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A UNIFIED VARIATIONAL FORMULATION FOR THE PARABOLIC-ELLIPTIC EDDY CURRENT EQUATIONS*

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Abstract. Transient excitation currents generate electromagnetic fields which, in turn, induce electric currents in proximal conductors. For slowly varying fields, this can be described by the eddy current equations, which are obtained by neglecting the dielectric displacement currents in Maxwell's equations. The eddy current equations are of parabolic-elliptic type: In insulating regions, the field instantaneously adapts to the excitation (quasistationary elliptic behavior), while in conducting regions, this adaptation takes some time due to the induced eddy currents (parabolic behavior). For fixed conductivity, the equations are well studied. However, little rigorous mathematical results are known for the solution's dependence on the conductivity, in particular for the solution's sensitivity with respect to the equation changing from elliptic to parabolic type. In this work, we derive a new unified variational formulation for the eddy current equations that is uniformly coercive with respect to the conductivity. We then apply our new unified formulation to study the case when the conductivity approaches zero and rigorously linearize the eddy current equations around a non-conducting domain with respect to the introduction of a conducting object.

Key words. eddy current problem, parabolic-elliptic equation, unified variational formulation

AMS subject classifications. 35M10, 35Q60, 35R05

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1. Introduction.

Transient excitation currents $J(x,t)$ generate electromagnetic fields, $E(x,t)$ and $H(x,t)$, which can be described by Maxwell's equations

$$\begin{aligned}\operatorname{curl} H &= \epsilon\partial_t E + \sigma E + J,\\ \operatorname{curl} E &= -\mu\partial_t H,\end{aligned}$$

where the curl-operator acts on the three spatial coordinates, ∂_t denotes the time-derivative, and (under the assumption of linear and isotropic time-independent material laws) $\sigma(x)$, $\epsilon(x)$, and $\mu(x)$ are the conductivity, permittivity, and permeability of the considered domain, respectively, material.

For slowly varying electromagnetic fields, the displacement currents $\epsilon\frac{\partial E}{\partial t}$ can be neglected. This leads to the *transient eddy current equations*

$$\operatorname{curl} H = \sigma E + J, \tag{1.1}$$

$$\operatorname{curl} E = -\mu\partial_t H, \tag{1.2}$$

respectively, after eliminating H,

$$\partial_t(\sigma E) + \operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E\right) = -\partial_t J. \tag{1.3}$$

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The eddy current model is well established in the engineering literature; cf., e.g., Albanese and Rubinacci [2] or Dirks [14]. A rigorous mathematical justification has been derived by Alonso [3] and Ammari, Buffa, and Nédélec [4] for the (low-frequency) time-harmonic case. This also justifies the transient model when the excitation is composed of low-frequency components; cf. [4, sect. 8].

In a typical application the domain under consideration consists of both conducting regions ($\sigma(x) > 0$) and nonconducting regions ($\sigma(x) = 0$), which has two interesting consequences. The first one is that (1.3) is of parabolic-elliptic type. The physical interpretation is that the time-scale is different in the conducting and the insulating regions. In the insulating regions, the field instantaneously adapts to the excitation (quasi-stationary behavior), while, in the conducting regions, this adaptation takes some time (due to eddy currents induced by the varying electromagnetic fields). A particular consequence is that initial values are meaningful only in the conducting region.

The second consequence is that (1.3), together with initial values, does only determine E up to the addition of a *gauge field*, which is a curl-free field that vanishes inside the conductor. However, in many applications, one is interested only in σE, or $\operatorname{curl} E$, anyway.

Several well-posed variational formulations have been proposed for the eddy current equations and used for the numerical solution; cf., e.g., [8, 7, 6, 15, 5, 18, 19, 20, 1]. These approaches concentrate on solving the eddy current equations with a fixed conducting region in which the conductivity is assumed to be bounded from below by some positive constant. Accordingly, the variational formulations, with their underlying solution spaces and coercivity constants, depend, in some form or another, on this lower bound, on the support of the conductivity, or on both. A noteworthy exemption appears when σ is constant inside the conductor. Then, a solution of (1.3) can be found by considering the standard variational formulation of (1.3) (see (2.4) below) in the space of divergence-free functions, where it is coercive. In the general case of spatially varying σ, however, the solution of (1.3) will not be divergence-free in the conductor. In that case, restricting the standard variational formulation of (1.3) to the space of divergence-free functions will *not* yield the true solution up to a gauge field.

Inverse problems, as well as sensitivity considerations, require a unified formulation with respect to σ. An important application is landmine detection, where a source current in an inductor coil is used to generate electromagnetic fields that, in turn, induce currents in a buried conductor. The resulting change in the magnetic field can then be measured by a receiver coil, so that, from one or several such measurements, one may try to reconstruct information about the buried object; cf., e.g., [17] for a formulation of the corresponding measurement operators. A natural approach to these inverse problems is to start by linearizing the problem with respect to σ around a homogeneous nonconducting state. Roughly speaking, this leads to the following question: *How does the solution of the elliptic magnetostatic problem* (i.e., (1.3) with $\sigma = 0$) *change if the problem becomes a little bit parabolic*? For a scalar analogue this question has been answered in [16]. To the best knowledge of the authors, no rigorous linearization results are known for the eddy current model so far.

In this work, we derive a new unified variational formulation for the eddy current equations that is uniformly coercive with respect to σ. To be more precise, what we present is a variational formulation that is uniformly coercive (and hence uniquely solvable) in the space of divergence-free functions and whose solution agrees with the true solution up to the addition of a gradient field. At this point, let us stress again that, for spatially varying σ, the standard variational formulation of (1.3) restricted to divergence-free functions does not determine the solution up to a curl-free field.

We apply our new unified formulation to study the limit of the solutions of (1.3) for $\sigma \to 0$ and prove convergence against their magnetostatic counterparts. We then turn to the above-mentioned question and rigorously determine the directional derivative of the solutions of (1.3) with $\sigma = 0$, with respect to σ; i.e., we linearize the solutions of the elliptic (magnetostatic) problem with respect to the problem becoming parabolic in some parts.

This paper is organized as follows. In section 2 we introduce the necessary notation, derive the standard variational formulation for (1.3), characterize well-posed initial conditions, and prove uniqueness of solutions (up to gauge fields). Section 3 contains our main theoretical tool, which is a new uniformly coercive variational formulation that determines the solution up to the addition of a gradient field. This also proves solvability of the eddy current equations. Finally, in section 4, we apply our new unified formulation to study the dependence of the solutions when the conductivity approaches zero and rigorously linearize (1.3) around a nonconducting domain with respect to the introduction of a conducting object.

2. The eddy current equation.

2.1. Notation and general assumptions. Let $T > 0$. Let $\mu \in L^\infty_+(\mathbb{R}^3)$, where we denote by $L^\infty_+(\mathbb{R}^3)$ the space of $L^\infty(\mathbb{R}^3)$-functions with positive (essential) infima (denoted by $\inf \mu$). Let $\sigma \in L^\infty_\geq(\mathbb{R}^3)$ have bounded support Ω, where $L^\infty_\geq(\mathbb{R}^3)$ is the space of $L^\infty(\mathbb{R}^3)$-functions that are almost everywhere nonnegative.

$\mathcal{D}(\mathbb{R})$, $\mathcal{D}(\mathbb{R}^3)$, $\mathcal{D}(]0,T[)$, respectively, $\mathcal{D}(\mathbb{R}^3\times]0,T[)$, denote the space of C^∞-functions in x, t, respectively, (x,t), which are compactly supported in $\mathbb{R}$, $\mathbb{R}^3$, $]0,T[$, respectively, $\mathbb{R}^3\times]0,T[$. We will also use the notation $\mathcal{D}([0,T[)$, respectively, the notation $\mathcal{D}(\mathbb{R}^3 \times [0,T[)$, for the space of restrictions of functions from $\mathcal{D}(]-\infty,T[)$, respectively, $\mathcal{D}(\mathbb{R}^3\times]-\infty,T[)$, to $[0,T]$, respectively, $\mathbb{R}^3 \times [0,T]$.

$\mathcal{D}'(\mathbb{R}^3)$ denotes the space of distributions, i.e., continuous linear mappings from $\mathcal{D}(\mathbb{R}^3)$ to $\mathbb{R}$. $\mathcal{D}'(\mathbb{R}^3\times]0,T[)^3$ is defined likewise.

Let $L^2_\rho(\mathbb{R}^3)$ and $W(\mathrm{curl})$ denote the distributional spaces

$$\begin{aligned} L^2_\rho(\mathbb{R}^3) &:= \{E \in \mathcal{D}'(\mathbb{R}^3) \,|\, (1+|x|^2)^{-\frac{1}{2}} E \in L^2(\mathbb{R}^3)\}, \\ W(\mathrm{curl}) &:= \{E \in L^2_\rho(\mathbb{R}^3)^3 \mid \mathrm{curl}\, E \in L^2(\mathbb{R}^3)^3\}. \end{aligned}$$

$L^2_\rho(\mathbb{R}^3)^n$, $n = 1, 3$, and $W(\mathrm{curl})$ are Hilbert spaces with norms

$$\|\cdot\|_\rho := \|(1+|x|^2)^{-\frac{1}{2}} \cdot\|_{L^2(\mathbb{R}^3)^n} \quad \text{and} \quad \|\cdot\|^2_{W(\mathrm{curl})} = \|\cdot\|^2_\rho + \|\,\mathrm{curl}\cdot\|^2_{L^2(\mathbb{R}^3)^3}.$$

We introduce the Beppo–Levi spaces

$$\begin{aligned} W^1(\mathbb{R}^3) &:= \{E \in L^2_\rho(\mathbb{R}^3) \,|\, \nabla E \in L^2(\mathbb{R}^3)^3\}, \\ W^1 := W^1(\mathbb{R}^3)^3 &= \{E \in L^2_\rho(\mathbb{R}^3)^3 \,|\, \nabla E \in L^2(\mathbb{R}^3)^{3\times 3}\}. \end{aligned}$$

In the latter space, ∇E denotes the 3×3 Jacobian of E. For a bounded domain $\mathcal{O} \subset \mathbb{R}^3$ with smooth boundary and connected complement, $W^1(\mathbb{R}^3 \setminus \overline{\mathcal{O}})$ is defined likewise. These spaces are Hilbert spaces with respect to the norms

$$\|\cdot\|_{W^1(\mathbb{R}^3)} := \|\nabla\cdot\|_{L^2(\mathbb{R}^3)^3}, \quad \|\cdot\|_{W^1(\mathbb{R}^3\setminus\overline{\mathcal{O}})} := \|\nabla\cdot\|_{L^2(\mathbb{R}^3\setminus\overline{\mathcal{O}})^3}, \quad \|\cdot\|_{W^1} := \|\nabla\cdot\|_{L^2(\mathbb{R}^3)^{3\times 3}};$$

cf., e.g., [11, IX.A, §1] and [12, XI.B, §1].

Note that $\mathcal{D}(\mathbb{R}^3)$ is dense in $L^2_\rho(\mathbb{R}^3)$ and in $W^1(\mathbb{R}^3)$ and that $\mathcal{D}(\mathbb{R}^3)^3$ is dense in $L^2_\rho(\mathbb{R}^3)^3$, $W(\mathrm{curl})$, and W^1.

We denote the dual space of a space H by H' and the dual pairing on $H' \times H$ by $\langle \cdot, \cdot \rangle_{H' \times H}$. Throughout this work, we frequently use the dual pairing between $W(\mathrm{curl})'$ and $W(\mathrm{curl})$; hence we write in this case

$$\langle G, E \rangle := \langle G, E \rangle_{W(\mathrm{curl})' \times W(\mathrm{curl})} \qquad \text{for } G \in W(\mathrm{curl})', \ E \in W(\mathrm{curl}).$$

We also write $\mathbb{R}^3_T := \mathbb{R}^3 \times]0, T[$ and $L^2(\mathbb{R}^3_T)$ instead of $L^2(\mathbb{R}^3 \times]0, T[)$ and usually omit the arguments x and t; we use them only where we expect them to improve readability.

For a Banach space X, $C(0, T, X)$, and $L^2(0, T, X)$ denote the spaces of vector-valued functions

$$E : \ [0, T] \to X,$$

which are continuous on $[0, T]$, respectively, and square integrable; cf., e.g., [13, XVIII, §1]. Spaces of functions with vector-valued time-derivatives will be introduced in some detail in subsection 2.2.

We consider the space $L^2(0, T, W(\mathrm{curl}))$ as the space to look for a solution of (1.3). Generally, it is not the case that every $E \in L^2(0, T, W(\mathrm{curl}))$ has some well-defined initial values. However, in the following we show that at least every solution of (1.3) has well-defined initial values. Then, we derive a standard variational formulation and discuss in what sense uniqueness can be expected.

In this paper, we assume that we are given the time-derivative of the excitation currents

$$\begin{aligned} &J_t \in L^2(0, T, W(\mathrm{curl})') \text{ with } \operatorname{div} J_t = 0 \\ &\text{and } \ E_0 \in L^2(\mathbb{R}^3)^3 \text{ with } \operatorname{div}(\sigma E_0) = 0. \end{aligned} \tag{2.1}$$

THEOREM 2.1. *Let $E \in L^2(0, T, W(\mathrm{curl}))$. The eddy current problem reads*

$$\partial_t(\sigma(x) E(x, t)) + \mathrm{curl}\left(\frac{1}{\mu(x)} \mathrm{curl}\, E(x, t) \right) = -J_t(x, t) \qquad \text{in } \mathbb{R}^3 \times]0, T[, \tag{2.2}$$

$$\sqrt{\sigma(x)} E(x, 0) = \sqrt{\sigma(x)} E_0(x) \qquad \text{in } \mathbb{R}^3. \tag{2.3}$$

The following holds.

(a) *For every solution $E \in L^2(0, T, W(\mathrm{curl}))$ of (2.2) we have*

$$\sqrt{\sigma} E \in C(0, T, L^2(\mathbb{R}^3)^3).$$

(b) *$E \in L^2(0, T, W(\mathrm{curl}))$ solves (2.2)–(2.3) if and only if E solves*

$$\begin{aligned} -\int_0^T \int_{\mathbb{R}^3} \sigma E \cdot \partial_t \Phi \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \mathrm{curl}\, E \cdot \mathrm{curl}\, \Phi \,\mathrm{d}x \,\mathrm{d}t \\ = -\int_0^T \langle J_t, \Phi \rangle \,\mathrm{d}t + \int_{\mathbb{R}^3} \sigma E_0 \cdot \Phi(0) \,\mathrm{d}x \end{aligned} \tag{2.4}$$

for all $\Phi \in \mathcal{D}(\mathbb{R}^3 \times [0, T[)^3$.

(c) *Equations* (2.2)–(2.3) *uniquely determine* $\operatorname{curl} E$ *and* $\sqrt{\sigma}E$. *Moreover, if* $E \in L^2(0,T,W(\operatorname{curl}))$ *solves* (2.2)–(2.3), *then every function* $F \in L^2(0,T,W(\operatorname{curl}))$ *with* $\operatorname{curl} F = \operatorname{curl} E$ *and* $\sqrt{\sigma}F = \sqrt{\sigma}E$ *also solves* (2.2)–(2.3).

Before we prove Theorem 2.1 in the following subsection, let us stress again the somewhat subtle point that the initial condition (2.3) is meaningful only for solutions of (2.2). When we speak of a solution $E \in L^2(0,T,W(\operatorname{curl}))$ of (2.2)–(2.3), this is to be understood in the following order: First, $E \in L^2(0,T,W(\operatorname{curl}))$ has to solve (2.2) so that $\sqrt{\sigma}E \in C(0,T,L^2(\mathbb{R}^3)^3)$, and, second, this continuous function $\sqrt{\sigma}E$ has to fulfill the initial condition (2.3). Note that this is similar to the interpretation of Neumann boundary values for second-order elliptic equations.

The multiplication with $\sqrt{\sigma}$ in the initial condition (2.3) can be interpreted as stating that, wherever it makes sense to speak of initial values, they must agree with E_0. In $\Omega = \operatorname{supp}\sigma$, the equation is parabolic, and initial values are meaningful and necessary. Outside of Ω, where the equation is elliptic, initial conditions are meaningless, and (2.3) does not contain any information.

Let us stress that, in this section, we require only that σ be nonnegative, be bounded, and have bounded support.

2.2. Initial values, a standard variational formulation, and uniqueness. For $E \in L^2(0,T,W(\operatorname{curl}))$ we have that $E(t), \operatorname{curl} E(t) \in L^2(\mathbb{R}^3)^3$ for $t \in]0,T[$ a.e. and consequently the products

$$\frac{1}{\mu}\operatorname{curl} E(t),\ \sigma E(t) \in L^2(\mathbb{R}^3)^3$$

are well defined. Moreover, the assumption $\operatorname{div}(\sigma E_0) = 0$ is well defined in the sense of distributions since $E_0 \in L^2(\mathbb{R}^3)^3$. Since $\mathcal{D}(\mathbb{R}^3)^3$ is dense in $W(\operatorname{curl})$, we can also regard $L^2(0,T,W(\operatorname{curl})')$ as a subspace of $\mathcal{D}'(\mathbb{R}^3\times]0,T[)^3$. Hence, $\operatorname{div} J_t$ is also well defined in the sense of distributions.

Now, the transient eddy current equation (2.2) is equivalent to

$$-\int_0^T \int_{\mathbb{R}^3} \sigma E \cdot \partial_t \Phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl}\Phi \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \langle J_t, \Phi\rangle \,\mathrm{d}t \tag{2.5}$$

for all $\Phi \in \mathcal{D}(\mathbb{R}^3\times]0,T[)^3$.

In the rest of this subsection, we continue along the lines of [16, section 2].

We first recall the definition of the time-derivative in the sense of vector-valued distributions: For two Banach spaces X, Y and a continuous injection $\iota : X \hookrightarrow Y$, $E \in L^2(0,T,X)$ has a time-derivative in $L^2(0,T,Y)$ in the sense of vector-valued distributions if there exists $\dot{E} \in L^2(0,T,Y)$ which fulfills

$$\int_0^T \dot{E}\,\varphi \,\mathrm{d}t = -\int_0^T \iota E\,\partial_t\varphi \,\mathrm{d}t \quad \text{for all } \varphi \in \mathcal{D}(]0,T[)$$

(cf., e.g., [13, XVIII, §1]). For a Gelfand triple $\mathcal{V} \overset{\iota}{\hookrightarrow} \mathcal{H} \overset{\iota'}{\hookrightarrow} \mathcal{V}'$ of real separable Hilbert spaces $\mathcal{V}$ and $\mathcal{H}$, the space

$$\mathcal{W}(0,T,\mathcal{V},\mathcal{V}') := \left\{ E \in L^2(0,T,\mathcal{V}) \,|\, \dot{E} \in L^2(0,T,\mathcal{V}') \right\}$$

is defined by taking the time-derivative with respect to the injection $\iota'\iota : \mathcal{V} \hookrightarrow \mathcal{V}'$. The image of the space $\mathcal{W}(0,T,\mathcal{V},\mathcal{V}')$ under ι is continuously embedded in $C(0,T,\mathcal{H})$, and, for $E, F \in \mathcal{W}(0,T,\mathcal{V},\mathcal{V}')$, the following integration by parts formula holds:

$$\int_0^T \left[\langle \dot{E}(t), F(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + \langle \dot{F}(t), E(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\right] \mathrm{d}t = (\iota E(T), \iota F(T))_{\mathcal{H}} - (\iota E(0), \iota F(0))_{\mathcal{H}},$$

where $(\cdot,\cdot)_{\mathcal{H}}$ denotes the inner product of $\mathcal{H}$; cf., e.g., [13, XVIII, §1, Thms. 1,2]. As a special case we have

$$H^1(0,T,\mathcal{V}) = \mathcal{W}(0,T,\mathcal{V},\mathcal{V}),$$

where $\mathcal{V} = \mathcal{H}$ is identified with its dual and ι is the identity mapping.

In view of (2.2), we introduce the space

$$\mathcal{W}_\sigma := \left\{E \in L^2(0,T,W(\mathrm{curl}))\,|\,(\sigma E)^{\cdot} \in L^2(0,T,W(\mathrm{curl})')\right\},$$

where $(\sigma E)^{\cdot}$ denotes the time-derivative of $\sigma E \in L^2(\mathbb{R}^3_T)^3$ in the sense of vector-valued distributions with respect to the canonical injection $L^2(\mathbb{R}^3)^3 \hookrightarrow W(\mathrm{curl})'$. Note that for every $E \in H^1(0,T,W(\mathrm{curl}))$, $\sigma E \in L^2(\mathbb{R}^3_T)^3$, and, in that sense, $E \in \mathcal{W}_\sigma$ with $(\sigma E)^{\cdot} = \sigma\dot{E}$.

LEMMA 2.2. *If $E \in \mathcal{W}_\sigma$, then $\sqrt{\sigma}E \in C(0,T,L^2(\mathbb{R}^3)^3)$. Additionally, for two fields $E, F \in \mathcal{W}_\sigma$ the following integration by parts formula holds:*

$$\int_0^T \langle(\sigma E)^{\cdot}, F\rangle\,\mathrm{d}t + \int_0^T \langle(\sigma F)^{\cdot}, E\rangle\,\mathrm{d}t = \int_{\mathbb{R}^3} \sigma\left(E(T)\cdot F(T) - E(0)\cdot F(0)\right)\mathrm{d}x. \tag{2.6}$$

Proof. In [16, section 2] this lemma is proven for a scalar analogue. We repeat the proof for the convenience of the reader.

We define the space L^2_σ by taking the closure of

$$\left\{\sqrt{\sigma}E\,|\,E \in L^2(\mathbb{R}^3)^3\right\} \subseteq L^2(\mathbb{R}^3)^3$$

with respect to the $L^2(\mathbb{R}^3)^3$-norm. L^2_σ is a separable Hilbert space equipped with the standard $L^2(\mathbb{R}^3)^3$–inner product.

Then we define a mapping I by

$$I:\ W(\mathrm{curl}) \to L^2_\sigma, \quad E \mapsto \sqrt{\sigma}E,$$

which is continuous and has dense range. We identify the Hilbert space L^2_σ with its dual. Then, after factoring out the nullspace N of I, we obtain that

$$\iota:\ W(\mathrm{curl})/N \to L^2_\sigma, \quad E + N \mapsto IE$$

defines an injective, continuous mapping and hence a Gelfand triple

$$W(\mathrm{curl})/N \overset{\iota}{\hookrightarrow} L^2_\sigma \overset{\iota'}{\hookrightarrow} (W(\mathrm{curl})/N)'.$$

For all $G \in L^2_\sigma$ the dual mapping ι' is given by

$$\langle \iota' G, F + N\rangle_{(W(\mathrm{curl})/N)'\times W(\mathrm{curl})/N} = \int_{\mathbb{R}^3} G\cdot\sqrt{\sigma}F\,\mathrm{d}x \quad \text{for all } F \in W(\mathrm{curl}). \tag{2.7}$$

Let $E \in \mathcal{W}_\sigma$ and let $G = (\sigma E)^{\cdot} \in L^2(0,T,W(\mathrm{curl})')$ be the time-derivative of $\sigma E \in L^2(\mathbb{R}^3_T)^3$ with respect to the canonical injection $L^2(\mathbb{R}^3)^3 \hookrightarrow W(\mathrm{curl})'$. Now we show that G is the time-derivative of $E+N \in L^2(0,T,W(\mathrm{curl})/N)$ with respect to $\iota'\iota$. For $\varphi \in \mathcal{D}(]0,T[)$ and $F \in N$ we have

$$\int_0^T \langle G(t), F\rangle \varphi(t)\,\mathrm{d}t = -\int_0^T \int_{\mathbb{R}^3} \sigma E(t)\cdot F\,\mathrm{d}x\,\partial_t\varphi(t)\,\mathrm{d}t = 0,$$

and thus $\langle G(t), F\rangle = 0$ for $t \in]0,T[$ a.e. Hence, $G(t) \in N^\perp$ and we can identify G with an element of $L^2(0,T,(W(\mathrm{curl})/N)')$. Then, for $F+N \in W(\mathrm{curl})/N$ it follows that

$$\begin{aligned}\int_0^T \langle G(t), F+N\rangle_{(W(\mathrm{curl})/N)'\times W(\mathrm{curl})/N}\varphi(t)\,\mathrm{d}t &= \int_0^T \langle G(t), F\rangle\varphi(t)\,\mathrm{d}t\\ &= -\int_0^T\int_{\mathbb{R}^3}\sigma E(t)\cdot F\,\mathrm{d}x\,\partial_t\varphi(t)\,\mathrm{d}t\\ &= -\int_0^T \langle \iota'\iota(E(t)+N), F+N\rangle_{(W(\mathrm{curl})/N)'\times W(\mathrm{curl})/N}\partial_t\varphi(t)\,\mathrm{d}t\end{aligned}$$

and, accordingly, $G = (E+N)^\cdot$ and $E+N \in \mathcal{W}(0,T,W(\mathrm{curl})/N,(W(\mathrm{curl})/N)')$. Now, it follows that $\sqrt{\sigma}E = \iota(E+N) \in C(0,T,L^2_\sigma) \subseteq C(0,T,L^2(\mathbb{R}^3)^3)$, and using (2.7) we obtain the integration by parts formula (2.6). ∎

For the next lemma recall that for $E \in L^2(0,T,W(\mathrm{curl}))$ the equation (1.3) is to be understood in the sense of distributions; cf. the beginning of this subsection.

Lemma 2.3. *Every solution $E \in L^2(0,T,W(\mathrm{curl}))$ of (1.3) is in $\mathcal{W}_\sigma$ and thus has well-defined initial values*

$$\sqrt{\sigma(x)}E(x,0) \in L^2(\mathbb{R}^3)^3.$$

For $t \in]0,T[$ a.e., $(\sigma E)^\cdot(t) \in W(\mathrm{curl})'$ is given by

$$\langle(\sigma E)^\cdot(t), F\rangle = -\langle J_t(t), F\rangle - \int_{\mathbb{R}^3}\frac{1}{\mu}\operatorname{curl}E(t)\cdot\operatorname{curl}F\,\mathrm{d}x \quad \textit{for all } F \in W(\mathrm{curl}). \tag{2.8}$$

Proof. Let E be a solution of (1.3). Define $G(t) \in W(\mathrm{curl})'$ by

$$\langle G(t), \Psi\rangle := -\langle J_t(t),\Psi\rangle - \int_{\mathbb{R}^3}\frac{1}{\mu}\operatorname{curl}E(t)\cdot\operatorname{curl}\Psi\,\mathrm{d}x \quad \text{for all } \Psi \in W(\mathrm{curl}).$$

Then $G \in L^2(0,T,W(\mathrm{curl})')$, and, due to the fact that E solves (2.5) with $\Phi = \Psi\varphi$ for all $\varphi \in \mathcal{D}(]0,T[)$ and all $\Psi \in \mathcal{D}(\mathbb{R}^3)^3$, it holds that

$$\int_0^T \langle G(t),\Psi\rangle\varphi(t)\,\mathrm{d}t = -\int_0^T\int_{\mathbb{R}^3}\sigma E\cdot\Psi\,\mathrm{d}x\partial_t\varphi\,\mathrm{d}t = -\int_0^T\langle\sigma E(t),\Psi\rangle\partial_t\varphi(t)\,\mathrm{d}t. \tag{2.9}$$

Since $\mathcal{D}(\mathbb{R}^3)^3$ is dense in $W(\mathrm{curl})$ and both sides depend continuously on Ψ, we obtain that (2.9) holds for all $\Psi \in W(\mathrm{curl})$. Now it follows from the fact that $W(\mathrm{curl}) \otimes \mathcal{D}(]0,T[)$ is dense in $L^2(0,T,W(\mathrm{curl}))$ that $G = (\sigma E)^\cdot$ with respect to the canonical injection $L^2(\mathbb{R}^3)^3 \hookrightarrow W(\mathrm{curl})'$. This shows that $E \in \mathcal{W}_\sigma$. ∎

Lemma 2.3 shows that the initial condition (2.3) makes sense for solutions of (2.2), and, in that sense, we can speak of solutions $E \in L^2(0,T,W(\mathrm{curl}))$ of (2.2)–(2.3). Now, we give an equivalent variational formulation.

Lemma 2.4. *The following problems are equivalent:*

(a) *Find* $E \in L^2(0,T,W(\mathrm{curl}))$ *that solves* (2.2) *and* (2.3).

(b) *Find* $E \in \mathcal{W}_\sigma$ *that solves* (2.3) *and*

$$\int_0^T \langle (\sigma E)^\cdot, F\rangle \,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} F \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \langle J_t, F\rangle \,\mathrm{d}t \tag{2.10}$$

for all $F \in L^2(0,T,W(\mathrm{curl}))$.

(c) *Find* $E \in L^2(0,T,W(\mathrm{curl}))$ *that solves*

$$-\int_0^T \langle (\sigma F)^\cdot, E\rangle \,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} F \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \langle J_t, F\rangle \,\mathrm{d}t + \int_{\mathbb{R}^3} \sigma E_0 \cdot F(0)\,\mathrm{d}x$$

for all $F \in \mathcal{W}_\sigma$ *with* $\sqrt{\sigma}F(T) = 0$.

(d) *Find* $E \in L^2(0,T,W(\mathrm{curl}))$ *that solves*

$$-\int_0^T \int_{\mathbb{R}^3} \sigma E \cdot \partial_t \Phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \langle J_t, \Phi\rangle \,\mathrm{d}t + \int_{\mathbb{R}^3} \sigma E_0 \cdot \Phi(0)\,\mathrm{d}x$$

for all $\Phi \in \mathcal{D}(\mathbb{R}^3 \times [0,T[)^3$.

Proof. We start by showing (a) $\Longrightarrow$ (b). If $E \in L^2(0,T,W(\mathrm{curl}))$ solves the eddy current equations (2.2)–(2.3), it follows from Lemma 2.3 that $E \in \mathcal{W}_\sigma$ and (2.10) holds for all $F(x,t) = G(x)\varphi(t)$ with $G \in W(\mathrm{curl})$ and $\varphi \in \mathcal{D}(]0,T[)$. Since $W(\mathrm{curl}) \otimes \mathcal{D}(]0,T[)$ is dense in $L^2(0,T,W(\mathrm{curl}))$, and both sides of (2.10) depend continuously on $F \in L^2(0,T,W(\mathrm{curl}))$, (b) follows.

(b) $\Longrightarrow$ (c) follows from the integration by parts formula (2.6).

(c) $\Longrightarrow$ (d) follows from the fact that for $\Phi \in \mathcal{D}(\mathbb{R}^3 \times [0,T[)^3$ the time-derivative $(\sigma\Phi)^\cdot \in L^2(0,T,W(\mathrm{curl})')$ of $\sigma\Phi \in L^2(\mathbb{R}^3_T)^3$ with respect to the canonical injection $L^2(\mathbb{R}^3)^3 \hookrightarrow W(\mathrm{curl})'$ is the image of the classical time-derivative $\sigma\partial_t\Phi(t)$ under this injection, i.e.,

$$\langle (\sigma\Phi)^\cdot(t), E(t)\rangle = \int_{\mathbb{R}^3} \sigma\partial_t\Phi(t)\cdot E(t)\,\mathrm{d}x \qquad \text{for } t \in]0,T[\text{ a.e.}$$

Finally, to show the implication (d) $\Longrightarrow$ (a) we use (d) with $\Phi \in \mathcal{D}(\mathbb{R}^3 \times]0,T[)^3$. Then $E \in L^2(0,T,W(\mathrm{curl}))$ solves (2.3), and Lemma 2.3 yields $E \in \mathcal{W}_\sigma$. Now, the integration by parts formula (2.6) applied to (d) with $\Phi = \Psi\varphi$, $\Psi \in \mathcal{D}(\mathbb{R}^3)^3$, $\varphi \in \mathcal{D}([0,T[)$ with $\varphi(0) = 1$, and using Lemma 2.3 implies that $\sqrt{\sigma}E_0 = \sqrt{\sigma}E(0)$. ∎

Now, the proof of Theorem 2.1 reads as follows.

Proof of Theorem 2.1.

(a) This follows from Lemmas 2.2 and 2.3.

(b) This follows from the equivalence of (a) and (d) in Lemma 2.4.

(c) Assume that $E \in \mathcal{W}_\sigma$ is a solution of (2.2)–(2.3) with $\sqrt{\sigma}E(0) = 0$ and $J_t = 0$. Using Lemma 2.4(b) and the integration by parts formula (2.6) implies

$$\begin{aligned} 0 &= \int_0^T \langle (\sigma E)^\cdot, E\rangle \,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} E \,\mathrm{d}x\,\mathrm{d}t \\ &\geq \frac{1}{2}\|\sqrt{\sigma}E(T)\|^2_{L^2(\mathbb{R}^3)^3} + \frac{1}{\|\mu\|_\infty}\|\operatorname{curl} E\|^2_{L^2(\mathbb{R}^3_T)^3}. \end{aligned}$$

We obtain $\operatorname{curl} E = 0$ and $\sqrt{\sigma} E = 0$. The second assertion is obvious. ∎

3. A unified variational formulation. In this section we present a new, uniquely solvable, and uniformly coercive variational formulation that determines the solution of the eddy current equations, (2.2) and (2.3), up to the addition of a gradient field. From this we obtain solvability of (2.2) and (2.3) and a continuity result that is uniform with respect to the conductivity σ.

Our general approach is as follows. We write

$$E = \tilde{E} + \nabla u$$

with a divergence-free field $\tilde{E}$ and a gradient field ∇u. Note that this is very similar to the classical (A, φ)-formulation with Coulomb gauge (cf., e.g., [9, I.A, 4, §3]), where A is a divergence-free magnetic vector potential and φ a scalar function with

$$E = -\partial_t (A + \nabla \varphi).$$

The crucial point is to consider $\nabla u = \nabla u_{\tilde{E}}$ as a continuous linear function of $\tilde{E}$; cf. Lemma 3.1. This allows us to rewrite the eddy current equations (2.2)–(2.3) as a variational equation for $\tilde{E}$, which is uniformly coercive on the space of divergence-free functions and thus uniquely determines the field $\tilde{E}$. Note that $\tilde{E}$ does not solve the eddy current equations. Our new variational formulation enables us to study the asymptotic behavior of $\tilde{E}$ for $\sigma \to 0$. From this we can then deduce properties of the asymptotic behavior of any solution E of the eddy current equations.

For our results we need stronger assumptions on σ. Let $R > 0$ and B_R denote the open ball with radius R centered at the origin. In the following, we assume that

$$\begin{aligned}\sigma \in L^\infty_R(\mathbb{R}^3) := \{\sigma \in L^\infty_{\geq}(\mathbb{R}^3) \,|\, \Omega := \operatorname{supp} \sigma \subset B_R,\ \sigma \in L^\infty_+(\Omega),\ \text{and } \exists\, s \in \mathbb{N}:\\ \Omega = \cup_{i=1}^s \Omega_i,\ \text{where } \Omega_i,\ \text{are bounded Lipschitz domains with}\\ \overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset,\ i,j = 1,\dots,s,\ i \neq j,\ \text{and } \mathbb{R}^3 \setminus \overline{\Omega} \text{ is connected}\}.\end{aligned}$$

Note that our continuity results do not depend on the lower bound of σ.

The case of $\sigma \equiv 0$ is treated separately.

LEMMA 3.1. *There is a continuous linear map*

$$L^2_\rho(\mathbb{R}^3)^3 \to H(\operatorname{curl} 0, \mathbb{R}^3) := \{E \in L^2(\mathbb{R}^3)^3 \mid \operatorname{curl} E = 0\}, \quad E \mapsto \nabla u_E,$$

with

$$\operatorname{div}(\sigma(E + \nabla u_E)) = 0 \quad \text{in } \mathbb{R}^3, \tag{3.1}$$

and which extends (by setting $\nabla u_E(t) := \nabla u_{E(t)}$ for $t \in]0,T[$ a.e.) to a continuous linear map

$$L^2(0,T,L^2_\rho(\mathbb{R}^3)^3) \to L^2(0,T,H(\operatorname{curl} 0, \mathbb{R}^3)), \quad E \mapsto \nabla u_E,$$

for which $E \in H^1(0,T,L^2_\rho(\mathbb{R}^3)^3)$ implies

$$\nabla u_E \in H^1(0,T,H(\operatorname{curl} 0,\mathbb{R}^3)) \quad \text{and} \quad (\nabla u_E)^\cdot = \nabla u_{\dot{E}}.$$

Proof. Let $E \in L^2_\rho(\mathbb{R}^3)^3$. Due to Poincaré's inequality (cf., e.g., [10, IV, §7, Prop. 2]), the fact that σ is positively bounded from below on Ω, and the Lax–Milgram theorem (cf., e.g., [22, section 8, Thm. 8.14]), there exists a unique $u_E \in H^1_\square(\Omega)$ that solves

$$\int_\Omega \sigma \nabla u \cdot \nabla v \,\mathrm{d}x = -\int_\Omega \sigma E \cdot \nabla v \,\mathrm{d}x \quad \text{for all } v \in H^1(\Omega). \tag{3.2}$$

Here, $H^1_\square(\Omega) := \{v \in H^1(\Omega) \mid \int_{\Omega_i} v \,\mathrm{d}x = 0,\ i = 1, \dots, s\}$. Furthermore, u_E depends continuously on $E|_\Omega \in L^2(\Omega)^3$.

We extend u_E to an element of $W^1(\mathbb{R}^3)$ by solving $\Delta u = 0$ on $\mathbb{R}^3 \setminus \overline{\Omega}$ with $u|_{\partial\Omega} = u_E|_{\partial\Omega}$ for $u \in W^1(\mathbb{R}^3 \setminus \overline{\Omega})$. Again, the Lax–Milgram theorem provides a unique solution which depends continuously on $u_E|_{\partial\Omega}$ and thus on E.

Let u_E again denote its extension. Then $u_E \in W^1(\mathbb{R}^3)$, and the mapping $E \mapsto \nabla u_E$ is well defined, linear, and continuous with a continuity constant that depends on the lower and upper bounds of σ. Moreover, (3.1) is fulfilled.

The remaining assertions follow from standard time regularity arguments; cf., e.g., the proof of Lemma 3.7(a) below. ∎

For the rest of this paper, let ∇u_E denote the image of E under this mapping. Note that there are different possibilities for constructing this map, but $\sqrt{\sigma}\nabla u_E$ is uniquely determined by the condition (3.1). Moreover, it holds that

$$\|\sqrt{\sigma}\nabla u_E\|_{L^2(\mathbb{R}^3)^3} \le \|\sqrt{\sigma} E\|_{L^2(\mathbb{R}^3)^3}, \tag{3.3}$$

and, obviously, for all $E \in W^1$, we have $E + \nabla u_E \in W(\mathrm{curl})$.

The fact that the curl of a solution is unique, but not the solution itself, leads to the idea to work with spaces where $\|\operatorname{curl} \cdot\|_{L^2(\mathbb{R}^3)^3}$ defines a norm. Therefore, let

$$W^1_\diamondsuit := \{E \in W^1 = W^1(\mathbb{R}^3)^3 \mid \operatorname{div} E = 0\}, \quad \|\cdot\|_\diamondsuit := \|\operatorname{curl} \cdot\|_{L^2(\mathbb{R}^3)^3}.$$

On $W^1_\diamondsuit$, we have that $\|\cdot\|_{W^1} = \|\nabla \cdot\|_{L^2(\mathbb{R}^3)^{3\times 3}} = \|\cdot\|_\diamondsuit$ (cf., e.g., the proof of [11, IX.A, §1, Thm. 3]), so that $W^1_\diamondsuit$ equipped with the norm $\|\cdot\|_\diamondsuit$ is a Hilbert space.

We define the bilinear form $a : L^2(0,T,W^1) \times H^1(0,T,W^1) \to \mathbb{R}$ by

$$a(E,\Phi) := -\int_0^T \int_{\mathbb{R}^3} \sigma(E + \nabla u_E) \cdot \dot{\Phi} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} E \cdot \operatorname{curl} \Phi \,\mathrm{d}x\,\mathrm{d}t, \tag{3.4}$$

and, motivated by Lemma 2.4(d), the linear form $l : H^1(0,T,W^1) \to \mathbb{R}$:

$$l(\Phi) := -\int_0^T \langle J_t, \Phi\rangle \,\mathrm{d}t + \int_{\mathbb{R}^3} \sigma E_0 \cdot \Phi(0) \,\mathrm{d}x.$$

Now we can state the main result of this section. Let

$$H^1_{T0}(0,T,W^1_\diamondsuit) := \{\Psi \in H^1(0,T,W^1_\diamondsuit) \mid \Psi(T) = 0\}.$$

THEOREM 3.2.

(a) *If $\tilde{E} \in L^2(0,T,W^1_\diamondsuit)$ solves*

$$a(\tilde{E},\Phi) = l(\Phi) \quad \textit{for all } \Phi \in H^1_{T0}(0,T,W^1_\diamondsuit), \tag{3.5}$$

then $\tilde{E} + \nabla u_{\tilde{E}} \in L^2(0,T,W(\mathrm{curl}))$ solves (2.2)–(2.3). $a|_{H^1_{T0}(0,T,W^1_\diamondsuit)^2}$ is uniformly coercive with respect to $\|\cdot\|_{L^2(0,T,W^1_\diamondsuit)}$:

$$a(\Phi,\Phi) \ge \frac{1}{\|\mu\|_\infty} \|\Phi\|^2_{L^2(0,T,W^1_\diamondsuit)} \quad \textit{for all } \Phi \in H^1_{T0}(0,T,W^1_\diamondsuit).$$

(b) *There is a unique solution* $\tilde{E} \in L^2(0,T,W^1_\diamond)$ *of* (3.5)*.* $\tilde{E}$ *depends continuously on* J_t *and* $\sqrt{\sigma}E_0$*:*

(3.6)
$$\|\tilde{E}\|_{L^2(0,T,W^1_\diamond)} \leq \max(\|\mu\|_\infty, 2)\max(\sqrt{5}\|J_t\|_{L^2(0,T,W(\mathrm{curl})')}, \|\sqrt{\sigma}E_0\|_{L^2(\mathbb{R}^3)^3}).$$

$\tilde{E}+\nabla u_{\tilde{E}}$ *solves the eddy current equations* (2.2)–(2.3)*, and any other solution* $E \in L^2(0,T,W(\mathrm{curl}))$ *of* (2.2)–(2.3) *fulfills*

$$\mathrm{curl}\, E = \mathrm{curl}\, \tilde{E}, \quad \sqrt{\sigma}E = \sqrt{\sigma}(\tilde{E} + \nabla u_{\tilde{E}}). \tag{3.7}$$

$\mathrm{curl}\, E$ *and* $\sqrt{\sigma}E$ *depend continuously on* J_t *and* $\sqrt{\sigma}E_0$*:*

$$\|\mathrm{curl}\, E\|_{L^2(\mathbb{R}^3_T)^3} \leq \max(\|\mu\|_\infty, 2)\max(\sqrt{5}\|J_t\|_{L^2(0,T,W(\mathrm{curl})')}, \|\sqrt{\sigma}E_0\|_{L^2(\mathbb{R}^3)^3}),$$
$$\|\sqrt{\sigma}E\|_{L^2(\mathbb{R}^3_T)^3} \leq 4\sqrt{1+R^2}\|\sqrt{\sigma}\|_\infty \|\mathrm{curl}\, E\|_{L^2(\mathbb{R}^3_T)^3}.$$

If σ equals zero, we have the following result.

THEOREM 3.3. *For* $\sigma \equiv 0$*,* $E \in L^2(0,T,W^1_\diamond)$ *is a solution of* (2.2) *if and only if* E *solves*

$$a_0(E,F) = l_0(F) \quad \textit{for all } F \in L^2(0,T,W^1_\diamond), \tag{3.8}$$

where a_0 *and* l_0 *denote* $a(\cdot,\cdot)$ *and* $l(\cdot)$ *with* $\sigma \equiv 0$*. There exists a unique solution* $E \in L^2(0,T,W^1_\diamond)$*, and this solution depends continuously on* J_t*:*

$$\|E\|_{L^2(0,T,W^1_\diamond)} \leq \sqrt{5}\|\mu\|_\infty \|J_t\|_{L^2(0,T,W(\mathrm{curl})')}.$$

The proofs can be found in the following subsection.

COROLLARY 3.4. *Let* $(\sigma_n)_{n\in\mathbb{N}} \subset L^\infty_R(\mathbb{R}^3)$ *be a bounded sequence and* $\tilde{E}_n$*,* $n \in \mathbb{N}$*, be the corresponding unique solutions of* (3.5)*. Then the sequences*

$$(\tilde{E}_n)_{n\in\mathbb{N}} \subset L^2(0,T,W^1_\diamond) \quad \textit{and} \quad (\sqrt{\sigma_n}\tilde{E}_n)_{n\in\mathbb{N}},\ (\sqrt{\sigma_n}\nabla u_{\tilde{E}_n})_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^3_T)^3$$

are bounded. The bounds depend on the bound of $(\sigma_n)_{n\in\mathbb{N}}$*.*

In particular, for any sequence $(E_n)_{n\in\mathbb{N}} \subset L^2(0,T,W(\mathrm{curl}))$ *of corresponding solutions of* (2.2)–(2.3) *the sequences*

$$(\mathrm{curl}\, E_n)_{n\in\mathbb{N}},\ (\sqrt{\sigma}_n E_n)_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^3_T)^3$$

are bounded.

3.1. Existence. To show the first part of Theorem 3.2(a), we will make use of the following simple decomposition.

LEMMA 3.5.

(a) *Every* $\Phi \in \mathcal{D}(\mathbb{R}^3)^3$ *can be written as*

$$\Phi = \Psi + \nabla\phi,$$

with $\Psi \in W^1_\diamond$*,* $\phi \in W^1(\mathbb{R}^3)$*, and* $\nabla\phi \in W^1$*.*

(b) *Every* $\Phi \in \mathcal{D}(\mathbb{R}^3 \times [0,T[)^3$ *can be written as*

$$\Phi = \Psi + \nabla\phi,$$

with $\Psi \in H^1_{T0}(0,T,W^1_\diamond)$*,* $\phi \in H^1(0,T,W^1(\mathbb{R}^3))$*,* $\nabla\phi \in H^1(0,T,W^1)$*, and* $\nabla\phi(T) = 0$*.*

Proof. Let $\Phi \in \mathcal{D}(\mathbb{R}^3)^3$. Then the Lax–Milgram theorem yields a unique solution $\phi \in W^1(\mathbb{R}^3)$ of

$$\Delta\phi = \operatorname{div}\Phi \quad \text{ in } \mathbb{R}^3.$$

By standard regularity results $\phi \in C^\infty(\mathbb{R}^3)$. For a centered ball $B \subset \mathbb{R}^3$ containing the support of Φ, ϕ solves the exterior Dirichlet problem

$$\Delta\phi = 0 \quad \text{ in } \mathbb{R}^3 \setminus \overline{B}, \quad \phi|_{\partial B} \in H^{3/2}(\partial B)$$

so that it follows from, e.g., [21, Thm. 2.5.1] that $\nabla\phi \in W^1(\mathbb{R}^3 \setminus \overline{B})^3$, and hence $\nabla\phi \in W^1$. With $\Psi := \Phi - \nabla\phi \in W^1_\diamondsuit$ we obtain assertion (a).

Assertion (b) follows from standard time regularity arguments; cf., e.g., the proof of Lemma 3.7(a) below. ▯

We will prove the existence result in Theorem 3.2(b) using the Lions–Lax–Milgram theorem.

LEMMA 3.6 (Lions–Lax–Milgram theorem). *Let $\mathcal{H}$ be a Hilbert space and V be a normed (not necessarily complete) vector space. Let $a : \mathcal{H} \times V \to \mathbb{R}$ be a bilinear form satisfying the following properties:*

(a) *For every $\Phi \in V$, the linear form $E \mapsto a(E, \Phi)$ is continuous on $\mathcal{H}$.*

(b) *There exists $\alpha > 0$ such that*

$$\inf_{\|\Phi\|_V=1} \sup_{\|E\|_{\mathcal{H}}\le 1} |a(E,\Phi)| \ge \alpha.$$

Then for each continuous linear form $l \in V'$, there exists $E_l \in \mathcal{H}$ such that

$$a(E_l, \Phi) = \langle l, \Phi\rangle \text{ for all } \Phi \in V \quad \text{and} \quad \|E_l\|_{\mathcal{H}} \le \frac{1}{\alpha}\|l\|_{V'}.$$

The proof of Lemma 3.6 can be found, for example, in [23, III.2, Thm. 2.1, Cor. 2.1].

Proof of Theorem 3.2.

(a) Obviously, for gradient fields $\nabla\phi \in H^1(0,T,W^1)$ with $\phi \in H^1(0,T,W^1(\mathbb{R}^3))$, $a(\cdot, \nabla\phi)$ as well as $l(\nabla\phi)$ vanish. (For the latter, recall that $\operatorname{div} J_t = 0$ and $\operatorname{div}(\sigma E_0) = 0$.) Hence, it follows from the decomposition in Lemma 3.5, and from the linearity of a and l, that (for any $\tilde{E} \in L^2(0,T,W^1_\diamondsuit)$)

$$a(\tilde{E}, \Phi) = l(\Phi)$$

holds for all $\Phi \in \mathcal{D}(\mathbb{R}^3 \times [0,T[)^3$ if it holds for all $\Phi \in H^1_{T0}(0,T,W^1_\diamondsuit)$. Lemma 2.4 yields the first assertion.

For $\Phi \in H^1_{T0}(0,T,W^1_\diamondsuit)$, Lemma 3.1 and the integration by parts formula (2.6) yield

$$\begin{aligned} a(\Phi,\Phi) &= -\int_0^T \int_{\mathbb{R}^3} \sigma(\Phi + \nabla u_\Phi)\cdot\dot{\Phi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu}\,|\operatorname{curl}\Phi|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\ge \frac{1}{2}\|\sqrt{\sigma}(\Phi + \nabla u_\Phi)(0)\|^2_{L^2(\mathbb{R}^3)^3} + \frac{1}{\|\mu\|_\infty}\|\Phi\|^2_{L^2(0,T,W^1_\diamondsuit)} \end{aligned} \tag{3.9}$$

and thus the second assertion.

(b) We will apply the Lions–Lax–Milgram theorem. We use the Hilbert space $\mathcal{H} := L^2(0,T,W^1_\diamondsuit)$ and equip its subspace $V := H^1_{T0}(0,T,W^1_\diamondsuit)$ with the norm

$$\|\Phi\|_V^2 := \|\Phi\|^2_{L^2(0,T,W^1_\diamondsuit)} + \|\sqrt{\sigma}(\Phi + \nabla u_\Phi)(0)\|^2_{L^2(\mathbb{R}^3)^3}.$$

Then (3.9) implies that

$$\inf_{\|\Phi\|_V=1} \sup_{\|E\|_{\mathcal{H}}\leq 1} |a(E,\Phi)| \geq \inf_{\|\Phi\|_V=1} |a(\Phi,\Phi)| \geq \frac{1}{\max(\|\mu\|_\infty, 2)}.$$

Given $\Phi \in V$ we set $C := \max(\|\Phi\|_{L^2(0,T,W^1_\diamondsuit)}, \|\dot{\Phi}\|_{L^2(0,T,L^2(B_R)^3)})$. Then it follows from (3.3) and $\mu \in L^\infty_+(\mathbb{R}^3)$ that for all $E \in \mathcal{H}$

$$\begin{aligned}
|a(E,\Phi)| &= \left| \int_0^T \int_{\mathbb{R}^3} \left[-\sigma(E + \nabla u_E)\cdot\dot{\Phi} + \frac{1}{\mu}\operatorname{curl} E \cdot \operatorname{curl}\Phi \right] \mathrm{d}x\,\mathrm{d}t \right| \\
&\leq C\left[2\|\sqrt{\sigma}\|_\infty \|\sqrt{\sigma}E\|_{L^2(\mathbb{R}^3_T)^3} + \frac{1}{\inf\mu}\|E\|_{L^2(0,T,W^1_\diamondsuit)} \right] \\
&\leq C\left[\|\sigma\|_\infty 2\|E\|_{L^2(0,T,L^2(B_R)^3)} + \frac{1}{\inf\mu}\|E\|_{L^2(0,T,W^1_\diamondsuit)} \right] \\
&\leq C\left[\|\sigma\|_\infty 2\sqrt{1+R^2}\|E\|_{L^2(0,T,L^2_\rho(\mathbb{R}^3)^3)} + \frac{1}{\inf\mu}\|E\|_{L^2(0,T,W^1_\diamondsuit)} \right].
\end{aligned}$$

Similarly to the proof of [12, XI.B, §1, Lemma 1], it holds that

$$\|F\|_\rho \leq 2\|\nabla F\|_{L^2(\mathbb{R}^3)^3} = 2\|F\|_\diamondsuit \quad \text{for all } F \in W^1_\diamondsuit, \tag{3.10}$$

and thus

$$|a(E,\Phi)| \leq C\left[4\|\sigma\|_\infty\sqrt{1+R^2} + \frac{1}{\inf\mu} \right] \|E\|_{\mathcal{H}}.$$

Hence, for fixed $\Phi \in V$, $a(\cdot,\Phi)$ is continuous on $\mathcal{H}$.
Equation (3.10) also yields

$$\|F\|^2_{W(\mathrm{curl})} = \|F\|_\rho^2 + \|\operatorname{curl} F\|^2_{L^2(\mathbb{R}^3)^3} \leq 5\|F\|_\diamondsuit^2 \quad \text{for all } F \in W^1_\diamondsuit, \tag{3.11}$$

so that we obtain, for all $\Phi \in V$,

$$\begin{aligned}
|l(\Phi)| &= \left| -\int_0^T \langle J_t, \Phi\rangle\,\mathrm{d}t + \int_{\mathbb{R}^3} \sigma E_0 \cdot \Phi(0)\,\mathrm{d}x \right| \\
&\leq \|J_t\|_{L^2(0,T,W(\mathrm{curl})')}\|\Phi\|_{L^2(0,T,W(\mathrm{curl}))} \\
&\quad + \|\sqrt{\sigma}E_0\|_{L^2(\mathbb{R}^3)^3}\|\sqrt{\sigma}(\Phi + \nabla u_\Phi)(0)\|_{L^2(\mathbb{R}^3)^3} \\
&\leq \max(\sqrt{5}\|J_t\|_{L^2(0,T,W(\mathrm{curl})')}, \|\sqrt{\sigma}E_0\|_{L^2(\mathbb{R}^3)^3})\|\Phi\|_V.
\end{aligned}$$

Hence, $l \in V'$ and

$$\|l\|_{V'} \leq \max(\sqrt{5}\|J_t\|_{L^2(0,T,W(\mathrm{curl})')}, \|\sqrt{\sigma}E_0\|_{L^2(\mathbb{R}^3)^3}).$$

Now, Lemma 3.6 yields the existence of an $\tilde{E} \in \mathcal{H} = L^2(0,T,W^1_\diamond)$ that fulfills (3.5) and depends continuously on l, i.e.,

$$\|\tilde{E}\|_{L^2(0,T,W^1_\diamond)} \leq \max(\|\mu\|_\infty, 2)\max(\sqrt{5}\|J_t\|_{L^2(0,T,W(\mathrm{curl})')}, \|\sqrt{\sigma}E_0\|_{L^2(\mathbb{R}^3)^3}).$$

Part (a) yields that $\tilde{E} + \nabla u_{\tilde{E}} \in L^2(0,T,W(\mathrm{curl}))$ is a solution of the eddy current equations (2.2)–(2.3).

To show uniqueness, let $\tilde{E}_1, \tilde{E}_2 \in L^2(0,T,W^1_\diamond)$ be two solutions of (3.5). Then, $\tilde{E}_1 + \nabla u_{\tilde{E}_1}, \tilde{E}_2 + \nabla u_{\tilde{E}_2} \in L^2(0,T,W(\mathrm{curl}))$ both solve the eddy current equations (2.2)–(2.3). Now, Theorem 2.1(c) implies

$$\mathrm{curl}\,\tilde{E}_1 = \mathrm{curl}(\tilde{E}_1 + \nabla u_{\tilde{E}_1}) = \mathrm{curl}(\tilde{E}_2 + \nabla u_{\tilde{E}_2}) = \mathrm{curl}\,\tilde{E}_2,$$

and it follows that

$$0 = \|\,\mathrm{curl}(\tilde{E}_1 - \tilde{E}_2)\|_{L^2(\mathbb{R}^3)^3} = \|\tilde{E}_1 - \tilde{E}_2\|_\diamond.$$

The remaining assertions of (b) follow similarly from Theorem 2.1(c). ☐

Proof of Theorem 3.3. Theorem 3.3 follows from $\mu \in L^\infty_+(\mathbb{R}^3)$, (3.11), and the Lax–Milgram theorem. ☐

3.2. On time regularity. We close this section by showing a result on time regularity of the solutions.

LEMMA 3.7. *Let* $J_t \in H^1(0,T,W(\mathrm{curl})')$ *and* $E_0 \in W(\mathrm{curl})$ *such that*

$$\mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\,E_0\right) = -J_t(0)$$

in addition to the general assumptions (2.1) *on* J_t *and* E_0. *Let* $\tilde{E} \in L^2(0,T,W^1_\diamond)$ *be the solution of* (3.5). *Then, the following hold.*

(a) $\tilde{E} \in H^1(0,T,W^1_\diamond)$ *and* $\tilde{F} = (\tilde{E})^\cdot$ *is the solution of*

$$a(\tilde{F},\Phi) = -\int_0^T \langle (J_t)^\cdot, \Phi\rangle\,\mathrm{d}t \quad \textit{for all } \Phi \in H^1_{T0}(0,T,W^1_\diamond). \tag{3.12}$$

$F = \tilde{F} + \nabla u_{\tilde{F}} \in L^2(0,T,W(\mathrm{curl}))$ *solves*

$$\partial_t(\sigma F) + \mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\,F\right) = -(J_t)^\cdot \quad \textit{in } \mathbb{R}^3\times]0,T[$$

with zero initial conditions.

(b) *For any solution* $E \in L^2(0,T,W(\mathrm{curl}))$ *of the eddy current equations* (2.2)–(2.3) *we have that*

$$E|_\Omega \in H^1(0,T,L^2(\Omega)^3), \quad (E|_\Omega)^\cdot = F|_\Omega,$$
$$\mathrm{curl}\,E \in H^1(0,T,L^2(\mathbb{R}^3)^3), \quad (\mathrm{curl}\,E)^\cdot = \mathrm{curl}\,F = \mathrm{curl}\,\tilde{F}.$$

Proof.

(a) Theorem 3.2 yields that (3.12) has a unique solution $\tilde{F} \in L^2(0,T,W^1_\diamond)$, so it remains only to show that $\tilde{F} = (\tilde{E})^\cdot$, which, in turn, follows if

$$Z(t) = \int_0^t \tilde{F}(s)\,\mathrm{d}s + E_0 + \nabla v_{E_0} \in H^1(0,T,W^1_\diamond)$$

solves (3.5). Here, $v_{E_0} \in W^1(\mathbb{R}^3)$ is the unique solution of

$$\Delta v_{E_0} = -\operatorname{div} E_0 \quad \text{in } \mathbb{R}^3.$$

Let $\Phi \in H^1_{T0}(0,T,W^1_\diamondsuit)$. We define

$$\Psi(t) = \int_0^t \Phi(s)\,\mathrm{d}s - \int_0^T \Phi(s)\,\mathrm{d}s \in H^1_{T0}(0,T,W^1_\diamondsuit).$$

Note that our general assumption $\operatorname{div}(\sigma E_0) = 0$ together with Lemma 3.1 implies that

$$\sigma \nabla u_{Z(0)} = \sigma \nabla u_{(E_0 + \nabla v_{E_0})} = -\sigma \nabla v_{E_0},$$

so that we obtain

$$\begin{aligned}
a(Z,\Phi) &= \int_0^T \int_{\mathbb{R}^3} \left(-\sigma (Z + \nabla u_Z) \cdot \dot{\Phi} + \frac{1}{\mu} \operatorname{curl} Z \cdot \operatorname{curl} \Phi \right) \mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}^3} \left(\sigma (Z + \nabla u_Z)^{\cdot} \cdot \dot{\Psi} - \frac{1}{\mu} \operatorname{curl} \dot{Z} \cdot \operatorname{curl} \Psi \right) \mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}^3} \left(\sigma \left(Z(0) + \nabla u_{Z(0)} \right) \cdot \dot{\Psi}(0) - \frac{1}{\mu} \operatorname{curl} Z(0) \cdot \operatorname{curl} \Psi(0) \right) \mathrm{d}x \\
&= -a(\dot{Z},\Psi) + \int_{\mathbb{R}^3} \left(\sigma E_0 \cdot \Phi(0) - \frac{1}{\mu} \operatorname{curl} E_0 \cdot \operatorname{curl} \Psi(0) \right) \mathrm{d}x \\
&= \int_0^T \langle (J_t)^{\cdot}, \Psi \rangle \,\mathrm{d}t + \int_{\mathbb{R}^3} \sigma E_0 \cdot \Phi(0)\,\mathrm{d}x + \langle J_t(0), \Psi(0) \rangle \\
&= -\int_0^T \langle J_t, \dot{\Psi} \rangle \,\mathrm{d}t + \int_{\mathbb{R}^3} \sigma E_0 \cdot \Phi(0)\,\mathrm{d}x = l(\Phi).
\end{aligned}$$

(b) Part (b) follows immediately from (a) and Theorem 2.1(c). ∎

The analogous assertion holds for $\sigma \equiv 0$.

LEMMA 3.8. *Let $\sigma \equiv 0$, and let $J_t \in H^1(0,T,W(\mathrm{curl})')$ in addition to the general assumptions* (2.1) *on J_t.*

If $\tilde{E} \in L^2(0,T,W^1_\diamondsuit)$ is the solution of (3.8)*, then $\tilde{E} \in H^1(0,T,W^1_\diamondsuit)$ and $F = (\tilde{E})^{\cdot}$ is the solution of*

$$\operatorname{curl}\left(\frac{1}{\mu} \operatorname{curl} F \right) = -(J_t)^{\cdot} \quad \textit{in } \mathbb{R}^3 \times]0,T[.$$

The proof is analogous to the proof of Lemma 3.7(a).

4. Sensitivity analysis. In this section we keep E_0 and J_t fixed and analyze the solution(s) behavior if σ approaches zero. To this end, let $(\sigma_n)_{n\in\mathbb{N}} \subset L^\infty_R(\mathbb{R}^3)$ be a sequence such that

$$\lim_{n\to\infty} \sigma_n = 0 \quad \text{in } L^\infty(\mathbb{R}^3).$$

Corresponding to $(\sigma_n)_{n\in\mathbb{N}}$, let $(E_n)_{n\in\mathbb{N}} \subset L^2(0,T,W(\mathrm{curl}))$ denote any sequence of solutions of (2.2)–(2.3), and let $(\tilde{E}_n)_{n\in\mathbb{N}} \subset L^2(0,T,W^1_\diamondsuit)$ denote the sequence of unique solutions of (3.5). For $\sigma \equiv 0$, let $E \in L^2(0,T,W(\mathrm{curl}))$ denote any solution of (2.2), and let $\tilde{E} \in L^2(0,T,W^1_\diamondsuit)$ denote the solution of (3.8).

Our first result is that the solutions converge.

THEOREM 4.1. *It holds that*

$$\operatorname{curl} E_n \to \operatorname{curl} E,\ \sqrt{\sigma_n}E_n \to 0 \textit{ in } L^2(\mathbb{R}^3_T)^3,\ \textit{and}\ \ (\sigma_n E_n)^{\cdot} \to 0 \textit{ in } L^2(0,T,W(\operatorname{curl})').$$

Moreover, we show that (under some regularity assumptions) the directional derivative of E with respect to σ exists and can be characterized in the following way.

THEOREM 4.2. *Let* $J_t \in H^1(0,T,W(\operatorname{curl})')$ *and* $E_0 \in W(\operatorname{curl})$ *such that*

$$\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} E_0\right) = -J_t(0)$$

in addition to our general assumptions (2.1) *on* J_t *and* E_0. *Let* $d \in L^\infty_R(\mathbb{R}^3)$ *and* $h > 0$. *Let* $E_d \in H^1(0,T,W(\operatorname{curl}))$ *be a solution of* (2.2) *with* $\sigma \equiv 0$ *that fulfills* $\operatorname{div}(dE_d) = 0$ *and* $F \in L^2(0,T,W(\operatorname{curl}))$ *be a solution of*

$$\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl} F\right) = -d\dot{E}_d \quad \textit{in } \mathbb{R}^3\times]0,T[.$$

Let $E_h \in L^2(0,T,W(\operatorname{curl}))$ *be a solution of* (2.2)–(2.3) *with* $\sigma = hd$. *Then*

$$\frac{1}{h}(\operatorname{curl} E_h - \operatorname{curl} E) \to \operatorname{curl} F \quad \textit{in } L^2(\mathbb{R}^3_T)^3 \qquad (h \to 0^+).$$

Let us first comment on the existence of E_d and F. For instance, we can choose $E_d = \tilde{E} + \nabla u_{\tilde{E}}$, where $\nabla u_{\tilde{E}}$ is the image of $\tilde{E}$ under the mapping defined in Lemma 3.1 with $\sigma = d$. Then the time regularity of E_d and the existence of F follow from Lemmas 3.8 and 3.1 and Theorem 3.3. Note that E_d, F, and also E_h are not unique. Theorem 4.2 holds for every choice of E_h, E_d, and F.

The two theorems are proved in the following two subsections.

4.1. Convergence. Obviously, $\sqrt{\sigma_n}E_0 \to 0$ in $L^2(\mathbb{R}^3)^3$.

LEMMA 4.3. *It holds that*

$$\tilde{E}_n \to \tilde{E} \quad \textit{in } L^2(0,T,W^1_\diamond) \quad \textit{and} \quad \sqrt{\sigma_n}\tilde{E}_n,\ \sqrt{\sigma_n}\nabla u_{\tilde{E}_n} \to 0 \quad \textit{in } L^2(\mathbb{R}^3_T)^3.$$

Proof. First, we show that $\tilde{E}_n \rightharpoonup \tilde{E}$. To prove this it suffices to show that every subsequence of $(\tilde{E}_n)_{n\in\mathbb{N}}$ has a subsequence that converges weakly against $\tilde{E}$. From Corollary 3.4 we know that $(\tilde{E}_n)_{n\in\mathbb{N}} \subset L^2(0,T,W^1_\diamond)$ is bounded. Using that $\operatorname{supp}\sigma_n \subset B_R$ and Lemma 3.1, we obtain the second part of the assertion,

$$\sqrt{\sigma_n}\tilde{E}_n, \sqrt{\sigma_n}\nabla u_{\tilde{E}_n} \to 0 \quad \text{in } L^2(\mathbb{R}^3_T)^3.$$

Alaoglu's theorem (cf., e.g., [22, Thm. 6.62]) yields that every subsequence of $(\tilde{E}_n)_{n\in\mathbb{N}}$ contains a subsequence (that we still denote by $(\tilde{E}_n)_{n\in\mathbb{N}}$ for ease of notation) that converges weakly against some $\tilde{E}' \in L^2(0,T,W^1_\diamond)$. We show that all these weak limits are identical to $\tilde{E}$:

$\tilde{E}_n \rightharpoonup \tilde{E}'$ in $L^2(0,T,W^1_\diamond)$ implies that $\operatorname{curl}\tilde{E}_n \rightharpoonup \operatorname{curl}\tilde{E}'$ in $L^2(\mathbb{R}^3_T)^3$, so that for every $\Phi \in H^1_{T0}(0,T,W^1_\diamond)$ the left-hand side $a(\tilde{E}_n, \Phi)$ of (3.5) with $\sigma = \sigma_n$ converges against $a_0(\tilde{E}', \Phi)$. Clearly, the right-hand side of (3.5) with $\sigma = \sigma_n$ converges against $l_0(\Phi)$. Hence, $\tilde{E}'$ solves (3.8), and thus uniqueness provides $\tilde{E} = \tilde{E}'$, and hence

$$\tilde{E}_n \rightharpoonup \tilde{E} \quad \text{in } L^2(0,T,W^1_\diamond).$$

Since $\tilde{E}_n + \nabla u_{\tilde{E}_n}$ solves the eddy current equations (2.2)–(2.3) with $\sigma = \sigma_n$, we obtain using Lemma 2.4(b)

$$\begin{aligned}\|\mu^{-\frac{1}{2}} \operatorname{curl} \tilde{E}_n\|^2_{L^2(\mathbb{R}^3_T)^3} &= -\int_0^T \langle (\sigma_n(\tilde{E}_n + \nabla u_{\tilde{E}_n}))^{\cdot}, \tilde{E}_n + \nabla u_{\tilde{E}_n}\rangle \,\mathrm{d}t - \int_0^T \langle J_t, \tilde{E}_n\rangle \,\mathrm{d}t \\ &\leq \frac{1}{2}\|\sqrt{\sigma_n} E_0\|^2_{L^2(\mathbb{R}^3)^3} - \int_0^T \langle J_t, \tilde{E}_n\rangle \,\mathrm{d}t \\ &= \frac{1}{2}\|\sqrt{\sigma_n} E_0\|^2_{L^2(\mathbb{R}^3)^3} + \int_0^T \int_{\mathbb{R}^3} \frac{1}{\mu} \operatorname{curl} \tilde{E} \cdot \operatorname{curl} \tilde{E}_n \,\mathrm{d}x \,\mathrm{d}t,\end{aligned}$$

and hence

$$\limsup_{n\to\infty} \|\mu^{-\frac{1}{2}} \operatorname{curl} \tilde{E}_n\|_{L^2(\mathbb{R}^3_T)^3} \leq \|\mu^{-\frac{1}{2}} \operatorname{curl} \tilde{E}\|_{L^2(\mathbb{R}^3_T)^3},$$

which, together with $\tilde{E}_n \rightharpoonup \tilde{E}$, yields $\tilde{E}_n \to \tilde{E}$. ∎

Proof of Theorem 4.1. For any solutions E_n, respectively, E, of (2.2)–(2.3) with $\sigma = \sigma_n$, respectively, $\sigma \equiv 0$, we have that

$$\sqrt{\sigma_n} E_n = \sqrt{\sigma_n}(\tilde{E}_n + \nabla u_{\tilde{E}_n}), \quad \operatorname{curl} E_n = \operatorname{curl} \tilde{E}_n, \quad \text{and} \quad \operatorname{curl} E = \operatorname{curl} \tilde{E},$$

so that Lemma 4.3 provides $\operatorname{curl} E_n \to \operatorname{curl} E$ and $\sqrt{\sigma}_n E_n \to 0$.

From the explicit form (2.8) of $(\sigma_n E_n)^{\cdot}$ given in Lemma 2.3, we obtain for all $F \in L^2(0, T, W(\operatorname{curl}))$

$$\left| \int_0^T \langle (\sigma_n E_n)^{\cdot}, F\rangle \,\mathrm{d}t \right| \leq \frac{1}{\inf \mu} \|\operatorname{curl}(E - E_n)\|_{L^2(\mathbb{R}^3_T)^3} \|\operatorname{curl} F\|_{L^2(\mathbb{R}^3_T)^3},$$

and hence $(\sigma_n E_n)^{\cdot} \to 0$. ∎

4.2. Linearization results. To characterize the directional derivative of E with respect to σ, some more time regularity is needed. To this end, we assume in addition to (2.1) that $J_t \in H^1(0, T, W(\operatorname{curl})')$ and $E_0 \in W(\operatorname{curl})$ such that

$$\operatorname{curl}\left(\frac{1}{\mu} \operatorname{curl} E_0\right) = -J_t(0).$$

LEMMA 4.4. *For every $n \in \mathbb{N}$, $E_n - E \in L^2(0, T, W(\operatorname{curl}))$ solves*

$$\operatorname{curl}\left(\frac{1}{\mu} \operatorname{curl}(E_n - E)\right) = -\sigma_n \dot{E}_n \quad \textit{in } \mathbb{R}^3 \times]0, T[.$$

Moreover, there is a constant C such that

$$\limsup_{n\to\infty} \frac{\|\tilde{E}_n - \tilde{E}\|_{L^2(0,T,W^1_\diamond)}}{\|\sigma_n\|_\infty} \leq C.$$

Proof. From Lemmas 3.7 and 3.8 we know that the time derivatives of $\tilde{E}_n$, $\tilde{E}$, $u_{\tilde{E}_n}$, and $E_n|_{\Omega_n}$ exist. Then, it is easily checked that $\tilde{E}_n - \tilde{E}$ solves

$$a_0(\tilde{E}_n - \tilde{E}, \Phi) = -\int_0^T \int_{\mathbb{R}^3} \sigma_n(\dot{\tilde{E}}_n + \nabla u_{\dot{\tilde{E}}_n}) \cdot \Phi \,\mathrm{d}x \,\mathrm{d}t$$

for all $\Phi \in H^1_{T0}(0,T,W^1_\diamondsuit)$ and thus also for all $\Phi \in L^2(0,T,W^1_\diamondsuit)$. So the first assertion follows from the identity $(\tilde{E}_n + \nabla u_{\tilde{E}_n})|_{\Omega_n} = E_n|_{\Omega_n}$.

From Theorem 3.3 and (3.3) we now obtain a constant $C' > 0$ (depending on μ and R) so that

$$\begin{aligned}\|\tilde{E}_n - \tilde{E}\|_{L^2(0,T,W^1_\diamondsuit)} &\le C'\|\sqrt{\sigma_n}\|_\infty \|\sqrt{\sigma_n}\dot{E}_n\|_{L^2(\mathbb{R}^3_T)^3}\\ &\le 2C'\|\sigma_n\|_\infty \|(\tilde{E}_n)^{\cdot}\|_{L^2(0,T,L^2(B_R)^3)}.\end{aligned}$$

As every $(\tilde{E}_n)^{\cdot}$ solves (3.12) with $\sigma = \sigma_n$, Corollary 3.4 yields that $((\tilde{E}_n)^{\cdot})_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(0,T,W^1_\diamondsuit)$, and thus the second assertion follows. ∎

LEMMA 4.5. *Let $d \in L^\infty_R(\mathbb{R}^3)$ and $\tilde{E}_d = \tilde{E} + \nabla u_{\tilde{E}}$, where $\nabla u_{\tilde{E}}$ is the image of $\tilde{E}$ under the mapping defined in Lemma* 3.1 *with $\sigma = d$. Let $\tilde{F} \in L^2(0,T,W^1_\diamondsuit)$ be the solution of*

$$a_0(\tilde{F},\Phi) = -\int_0^T \int_{\mathbb{R}^3} d(\tilde{E}_d)^{\cdot} \cdot \Phi \,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } \Phi \in L^2(0,T,W^1_\diamondsuit). \tag{4.1}$$

Furthermore, for $h > 0$ let $\tilde{E}_h \in L^2(0,T,W^1_\diamondsuit)$ be the solution of (3.5) *corresponding to $\sigma = hd$. Then for $h \to 0^+$*

$$\frac{1}{h}(\tilde{E}_h - \tilde{E}) \to \tilde{F} \quad \text{in } L^2(0,T,W^1_\diamondsuit).$$

Proof. Lemmas 3.7, 3.8, and 4.3 yield that $(\tilde{E}_h)^{\cdot} \to (\tilde{E})^{\cdot}$ in $L^2(0,T,W^1_\diamondsuit)$. The mapping defined in Lemma 3.1 does not change if we take $\sigma = d$ instead of $\sigma = hd$. Hence, as d is fixed, the continuity of this mapping implies that $\nabla u_{\dot{\tilde{E}}_h} \to \nabla u_{\dot{\tilde{E}}}$ in $L^2(\mathbb{R}^3_T)^3$.

From Lemma 4.4 we obtain that for all $\Phi \in L^2(0,T,W^1_\diamondsuit)$

$$a_0\left(\frac{1}{h}(\tilde{E}_h - \tilde{E}) - \tilde{F}, \Phi\right) = -\int_0^T \int_{\mathbb{R}^3} d(\tilde{E}_h + \nabla u_{\tilde{E}_h} - \tilde{E} - \nabla u_{\tilde{E}})^{\cdot} \cdot \Phi \,\mathrm{d}x\,\mathrm{d}t.$$

The assertion now follows from setting $\Phi := \frac{1}{h}(\tilde{E}_h - \tilde{E}) - \tilde{F}$ and using the coercivity of a_0. ∎

Proof of Theorem 4.2. Let $E_d \in H^1(0,T,W(\mathrm{curl}))$ be a solution of (2.2) with $\sigma \equiv 0$ that fulfills $\mathrm{div}(dE_d) = 0$ and $F \in L^2(0,T,W(\mathrm{curl}))$ be a solution of

$$\mathrm{curl}\left(\frac{1}{\mu}\,\mathrm{curl}\,F\right) = -d\dot{E}_d \quad \text{in } \mathbb{R}^3\times]0,T[.$$

Let $\tilde{E}_d$ and $\tilde{F}$ be as in Lemma 4.5.

Since both E_d and $\tilde{E}_d$ solve (2.2) with $\sigma \equiv 0$, we have $\mathrm{curl}\,E_d = \mathrm{curl}\,\tilde{E}_d$. Hence, for $t \in]0,T[$ a.e., using the Poincaré lemma on B_R (cf., e.g., [11, IX.A, §1, Lemma 4]), we obtain a $p \in H^1(B_R)$ with $(E_d(t) - \tilde{E}_d(t))|_{B_R} = \nabla p$. Now $\mathrm{div}(d(E_d - \tilde{E}_d)) = 0$ implies that

$$\int_{\mathbb{R}^3} d\nabla p \cdot \nabla\varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^3)$$

so that $\sqrt{d}\nabla p = 0$. It follows that $d(E_d)^{\cdot} = d(\tilde{E}_d)^{\cdot}$ and hence $\mathrm{curl}\,F = \mathrm{curl}\,\tilde{F}$. Since also $\mathrm{curl}\,E_h = \mathrm{curl}\,\tilde{E}_h$, and $\mathrm{curl}\,E = \mathrm{curl}\,\tilde{E}$, the assertion follows from Lemma 4.5. ∎

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