ERRATUM: MONOTONICITY IN INVERSE MEDIUM SCATTERING ON UNBOUNDED DOMAINS
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Abstract. We correct a mistake in the proof of Theorem 5.3 in [R. Griesmaier and B. Harrach.

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1. An error in the proof of Theorem 5.3 in [3]. At the end of the proof of Theorem 5.3 in [3] “Applying Theorem 4.5 with
$D = B_R(0) \setminus \overline{D}$, $q_1 = 0$, and $q_2 = q \ldots$” is not possible, because the assumption of Theorem 4.5 in [3] that
$q_1(x) = q_2(x)$ for a.e. $x \in \mathbb{R}^d \setminus \overline{D}$ is not satisfied for this choice of $D$, $q_1$ and $q_2$.

To fix this issue we will extend the results on localized wave functions from Section 4 of [3] in Section 2 below. Then, in Section 3 we will reformulate Theorem 5.3 of [3], making stronger assumptions on the domains and on the index of refraction, and we will correct the final argument in the original proof in [3].

2. Simultaneously localized wave functions. We establish the existence of simultaneously localized wave functions that have arbitrarily large norm on some prescribed region $E \subseteq \mathbb{R}^d$ while at the same time having arbitrarily small norm in a different region $M \subseteq \mathbb{R}^d$, assuming among others that $\mathbb{R}^d \setminus (E \cup M)$ is connected. The result generalizes Theorem 4.1 in [3] in the sense that we not only control the total field but also the incident field. Similar results have recently been established for the Schrödinger equation in [4, Thm. 3.11] and for the Helmholtz obstacle scattering problem in [1, Thm. 4.5].

Theorem 2.1. Suppose that $q \in L^\infty_{0,+}(\mathbb{R}^d)$, and let $E, M \subseteq \mathbb{R}^d$ be open and Lipschitz bounded such that $\text{supp}(q) \subseteq \overline{E \cup M}$, $\mathbb{R}^d \setminus (E \cup M)$ is connected, and $E \cap M = \emptyset$. Assume furthermore that there is a connected subset $\Gamma \subseteq \partial E \setminus \overline{M}$ that is relatively open and $C^{1,1}$ smooth.

Then for any finite dimensional subspace $V \subseteq L^2(S^{d-1})$ there exists a sequence $(g_m)_{m \in \mathbb{N}} \subseteq V^\perp$ such that

$$
\int_E |u_{q,g_m}|^2 \, dx \to \infty \quad \text{and} \quad \int_M \left( |u_{q,g_m}|^2 + |u_{1,g_m}|^2 \right) \, dx \to 0 \quad \text{as } m \to \infty ,
$$

where $u_{1,g_m}, u_{q,g_m} \in H^1_{\text{loc}}(\mathbb{R}^d)$ are given by (2.8a)–(2.8b) in [3] with $g = g_m$.

The proof of Theorem 2.1 relies on the following three lemmas.

Lemma 2.2. Suppose that $q \in L^\infty_{0,+}(\mathbb{R}^d)$, let $n^2 = 1 + q$, and assume that $D \subseteq \mathbb{R}^d$
is open and bounded. We define

\[ L_{q,D} : L^2(S^{d-1}) \rightarrow H^1(D), \quad g \mapsto u_{q,g}|_D, \]

where \( u_{q,g} \in H^1_{\text{loc}}(\mathbb{R}^d) \) is given by (2.8b) in [3]. Then \( L_{q,D} \) is a linear operator and its adjoint is given by

\[ L^*_{q,D} : H^1(D)^* \rightarrow L^2(S^{d-1}), \quad f \mapsto S^*_q w^\infty, \]

where \( H^1(D)^* \) is the dual of \( H^1(D) \), \( S^*_q \) denotes the adjoint of the scattering operator from (2.7) in [3], and \( w^\infty \in L^2(S^{d-1}) \) is the far field pattern of the radiating solution \( w \in H^1_{\text{loc}}(\mathbb{R}^d) \) to

\[ \Delta w + k^2 n^2 w = -f \quad \text{in } \mathbb{R}^d. \]

Proof. This follows from the same arguments that have been used in the proof of Lemma 2.2 in [3]. \( \square \)

**Lemma 2.3.** Suppose that \( q \in L^\infty_0(\mathbb{R}^d) \), and let \( E, M \subseteq \mathbb{R}^d \) be open and Lipschitz bounded such that \( \text{supp}(q) \subseteq E \cup M, \mathbb{R}^d \setminus (E \cup M) \) is connected, and \( E \cap M = \emptyset \). Assume furthermore that there is a connected subset \( \Gamma \subseteq \partial E \setminus M \) that is relatively open and \( C^{1,1} \) smooth. Then,

\[ R(L^*_{q,E}) \subsetneq R((L^*_{q,M} L^*_{0,M})) \]

and there exists an infinite dimensional subspace \( Z \subseteq R(L^*_{q,E}) \) such that

\[ Z \cap R((L^*_{q,M} L^*_{0,M})) = \{0\}. \]

Proof. Let \( h \in R(L^*_{q,E}) \cap R((L^*_{q,M} L^*_{0,M})) \). Then Lemma 2.2 shows that there exist \( f_{q,E} \in H^1(E)^* \) and \( f_{q,M}, f_{0,M} \in H^1(M)^* \) such that the far field patterns \( w^\infty_{q,E}, w^\infty_{q,M}, w^\infty_{0,M} \) of the radiating solutions \( w_{q,E}, w_{q,M}, w_{0,M} \in H^1_{\text{loc}}(\mathbb{R}^d) \) to

\[
\begin{align*}
\Delta w_{q,E} + k^2 (1 + q) w_{q,E} &= -f_{q,E} & \text{in } \mathbb{R}^d, \\
\Delta w_{q,M} + k^2 (1 + q) w_{q,M} &= -f_{q,M} & \text{in } \mathbb{R}^d, \\
\Delta w_{0,M} + k^2 w_{0,M} &= -f_{0,M} & \text{in } \mathbb{R}^d,
\end{align*}
\]

satisfy

\[ h = S^*_q w^\infty_{q,E} = w^\infty_{0,M} + S^*_q w^\infty_{q,M}. \]

Here we used that \( S_0 \) is the identity operator. Accordingly, using the definition of the scattering operator in (2.7) of [3], we find that

\[
\begin{align*}
0 &= w^\infty_{q,E} - w^\infty_{q,M} - S_q w^\infty_{0,M} \\
&= w^\infty_{q,E} - w^\infty_{q,M} - w^\infty_{0,M} - 2ik|C|d^2 F_q w^\infty_{0,M} \\
&= w^\infty_{q,E} - (w^\infty_{q,M} + w^\infty_{0,M} + v^\infty_q),
\end{align*}
\]

where \( v^\infty_q \) is the far field of a radiating solution \( v_q \in H^1_{\text{loc}}(\mathbb{R}^d) \) to

\[ \Delta v_q + k^2 (1 + q) v_q = 0 \quad \text{in } \mathbb{R}^d. \]
Since $\text{supp}(q) \subseteq E \cup M$ and $\mathbb{R}^d \setminus (E \cup M)$ is connected, Rellich’s lemma and unique continuation guarantee that
\[(2.2) \quad w_{q,E} - (w_{q,M} + w_{0,M} + v_q) = 0 \quad \text{in } \mathbb{R}^d \setminus (E \cup M)\]
(cf., e.g., [2, Thm. 2.14]).

Next we discuss the regularity of the traces of $w_{q,E}$ and $w_{q,M} + w_{0,M} + v_q$ at the boundary segment $\Gamma \subseteq \partial E \setminus M$. W.l.o.g, we may assume that $\Gamma$ is bounded away from $M$. Since $\text{supp}(f_{q,M} + f_{0,M}) \subseteq M$, interior regularity results (see, e.g., [7, Thm. 4.18]) show that $(w_{q,M} + w_{0,M} + v_q)\big|_\Gamma \in H^{1/2}(\Gamma)$. Thus (2.2) implies that $w_{q,E}\big|_\Gamma^+ \in H^{1/2}(\Gamma)$ as well.

On the other hand, let $\tilde{H}^{1/2}(\Gamma)$ be the closure of $\mathcal{D}(\Gamma)$ in $H^{1/2}(\Gamma)$ (see, e.g., [7, p. 99]). We will construct sources $f \in H^1(\Gamma)^*$ such that $L_{q,E}^* f \notin \mathcal{R}(\langle L_{q,M}^*, L_{0,M}^* \rangle)$. Given any $g \in \tilde{H}^{1/2}(\Gamma)$, we denote by $\tilde{g} \in H^{1/2}(\partial E)$ its extension to $\partial E$ by zero. Accordingly, let $u^+ \in H^1_{\text{loc}}(\mathbb{R}^d \setminus E)$ be the radiating solution to the exterior Dirichlet problem
\[(2.3) \quad \Delta u^+ + k^2 n_2^2 u^+ = 0 \quad \text{in } \mathbb{R}^d \setminus E, \quad u^+ = \tilde{g} \quad \text{on } \partial E.\]

Similarly, we define $u^- \in H^1(\Gamma)$ as the solution to the interior Dirichlet problem
\[
\Delta u^- = 0 \quad \text{in } E, \quad u^- = \tilde{g} \quad \text{on } \partial E.
\]

Therewith we introduce $u \in L^2_{\text{loc}}(\mathbb{R}^d)$ by
\[
u := \begin{cases} u^- \quad \text{in } E, \\ u^+ \quad \text{in } \mathbb{R}^d \setminus E, \end{cases}
\]
and $f \in H^1(\Gamma)^*$ by
\[
f := -k^2 n_2^2 u^- - \gamma^* \left( \frac{\partial u}{\partial \nu} \bigg|^{+}_{\partial E} - \frac{\partial u}{\partial \nu} \bigg|^{-}_{\partial E} \right),
\]
where $\gamma^* : H^{-1/2}(\partial E) \to H^1(\Gamma)^*$ denotes the adjoint of the interior trace operator $\gamma : H^1(\Gamma) \to H^{1/2}(\partial E)$. Then $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ (see, e.g., [8, Lmm. 5.3]), and
\[
\Delta u + k^2 n_2^2 u = -f \quad \text{in } \mathbb{R}^d.
\]
(see, e.g., [7, Lmm. 6.9]). Accordingly, $L_{q,E}^* f = S_q u^\infty$, where $u^\infty \in L^2(S^{d-1})$ coincides with the far field of the radiating solution $u^+$. To the exterior Dirichlet problem (2.3). If $\tilde{g} \notin H^{1/2}(\partial E)$, then our regularity considerations above show that $L_{q,E}^* f \notin \mathcal{R}(\langle L_{q,M}^*, L_{0,M}^* \rangle)$.

Now let $X \subseteq \tilde{H}^{1/2}(\Gamma)$ be an infinite dimensional subspace of $\tilde{H}^{1/2}(\Gamma)$ such that $X \cap H^{1/2}(\Gamma) = \{0\}$ (e.g., the subspace of piecewise linear functions on $\Gamma$ that vanish on $\partial E$ as considered in the proof of Lemma 4.6 in [1]). Let $G_E : H^{1/2}(\Gamma) \to L^2(S^{d-1})$ be the operator that maps $g \in H^{1/2}(\Gamma)$ to the far field pattern of the radiating solution $u^+$ of (2.3), where $\tilde{g} \in H^{1/2}(\partial E)$ is again the extension of $g$ to $\partial E$ by zero. Then $G_E$ is one-to-one (see, e.g., [1, Thm. 3.2]), and thus $Z := S_q G_E(X)$ is infinite dimensional. Furthermore, we have just shown that
\[
Z \subseteq \mathcal{R}(L_{q,E}^*) \quad \text{and} \quad Z \cap \mathcal{R}(\langle L_{q,M}^*, L_{0,M}^* \rangle) = \{0\}.
\]
In the next lemma we quote a special case of Lemma 2.5 in [6].

**Lemma 2.4.** Let \( X, Y \) and \( Z \) be Hilbert spaces, and let \( A : X \to Y \) and \( B : X \to Z \) be bounded linear operators. Then,

\[
\exists C > 0 : \|Ax\| \leq C\|Bx\| \quad \forall x \in X \quad \text{if and only if} \quad \mathcal{R}(A^*) \subseteq \mathcal{R}(B^*).
\]

Now we give the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Let \( V \subseteq L^2(S^{d-1}) \) be a finite dimensional subspace. We denote by \( P_V : L^2(S^{d-1}) \to L^2(S^{d-1}) \) the orthogonal projection on \( V \). Combining Lemma 2.3 with a simple dimensionality argument (see [5, Lem. 4.7]) shows that

\[
Z \not\subseteq \mathcal{R} ((L^*_{q,M} L^*_{0,M})) + V = \mathcal{R} ((L^*_{q,M} L^*_{0,M} P_V)),
\]

where \( Z \subseteq \mathcal{R}(L^*_{q,E}) \) denotes the subspace in Lemma 2.3. Thus,

\[
\mathcal{R}(L^*_{q,E}) \not\subseteq \mathcal{R} ((L^*_{q,M} L^*_{0,M})) + V = \mathcal{R} ((L^*_{q,M} L^*_{0,M} P_V)),
\]

and accordingly Lemma 2.4 implies that there is no constant \( C > 0 \) such that

\[
\|L_{q,E}g\|_{L^2(E)}^2 \leq C^2 \left( \begin{array}{c}
L_{q,M}, L_{0,M} \\
L_{0,M} \end{array} \right) g \|_{L^2(M) \times L^2(M) \times L^2(S^{d-1})}^2
\]

\[
= C^2 (\|L_{q,M}g\|_{L^2(M)}^2 + \|L_{0,M}g\|_{L^2(M)}^2 + \|P_Vg\|_{L^2(S^{d-1})}^2)
\]

for all \( g \in L^2(S^{d-1}) \). Hence, there exists as sequence \((g_m)_{m \in \mathbb{N}} \subseteq L^2(S^{d-1})\) such that

\[
\|L_{q,E}g_m\|_{L^2(E)} \to \infty,
\]

\[
\|L_{q,M}g_m\|_{L^2(M)} + \|L_{0,M}g_m\|_{L^2(M)} + \|P_Vg_m\|_{L^2(S^{d-1})} \to 0 \quad \text{as} \quad m \to \infty.
\]

Setting \( g_m := g_m - P_Vg_m \in V^\perp \subseteq L^2(S^{d-1}) \) for any \( m \in \mathbb{N} \), we finally obtain

\[
\|L_{q,E}g_m\|_{L^2(E)} \geq \|L_{q,E}g_m\|_{L^2(E)} - \|L_{q,E}P_Vg_m\|_{L^2(S^{d-1})} \to \infty \quad \text{as} \quad m \to \infty,
\]

and

\[
\|L_{q,M}g_m\|_{L^2(M)} + \|L_{0,M}g_m\|_{L^2(M)} \leq \|L_{q,M}g_m\|_{L^2(M)} + \|L_{0,M}g_m\|_{L^2(M)} + \|P_Vg_m\|_{L^2(S^{d-1})} \to 0 \quad \text{as} \quad m \to \infty.
\]

Since \( L_{q,E}g_m = u_{q,g_m}|_E, L_{q,M}g_m = u_{q,g_m}|_M, \) and \( L_{0,M}g_m = u_{g_m}|_M \), this ends the proof. \( \square \)

### 3. Correction of the statement and of the proof of Theorem 5.3 in [3].

**Theorem 3.1.** Let \( B, D \subseteq \mathbb{R}^d \) be open and Lipschitz bounded such that \( \partial D \) is piecewise \( C^{1,1} \) smooth, and \( \mathbb{R}^d \setminus \overline{B} \) as well as \( \mathbb{R}^d \setminus \overline{D} \) are connected. Let \( q \in L^\infty_0(\mathbb{R}^d) \) with \( \text{supp}(q) = \overline{D} \), and suppose that \(-1 < q_{\min} \leq q \leq q_{\max} < \infty \) a.e. on \( D \) for some constants \( q_{\min}, q_{\max} \in \mathbb{R} \).

Furthermore, we assume that for any point \( x \in \partial D \) on the boundary of \( D \), there exists a connected unbounded neighborhood \( O \subseteq \mathbb{R}^d \) of \( x \) such that, for \( E := O \cap D \),

\[
q|_E \geq q_{\min,E} > 0 \quad \text{or} \quad q|_E \leq q_{\max,E} < 0
\]

for some constants \( q_{\min,E}, q_{\max,E} \in \mathbb{R} \).
(a) If \( D \subseteq B \), then there exists a constant \( C > 0 \) such that
\[
\alpha T_B \leq_{\text{fin}} \Re(F_g) \leq_{\text{fin}} \beta T_B \quad \text{for all } \alpha \leq \min\{0, q_{\text{min}}\}, \beta \geq \max\{0, C q_{\text{max}}\}.
\]
(b) If \( D \nsubseteq B \), then
\[
\alpha T_B \leq_{\text{fin}} \Re(F_g) \quad \text{for any } \alpha \in \mathbb{R} \quad \text{or} \quad \Re(F_g) \leq_{\text{fin}} \beta T_B \quad \text{for any } \beta \in \mathbb{R}.
\]

Remark 3.2. The assumptions on \( B \) and \( D \) as well as the local definiteness assumption (3.1) in Theorem 3.1 are stronger than in the original version of Theorem 5.3 in [3].

Proof of Theorem 3.1. If \( D \subseteq B \), then Corollary 3.4 and Theorem 4.5 in [3] with \( q_1 = 0 \) and \( q_2 = q \) show that there exists a constant \( C > 0 \) and a finite dimensional subspace \( V \subseteq L^2(S^{d-1}) \) such that, for all \( g \in V^\perp \) and any \( \beta \geq \max\{0, C q_{\text{max}}\} \),
\[
\Re\left( \int_{S^{d-1}} g \overline{F_g g} \, ds \right) \leq k^2 \int_D g|u_{q,g}|^2 \, dx \leq k^2 q_{\text{max}} \int_D |u_{q,g}|^2 \, dx
\]
\[
\leq k^2 C q_{\text{max}} \int_D |u_{g}^i|^2 \, dx \leq k^2 \beta \int_B |u_{g}^i|^2 \, dx.
\]

Similarly, Theorem 3.2 in [3] with \( q_1 = 0 \) and \( q_2 = q \) shows that there exists a finite dimensional subspace \( V \subseteq L^2(S^{d-1}) \) such that, for all \( g \in V^\perp \) and any \( \alpha \leq \min\{0, q_{\text{min}}\} \),
\[
\Re\left( \int_{S^{d-1}} g \overline{F_g g} \, ds \right) \geq k^2 \int_D g|u_g^i|^2 \, dx \geq k^2 q_{\text{min}} \int_D |u_g^i|^2 \, dx \geq k^2 \alpha \int_B |u_g^i|^2 \, dx,
\]
and part (a) is proven.

We prove part (b) by contradiction. Since \( D \nsubseteq B \), \( U := D \setminus B \) is not empty, and there exists \( x \in \overline{U} \cap \partial D \) as well as a connected unbounded open neighborhood \( O \subseteq \mathbb{R}^d \) of \( x \) with \( O \cap D \subseteq U \) and \( O \cap B = \emptyset \), such that (3.1) is satisfied with \( E := O \cap D \). Furthermore, let \( R > 0 \) be large enough such that \( B, D \subseteq B_R(0) \). Without loss of generality we assume that \( O \cap B_R(0) \), and \( B_R(0) \setminus \overline{O} \) are connected.

We first assume that \( q|_E \geq q_{\text{min},E} > 0 \), and that \( \Re(F_g) \leq_{\text{fin}} \beta T_B \) for some \( \beta \in \mathbb{R} \). Using the monotonicity relation (3.1) in Theorem 3.2 of [3] with \( q_1 = 0 \) and \( q_2 = q \), we find that there exists a finite dimensional subspace \( V \subseteq L^2(S^{d-1}) \) such that, for any \( g \in V^\perp \),
\[
0 \geq \int_{S^{d-1}} g\overline{(\Re(F_g g) - \beta T_B g)} \, ds \geq k^2 \int_{B_R(0)} (q - \beta \chi_B) |u_g^i|^2 \, dx
\]
\[
= k^2 \int_{B_R(0) \setminus \overline{O}} (q - \beta \chi_B) |u_g^i|^2 \, dx + k^2 \int_{B_R(0) \cap O} (q - \beta \chi_B) |u_g^i|^2 \, dx
\]
\[
\geq -k^2 (||q||_{L^\infty(\mathbb{R}^d)} + |\beta|) \int_{B_R(0) \setminus \overline{O}} |u_g^i|^2 \, dx + k^2 q_{\text{min},E} \int_E |u_g^i|^2 \, dx.
\]

However, this contradicts Theorem 4.1 in [3] with \( B = E, D = B_R(0) \setminus \overline{O} \), and \( q = 0 \), which guarantees the existence of a sequence \( (g_m)_{m \in \mathbb{N}} \subseteq V^\perp \) with
\[
\int_E |u_{g_m}^i|^2 \, dx \to \infty \quad \text{and} \quad \int_{B_R(0) \setminus \overline{O}} |u_{g_m}^i|^2 \, dx \to 0 \quad \text{as } m \to \infty.
\]
Consequently, \( \text{Re}(F_q) \leq_{\text{fin}} \beta T_B \) for all \( \beta \in \mathbb{R} \).

On the other hand, if \( |q|_E \leq q_{\text{max},E} < 0 \), and if \( \alpha T_B \leq_{\text{fin}} \text{Re}(F_q) \) for some \( \alpha \in \mathbb{R} \), then the monotonicity relation (3.3) in Corollary 3.4 of [3] with \( q_1 = 0 \) and \( q_2 = q \) shows that there exists a finite dimensional subspace \( V \subseteq L^2(S^{d-1}) \) such that, for any \( g \in V^\perp \),

\[
0 \leq \int_{S^{d-1}} g (\text{Re}(F_q) g - \alpha T_B g) \, ds \leq k^2 \int_{B_R(0)} (q |u_{q,g}|^2 - \alpha \chi_B |u_g|^2) \, dx
\]

\[
= k^2 \int_{B_R(0) \setminus \mathcal{O}} (q |u_{q,g}|^2 - \alpha \chi_B |u_g|^2) \, dx + k^2 \int_{B_R(0) \cap \mathcal{O}} (q |u_{q,g}|^2 - \alpha \chi_B |u_g|^2) \, dx
\]

\[
\leq k^2 q_{\text{max}} \int_{B_R(0) \setminus \mathcal{O}} |u_{q,g}|^2 \, dx + k^2 |\alpha| \int_{B_R(0) \setminus \mathcal{O}} |u_g|^2 \, dx + k^2 q_{\text{max},E} \int_{E} |u_{q,g}|^2 \, dx.
\]

Let \( M := B_R(0) \setminus \mathcal{O} \). Since \( \partial D \) is piecewise \( C^{1,1} \) smooth, there is a connected subset \( \Gamma \subseteq \partial E \setminus \mathcal{O} \) that is relatively open and \( C^{1,1} \) smooth. Applying Theorem 2.1 we find that there exists a sequence \( (g_m)_{m \in \mathbb{N}} \subseteq V^\perp \) such that

\[
\int_{E} |u_{q,g_m}|^2 \, dx \to \infty \quad \text{and} \quad \int_{B_R(0) \setminus \mathcal{O}} |u_{q,g_m}|^2 + |u_g|^2 \, dx \to 0 \quad \text{as} \ m \to \infty.
\]

However, since \( q_{\text{max},E} < 0 \), this gives a contradiction. Consequently, \( \alpha T_B \leq_{\text{fin}} \text{Re}(F_q) \) for all \( \alpha \in \mathbb{R} \), which ends the proof of part (b).

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