0}} = 0$. Again, unique continuation of analytic functions yields $Z = 0$ in $\mathbb{R}^3 \setminus \left(\overline{\Omega} \cup \{z\}\right)$. It follows

$$\operatorname{curl} G_{z,d} = \operatorname{curl} E \quad \text{in } \mathbb{R}^3 \setminus \left(\overline{\Omega} \cup \{z\}\right).$$

If $z \notin \mathbb{R}^3 \setminus \overline{\Omega}$ then $\operatorname{curl} G_{z,d} \notin L^2(\mathbb{R}^3 \setminus \overline{\Omega})$, which contradicts to the fact that $\operatorname{curl} E \in L^2(\mathbb{R}^3 \setminus \overline{\Omega})^3$. It follows $z \in \Omega$. □

7.3. Proof of the main result

Proof of theorem 6.1. "$\Longrightarrow$": Assume $z \in \Omega$. Lemma 7.3 yields that there is a preimage (v,w) of $\gamma_S G_{z,d}$ under $\Xi R_1'$, i.e.,

$$\Xi R_1'(v,w) = \gamma_S G_{z,d}.$$

We use inequality (19) and conclude for all $I \in TL^2_\diamondsuit(S)$ that

$$(\gamma_S G_{z,d}, I)_{L^2(S)^3} = (\Xi R_1'(v,w), I)_{L^2(S)^3} = \int_0^T (R_1'(v,w), \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t$$
$$= \langle (v,w), R_1 \Xi^* I\rangle_{L^2(\Omega)^3_\diamondsuit \times L^2(0,T,X)} \leq \|(v,w)\|_{(L^2(\Omega)^3_\diamondsuit)' \times L^2(0,T,X')} \|R_1 \Xi^* I\|_{L^2(\Omega)^3_\diamondsuit \times L^2(0,T,X)}$$
$$\leq C \left[\int_0^T (\Lambda \Xi^* I, \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t\right]^{1/2}$$

with a constant C independent of I. Then inequality (15), i.e.,

$$\exists C > 0 : \quad (\gamma_S G_{z,d}, I)^2_{L^2(S)^3} \leq C \int_0^T (\Lambda \Xi^* I, \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t \quad \text{for all } I \in TL^2_\diamondsuit(S),$$

follows immediately.

"$\Longleftarrow$": Assume (15) holds. Then, equation (20) yields for all $I \in TL^2_\diamondsuit(S)$, that

$$(\gamma_S G_{z,d}, I)^2_{L^2(S)^3} \leq C \int_0^T (\Lambda \Xi^* I, \Xi^* I)_{L^2(S)^3}\,\mathrm{d}t$$
$$\leq \frac{C}{2} \|R_2 \Xi^* I\|^2_{L^2(0,T,TH^{-\frac{1}{2}}(\operatorname{div}_\Gamma))^2 \times L^2(0,T,TH^{-1/2}_\diamondsuit(\Gamma)')^2}$$

with a constant C independent of I. We use [18, Lemma 3.4], again, and conclude

$$\gamma_S G_{z,d} \in \mathcal{R}\left(\Xi R_2'\right).$$

Lemma 7.6 shows that $z \in \Omega$. □

8. Concluding remarks

We have extended the ideas of the factorization method to the problem of localizing conducting objects by electromagnetic measurements in the eddy-current regime. We have shown that the position and shape of conducting (diamagnetic) objects are uniquely determined by such measurements. We also showed how a subset of the object can be characterized using a linear sampling approach.

The criteria derived in this work are constructive and may be implemented as in the previous works on factorization and sampling methods, cf., e.g., [20, 21] for numerical results for the time-harmonic Maxwell equations and [18] for results on the scalar parabolic-elliptic analogue of the eddy current equation.

The linear sampling method in theorem 5.2 is closely related to the MUSIC-type imaging (introduced in [17]). This is shown in [5] for Electric Impedance Tomography in case of small conductors, where the measurement operator is expanded in terms of the size of the conductor. In [7], MUSIC-type imaging is used for corrosion detection. It might be interesting to apply the results of the paper to the problem of corrosion detection using eddy currents.

Let us remark, that our theoretical results in section 6 require only excitations, that are linear in time and only time integral measurements. Moreover, our results hold for every final time T. In practice, this final time might play an important role. For instance, in thermal imaging, the imaging functional is quite sensitive to the final time T, as pointed out in [6].

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