SIAM J. NUMER. ANAL.
Vol. 45, No. 2, pp. 810–836

DETECTING INTERFACES IN A PARABOLIC-ELLIPTIC PROBLEM FROM SURFACE MEASUREMENTS*

FLORIAN FRÜHAUF†, BASTIAN GEBAUER‡, AND OTMAR SCHERZER†

Abstract. Assuming that the heat capacity of a body is negligible outside certain inclusions the heat equation degenerates to a parabolic-elliptic interface problem. In this work we aim to detect these interfaces from thermal measurements on the surface of the body. We deduce an equivalent variational formulation for the parabolic-elliptic problem and give a new proof of the unique solvability based on Lions's projection lemma. For the case that the heat conductivity is higher inside the inclusions, we develop an adaptation of the factorization method to this time-dependent problem. In particular this shows that the locations of the interfaces are uniquely determined by boundary measurements. The method also yields to a numerical algorithm to recover the inclusions and thus the interfaces. We demonstrate how measurement data can be simulated numerically by a coupling of a finite element method with a boundary element method, and finally we present some numerical results for the inverse problem.

Key words. parabolic-elliptic equation, inverse problems, factorization method

AMS subject classifications. 65J20, 35K65

DOI. 10.1137/050641545

1. Introduction.

We consider the heat equation in a domain $B \subset \mathbb{R}^n$

$$\partial_t(c(x)u(x,t)) - \nabla \cdot (\kappa(x)\,\nabla u(x,t)) = 0 \quad \text{in } B\times]0,T[, \tag{1.1}$$

with (spatially dependent) heat capacity c and conductivity κ. The special case we are studying here is that the heat capacity $c(x)$ is bounded from below inside an inclusion $\overline{\Omega} \subset B$, and negligibly small on the outside $Q := B \setminus \overline{\Omega}$ (cf. Figure 1.1 for a sketch of the geometry). Throughout this work Ω is allowed to be disconnected; thus the case of multiple inclusions is covered as well.

If we assume for simplicity that $c(x) = \chi_\Omega(x)$ is the characteristic function of Ω, then the evolution equation (1.1) can be rewritten as a parabolic-elliptic equation,

$$\partial_t u(x,t) - \nabla \cdot (\kappa(x)\,\nabla u(x,t)) = 0 \text{ in } \Omega\times]0,T[, \tag{1.2}$$

$$\nabla \cdot (\kappa(x)\,\nabla u(x,t)) = 0 \text{ in } Q\times]0,T[, \tag{1.3}$$

together with appropriate interface conditions on $\partial\Omega$.

For the case $B = \mathbb{R}^2$ and $\kappa = 1$ this problem also arises in the study of two-dimensional eddy currents and was studied by MacCamy and Suri in [23] and by Costabel, Ervin, and Stephan in [9]. In both papers boundary integral operators are used to replace the Laplace equation in the exterior of Ω by a nonlocal boundary condition for the parabolic equation inside Ω. This problem is then solved by

*Received by the editors September 29, 2005; accepted for publication (in revised form) November 21, 2006; published electronically April 20, 2007.
http://www.siam.org/journals/sinum/45-2/64154.html

†Department of Computer Science, University of Innsbruck, Technikerstr. 21a, 6020 Innsbruck, Austria (florian.fruehauf@uibk.ac.at, otmar.scherzer@uibk.ac.at, http://informatik.uibk.ac.at/infmath). The research of these authors was supported by the Austria Science Foundation (FWF) Projects Y-123INF, FSP 9203-N12, and FSP 9207-N12.

‡Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenbergerstr. 69, 4040 Linz, Austria (bastian.gebauer@ricam.oeaw.ac.at, http://www.ricam.oeaw.ac.at).

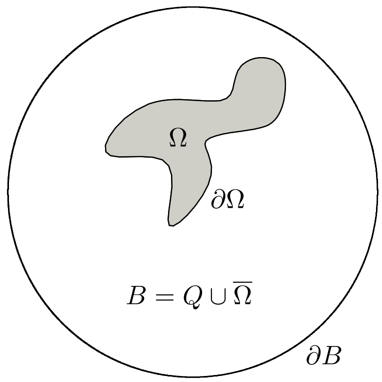

FIG. 1.1. *Sketch of geometry.*

a Galerkin method. In [8] Costabel uses boundary integral operators to solve the resulting interior problem also.

In this work we study the problem for general $\kappa \in L^\infty_+(B)$ in a bounded domain B with given Neumann boundary values on ∂B. By considering (1.1) in the sense of distributions we deduce (1.2), (1.3) together with natural interface conditions (that would otherwise have to be postulated). Moreover, we prove that the weak formulation in appropriate Sobolev spaces is equivalent to (1.1). We show existence of a unique solution using Lions's projection lemma; cf. section 2.

In section 3 we study the inverse problem of locating the interface $\partial\Omega$, resp., the inclusion Ω, from surface measurements on ∂B. If the conductivity is larger inside Ω than in the exterior Q, we show that the points belonging to Ω can be characterized using a variant of the so-called factorization method introduced by Kirsch in [16], generalized by Brühl and Hanke in [6, 5], and since then adapted to various stationary and time-harmonic problems; cf. [1, 2, 7, 15, 17, 18, 19] for more recent contributions. To our knowledge this is the first successful extension of this method to a time-dependent problem.

In section 4 we show how the direct problem can be solved numerically with a coupling of finite element methods and boundary element methods similar to [23]. Using simulated measurements we demonstrate the numerical realization of the factorization method following the ideas of Brühl and Hanke in [6, 5].

2. The direct problem.

2.1. A parabolic-elliptic problem. Let $T > 0$ and $\Omega, B \subset \mathbb{R}^n$, $n \geq 2$, be bounded domains with smooth boundaries, $\overline{\Omega} \subset B$, and connected complement $Q := B \setminus \overline{\Omega}$.

In this section we study the parabolic-elliptic problem

$$\partial_t(\chi_\Omega(x)u(x,t)) - \nabla \cdot (\kappa(x)\,\nabla u(x,t)) = 0 \quad \text{in } B\times]0,T[, \tag{2.1}$$

with $\kappa \in L^\infty_+(B)$, where we denote by L^∞_+ the space of L^∞-functions with positive essential infima, and χ_Ω is the characteristic function of Ω.

A standard way to treat an equation like (2.1) is to multiply both sides with a test function followed by a formal partial integration. Assuming additional (also

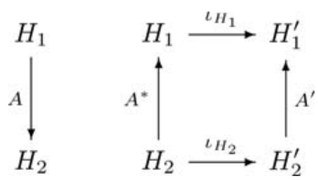

FIG. 2.1. *Relation between dual and adjoint operator.*

formal) boundary and initial conditions, this leads to a variational formulation, which is mathematically meaningful in some Sobolev spaces (and thus no longer formal). Instead of (2.1) one would then study this variational formulation, the so-called weak form of the equation.

In this work we proceed in a slightly different way. We start by noting that the left-hand side of (2.1) does have a mathematical meaning for every $u \in L^2(0, T, H^1(B))$ if the derivatives are interpreted in the sense of (scalar-valued) distributions.

We denote by $\mathcal{D}(B\times]0,T[)$ the space of infinitely often differentiable functions with support in $B\times]0,T[$ and by $\mathcal{D}'(B\times]0,T[)$ its dual space, i.e., the space of distributions on $B\times]0,T[$. By the definition of distributional derivatives, (2.1) is equivalent to

$$(2.2)\quad -\int_0^T \int_\Omega u(x,t)\partial_t\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_B \kappa(x)\,\nabla u(x,t)\cdot\nabla\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0$$

for all $\varphi \in \mathcal{D}(B\times]0,T[)$.

We will show in this section that (2.1) (together with appropriate boundary and initial conditions) has a unique solution in $L^2(0, T, H^1(B))$. In Theorem 2.6 we give an equivalent variational formulation in Sobolev spaces, using the time-derivative in the sense of vector-valued distributions (which we denote by u'). This variational formulation is the same that one would have obtained as the weak generalization of (2.1) using the above-mentioned formal arguments.

We denote by ν the exterior normal on ∂B, resp., the exterior normal on $\partial\Omega$, and by $\mathcal{D}(\overline{Q}\times]0,T[)$ the restrictions of functions from $\mathcal{D}(\mathbb{R}^n\times]0,T[)$ to $Q\times]0,T[$. Analogous notation is used for Ω and B, and $\mathcal{D}(\overline{B}\times[0,T[)$ is the space of restrictions of functions from $\mathcal{D}(\mathbb{R}^n\times]-\infty,T[)$ to $B\times]0,T[$.

We use the anisotropic Sobolev spaces from [22]. For $r, s \geq 0$ we write

$$H^{r,s}(\mathcal{X}) := L^2(0,T,H^r(\mathcal{X})) \cap H^s(0,T,L^2(\mathcal{X})) \text{ for } \mathcal{X} \in \{B, \Omega, Q, \partial B, \partial\Omega\},$$

and for $s < \frac{1}{2}$ and $\mathcal{X} \in \{\partial B, \partial\Omega\}$

$$H^{-r,-s}(\mathcal{X}) := (H^{r,s}(\mathcal{X}))'.$$

The inner product on a real Hilbert space H is denoted by $(\cdot,\cdot)$ and the dual pairing on $H' \times H$ by $\langle\cdot,\cdot\rangle$. They are related by the isometry $\iota_H : H \to H'$ that "identifies H with its dual"; i.e., $\langle \iota_H u, \cdot\rangle := (u, \cdot)$ for all $u \in H$. Throughout this work we rigorously distinguish between the dual operator (denoted by A') and the adjoint operator (denoted by A^*) of an operator $A \in \mathcal{L}(H_1, H_2)$ between real Hilbert spaces H_1, H_2. They satisfy the identity $A^* = \iota_{H_1}^{-1} A' \iota_{H_2}$; cf. Figure 2.1.

We summarize some known properties of the Dirichlet and Neumann traces for solutions of the Laplace, resp., heat equation. On the boundary $\partial\Omega$ we use the superscript $^-$ when the trace is taken from inside the inclusion Ω and the superscript $^+$ when it is taken from the outside.

THEOREM 2.1. (a) *The trace mapping*

$$v \mapsto v|_{\partial B}, \ \textit{resp.}, \ v \mapsto v^+|_{\partial\Omega}, \quad v \in \mathcal{D}(\overline{Q}\times]0,T[),$$

can be extended to a continuous mapping from $H^{1,0}(Q)$ *to* $H^{\frac{1}{2},0}(\partial B)$, *resp., to* $H^{\frac{1}{2},0}(\partial\Omega)$, *that has a continuous right inverse. The same holds for* $H^{1,0}(\Omega) \to H^{\frac{1}{2},0}(\partial\Omega)$, $v \mapsto v^-|_{\partial\Omega}$.

(b) *The Neumann traces* $\kappa\partial_\nu v|_{\partial B}$ *and* $\kappa\partial_\nu v^+|_{\partial\Omega}$ *are defined for every* $v \in H^{1,0}(Q)$ *that solves*

$$\nabla\cdot(\kappa\,\nabla v) = 0 \ \textit{in} \ Q\times]0,T[\tag{2.3}$$

by setting

$$\langle \kappa\partial_\nu v|_{\partial B}, f\rangle := \int_0^T \int_Q \kappa\,\nabla v\cdot\nabla v_f \,\mathrm{d}x\,\mathrm{d}t,$$
$$\langle \kappa\partial_\nu v^+|_{\partial\Omega}, \phi\rangle := -\int_0^T \int_Q \kappa\,\nabla v\cdot\nabla v_\phi \,\mathrm{d}x\,\mathrm{d}t$$

for every function f *on* ∂B *and every function* ϕ *on* $\partial\Omega$ *that have extensions* v_f, $v_\phi \in \mathcal{D}(\overline{Q}\times]0,T[)$ *with* $v_f|_{\partial B} = f$, $v_f|_{\partial\Omega} = 0$, *resp.*, $v_\phi|_{\partial B} = 0$, $v_\phi|_{\partial\Omega} = \phi$.

The Neumann traces can be extended to continuous mappings from the subspace of solutions of (2.3) *(equipped with the* $H^{1,0}(Q)$*-norm) to* $H^{-\frac{1}{2},0}(\partial B)$, *resp.*, $H^{-\frac{1}{2},0}(\partial\Omega)$.

(c) *The Neumann trace* $\kappa\partial_\nu v^-|_{\partial\Omega}$ *is defined for every* $v \in H^{1,0}(\Omega)$ *that solves*

$$\partial_t v - \nabla\cdot(\kappa\,\nabla v) = 0 \ \textit{in} \ \Omega\times]0,T[\tag{2.4}$$

by setting

$$\langle \kappa\partial_\nu v^-|_{\partial\Omega}, \phi\rangle := \int_0^T \int_\Omega \kappa\,\nabla v\cdot\nabla v_\phi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega v\,\partial_t v_\phi \,\mathrm{d}x\,\mathrm{d}t$$

for every function ϕ *on* $\partial\Omega$ *that has an extension* $v_\phi \in \mathcal{D}(\overline{\Omega}\times]0,T[)$ *with* $v_\phi|_{\partial\Omega} = \phi$.

The Neumann trace can be extended to a continuous mapping from the subspace of solutions of (2.4) *(equipped with the* $H^{1,0}(\Omega)$*-norm) to* $H^{-\frac{1}{2},-\frac{1}{4}}(\partial\Omega)$.

Proof. (a), (b) immediately follow from the classical trace theorems on H^1. For (c) we refer the reader to [8]. ◻

Denoting

$$[v]_{\partial\Omega} := v^+|_{\partial\Omega} - v^-|_{\partial\Omega} \text{ and } [\kappa\partial_\nu v]_{\partial\Omega} := \kappa\partial_\nu v^+|_{\partial\Omega} - \kappa\partial_\nu v^-|_{\partial\Omega}$$

we can write (2.1) as a diffraction problem.

LEMMA 2.2. $u \in H^{1,0}(B)$ *solves* (2.1) *if and only if* $u \in H^{1,0}(B\setminus\partial\Omega)$ *solves*

$$\partial_t u - \nabla\cdot(\kappa\,\nabla u) = 0 \ \textit{in} \ \Omega\times]0,T[, \tag{2.5}$$
$$\nabla\cdot(\kappa\,\nabla u) = 0 \ \textit{in} \ Q\times]0,T[, \tag{2.6}$$
$$[\kappa\partial_\nu u]_{\partial\Omega} = 0, \tag{2.7}$$
$$[u]_{\partial\Omega} = 0. \tag{2.8}$$

In particular, (2.6) *and* (2.7) *imply that* $\kappa\partial_\nu u^-|_{\partial\Omega}$ *can be extended by continuity to* $H^{-\frac{1}{2},0}(\partial\Omega)$.

Proof. Like in the stationary case we have $u \in H^{1,0}(B)$ if and only if $u \in H^{1,0}(B \setminus \partial\Omega)$ and u satisfies (2.8). The rest immediately follows from the definition of distributional derivatives and the Neumann traces. □

The next lemma shows uniqueness for the diffraction problem with a Neumann boundary condition and an initial condition on Ω. With respect to the Gelfand triple $H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^1(\Omega)'$ we denote by

$$W := W(0, T, H^1(\Omega), H^1(\Omega)')$$

the space of functions $u \in L^2(0, T, H^1(\Omega))$ with vector-valued distributional time derivative $u' \in L^2(0, T, H^1(\Omega)')$. From [10, Chp. XVIII], it follows that

$$W \subset C^0([0, T], L^2(\Omega)).$$

LEMMA 2.3. *Let* $u \in H^{1,0}(B \setminus \partial\Omega)$ *solve* (2.5), (2.6), *and*

$$[\kappa\partial_\nu u]_{\partial\Omega} = \psi \in H^{-\frac{1}{2},0}(\partial\Omega), \tag{2.9}$$

$$[u]_{\partial\Omega} = f \in H^{\frac{1}{2},0}(\partial\Omega), \tag{2.10}$$

$$\kappa\partial_\nu u|_{\partial B} = g \in H^{-\frac{1}{2},0}(\partial B). \tag{2.11}$$

Then $u|_\Omega \in W$ *and* u *is uniquely determined by* ψ, f, g, *and the initial condition*

$$u(x, 0) = 0 \text{ on } \Omega. \tag{2.12}$$

Proof. Again (2.9) implies that the Neumann trace $\kappa\partial_\nu u^-|_{\partial\Omega}$ can be extended by continuity to $H^{-\frac{1}{2},0}(\partial\Omega)$.

Thus we can define $w \in L^2(0, T, H^1(\Omega)')$ by setting for every $t \in]0, T[$ and $v \in H^1(\Omega)$

$$\langle w(t), v\rangle := \left\langle \kappa\partial_\nu u^-(t)|_{\partial\Omega}, v^-|_{\partial\Omega}\right\rangle - \int_\Omega \kappa\nabla u(t) \cdot \nabla v \,\mathrm{d}x.$$

We have

$$\begin{aligned}
&\int_\Omega \left(-\int_0^T u\,\partial_t\varphi \,\mathrm{d}t \right) v \,\mathrm{d}x \\
&= \int_0^T \langle \kappa\partial_\nu u^-|_{\partial\Omega}, v^-|_{\partial\Omega}\rangle \varphi \,\mathrm{d}t - \int_0^T \int_\Omega \kappa\nabla u \cdot \nabla v \,\mathrm{d}x \,\varphi \,\mathrm{d}t \\
&= \left\langle \int_0^T w\varphi \,\mathrm{d}t, v \right\rangle
\end{aligned}$$

for all $v(x)\varphi(t) \in \mathcal{D}(\overline{\Omega} \times]0, T[)$ and thus by continuous extension for all $v\varphi \in H^1(\Omega) \otimes \mathcal{D}(]0, T[)$. Thus in the sense of vector-valued distributions

$$w = (u|_\Omega)' \text{ with respect to } H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^1(\Omega)',$$

and hence $u|_\Omega \in W \subset C^0([0, T], L^2(\Omega))$.

To show uniqueness let $f = 0$, $\psi = 0$, $g = 0$, and (2.12) hold. Since Green's formula holds for functions in W we have

$$\begin{aligned}\frac{1}{2}\int_\Omega |u(T)|^2 \,\mathrm{d}x &= \int_0^T \langle u'(t), u(t)\rangle \,\mathrm{d}t \\ &= \int_0^T \langle \kappa\partial_\nu u^-|_{\partial\Omega}, u^-|_{\partial\Omega}\rangle \,\mathrm{d}t - \int_0^T\int_\Omega \kappa|\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T\int_B \kappa|\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

This implies that $u(x,t) = c(t)$, where $c \in C^0([0,T],\mathbb{R})$ solves $c' = 0$ and $c(0) = 0$. Thus $u = 0$. □

To show existence of a solution we proceed analogously to [8, Lemma 2.3] by using Lions's projection lemma.

LEMMA 2.4 (Lions's projection lemma). *Assume that H is a Hilbert space and Φ is a subspace of H. Moreover let $a :\ H \times \Phi \to \mathbb{R}$ be a bilinear form satisfying the following properties:*

(a) *For every $\varphi \in \Phi$, the linear form $u \mapsto a(u,\varphi)$ is continuous on H.*

(b) *There exists $\alpha > 0$ such that $a(\varphi,\varphi) \geq \alpha \|\varphi\|_H^2$ for all $\varphi \in \Phi$.*

Then for each continuous linear form $l \in H'$, there exists $u_0 \in H$ such that

$$a(u_0,\varphi) = \langle l,\varphi\rangle \text{ for all } \varphi\in\Phi \quad \text{and} \quad \|u_0\|_H \leq \frac{1}{\alpha}\|l\|_{H'}.$$

Proof. The lemma is proven in [20]. We repeat the proof for the sake of completeness.

From assumption (a) and the Riesz representation theorem it follows that for every $\varphi \in \Phi$ there exists $K\varphi \in H$ with

$$(u, K\varphi) = a(u,\varphi) \text{ for all } u \in H.$$

This defines a linear (possibly unbounded) operator $K :\ \Phi \to V := K(\Phi) \subseteq H$. From assumption (b) it follows that K is injective and thus possesses an inverse $R_0 :\ V \to \Phi$. Again using assumption (b) we have

$$\|R_0 v\|^2 \leq \frac{1}{\alpha}a(R_0 v, R_0 v) = \frac{1}{\alpha}(R_0 v, v) \leq \frac{1}{\alpha}\|R_0 v\|\,\|v\|,$$

which yields $\|R_0 v\| \leq \frac{1}{\alpha}\|v\|$. Thus R_0 can be extended by continuity to the closure $\overline{V}$ of V. If we denote this extension by $\overline{R}_0$ then we have $\overline{R}_0 :\ \overline{V} \to \overline{\Phi}$.

$\overline{\Phi}$ is a closed subspace of the Hilbert space H and thus also a Hilbert space. Using the Riesz representation theorem on $\overline{\Phi}$ we obtain a $\xi_l \in \overline{\Phi}$ with

$$l(\varphi) = (\xi_l, \varphi) \text{ for all } \varphi \in \overline{\Phi}.$$

Finally, let $P :\ H \to \overline{V}$ be the orthogonal projection onto $\overline{V}$; then $u_0 := P^*\overline{R}_0{}^*\xi_l$ has the desired properties. □

We prove existence of a solution of the parabolic-elliptic diffraction problem (2.5), (2.6), (2.9)–(2.12) under the additional assumption that g and ψ have vanishing integral mean. For $\mathcal{X} \in \{\partial B, \partial\Omega\}$ we define

$$H_\diamond^{-\frac{1}{2}}(\mathcal{X}) := \{g \in H^{-\frac{1}{2}}(\mathcal{X}) \ :\ \langle g, 1_{\mathcal{X}}\rangle = 0\} \quad \text{and} \quad H_\diamond^{-\frac{1}{2},0}(\mathcal{X}) := L^2(0,T,H_\diamond^{-\frac{1}{2}}(\mathcal{X})).$$

Again they are Hilbert spaces because they are closed subspaces of $H^{-\frac{1}{2}}(\mathcal{X})$, resp., $H^{-\frac{1}{2},0}(\mathcal{X})$.

LEMMA 2.5. *For every*

$$g \in H_\diamond^{-\frac{1}{2},0}(\partial B),\ f \in H^{\frac{1}{2},0}(\partial\Omega),\ \text{and } \psi \in H_\diamond^{-\frac{1}{2},0}(\partial\Omega),$$

there exists $u \in H^{1,0}(B \setminus \partial\Omega)$ *that solves* (2.5), (2.6), *and* (2.9)–(2.12).

u depends continuously on g, f, and ψ, and it fulfills

$$\int_\Omega u(x,t)\,\mathrm{d}x = 0 \qquad \text{for } t \in [0,T] \text{ a.e.}$$

Proof. Let $\gamma_{\partial\Omega}^- : H^{\frac{1}{2}}(\partial\Omega) \to H^1(Q)$ be a lifting operator, i.e., a continuous right inverse of the trace operator $\cdot|_{\partial\Omega}$ with $(\gamma_{\partial\Omega}^- h)|_{\partial B} = 0$ for all $h \in H^{\frac{1}{2}}(\partial\Omega)$, and set $u_f = \gamma_{\partial\Omega}^- f \in H^{1,0}(Q)$.

We define the spaces

$$H_\square^1(B) := \left\{ v \in H^1(B) \ :\ \int_\Omega v\,\mathrm{d}x = 0 \right\}, \qquad H := L^2(0,T,H_\square^1(B)),$$
$$\Phi := \left\{ \varphi \in \mathcal{D}([0,T[\times \overline{B}) \ :\ \int_\Omega \varphi\,\mathrm{d}x = 0 \right\},$$

and we set for all $v \in H$ and $\varphi \in \Phi$

$$a(v,\varphi) := \int_0^T \int_B \kappa \nabla v \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega v\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t,$$
$$\langle l, v\rangle := -\int_0^T \int_Q \kappa\nabla u_f \cdot \nabla v\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle g, v|_{\partial B}\rangle\,\mathrm{d}t - \int_0^T \langle \psi, v|_{\partial\Omega}\rangle\,\mathrm{d}t.$$

Since H is a closed subspace of $H^{1,0}(B)$, it is a Hilbert space. $\Phi \subset H$ and for every $\varphi \in \Phi$, the linear form $v \to a(v,\varphi)$ is continuous on H.

Poincaré's inequality yields that $\left(\int_B |\nabla v|^2\,\mathrm{d}x\right)^{1/2}$ is an equivalent norm on $H_\square^1(B)$; thus there exists $\alpha > 0$ such that for all $\varphi \in \Phi$

$$a(\varphi,\varphi) = \int_0^T \int_B \kappa|\nabla\varphi(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega |\varphi(0,x)|^2\,\mathrm{d}x$$
$$\geq \int_0^T \int_B \kappa|\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}t \geq \alpha \left\|\varphi\right\|_H^2.$$

Moreover, the continuity of the trace and lifting operators yields the existence of a constant C that does not depend on g, f, and ψ such that for all $v \in H$

$$\langle l, v\rangle \leq C\left(\|g\|_{H^{-\frac{1}{2},0}(\partial B)} + \|f\|_{H^{\frac{1}{2},0}(\partial\Omega)} + \|\psi\|_{H^{-\frac{1}{2},0}(\partial\Omega)} \right) \|v\|_{H^{1,0}(B)}$$
$$= C\left(\|g\|_{H_\diamond^{-\frac{1}{2},0}(\partial B)} + \|f\|_{H^{\frac{1}{2},0}(\partial\Omega)} + \|\psi\|_{H_\diamond^{-\frac{1}{2},0}(\partial\Omega)} \right) \|v\|_H,$$

and thus $l \in H'$ with $\|l\|_{H'} \leq C\big(\|g\|_{H_\diamond^{-\frac{1}{2},0}(\partial B)} + \|f\|_{H^{\frac{1}{2},0}(\partial\Omega)} + \|\psi\|_{H_\diamond^{-\frac{1}{2},0}(\partial\Omega)} \big)$.

Now Lemma 2.4 gives existence of $\tilde{u} \in H$ that solves

$$
\begin{aligned}
\int_0^T \int_B \kappa \nabla \tilde{u} \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega \tilde{u} \, \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t \\
= - \int_0^T \int_Q \kappa \nabla u_f \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \langle g, \varphi|_{\partial B} \rangle \, \mathrm{d}t - \int_0^T \langle \psi, \varphi|_{\partial\Omega} \rangle \, \mathrm{d}t
\end{aligned}
\tag{2.13}
$$

for all $\varphi \in \Phi$ and $\tilde{u}$ depends continuously on l (and therefore on g, f, and ψ).

We define $u \in H^{1,0}(B \setminus \partial\Omega)$ by setting $u|_\Omega := \tilde{u}|_\Omega$ and $u|_Q := \tilde{u}|_Q + u_f$. Then u solves (2.10) and there exist constants $C', C'' > 0$ such that

$$
\begin{aligned}
\|u\|_{H^{1,0}(B\setminus\partial\Omega)} &\leq C' \left(\|\tilde{u}|_\Omega\|_{H^{1,0}(\Omega)} + \|\tilde{u}|_Q\|_{H^{1,0}(Q)} + \|u_f\|_{H^{1,0}(Q)} \right) \\
&\leq C'' \left(\|\tilde{u}\|_H + \|u_f\|_{H^{1,0}(Q)} \right),
\end{aligned}
$$

and thus u depends continuously on g, f, and ψ.

Since $\int_\Omega \tilde{u}(x,t) \, \mathrm{d}x = 0$ for $t \in [0,T]$ a.e., the left side of (2.13) vanishes for all $\varphi(x,t) = c(t) \in \mathcal{D}([0,T[\times\overline{B})$. Due to our additional assumptions on g and ψ, the right side of (2.13) also vanishes for those φ. Thus (2.13) holds for all $\varphi \in \Phi$ and for all $\varphi(x,t) = c(t)$, which shows that (2.13) holds for all $\varphi \in \mathcal{D}([0,T[\times\overline{B})$, and we immediately obtain that u solves (2.5), (2.6), (2.9), and (2.11).

From Lemma 2.3 it follows that $\tilde{u}|_\Omega = u|_\Omega \in W$ and thus Green's formula holds. We obtain that for every $\varphi \in \mathcal{D}([0,T[\times B)$ with support in $[0,T[\times\Omega$

$$
\begin{aligned}
&- \int_\Omega u(0)\varphi(0) \, \mathrm{d}x \\
&= \int_0^T \int_\Omega u \, \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \langle u', \varphi \rangle \, \mathrm{d}t \\
&= \int_0^T \int_\Omega \tilde{u} \, \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \langle \kappa \partial_\nu u^-|_{\partial\Omega}, \varphi|_{\partial\Omega} \rangle \, \mathrm{d}t - \int_0^T \int_\Omega \kappa \nabla u \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_B \kappa \nabla \tilde{u} \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega \kappa \nabla u \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t \\
&= 0,
\end{aligned}
$$

where we used that the right side of (2.13) vanishes for $\operatorname{supp} \varphi \in [0,T[\times\Omega$. As $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$ this yields that $u|_\Omega(0) = 0$. □

We summarize the results of this section and give a useful variational formulation in Sobolev spaces.

THEOREM 2.6. *Let $g \in H_\diamond^{-\frac{1}{2},0}(\partial B)$, $f \in H^{\frac{1}{2},0}(\partial\Omega)$, and $\psi \in H_\diamond^{-\frac{1}{2},0}(\partial\Omega)$, and let $u_f \in H^{1,0}(B \setminus \partial\Omega)$ be such that $u_f|_{\partial B} = 0$, $u_f|_{\partial\Omega} = f$, and $u_f|_\Omega = 0$.*

For $u \in H^{1,0}(B \setminus \partial\Omega)$ the following three problems are equivalent and possess the same unique solution. The solution depends continuously on g, f, and ψ and it fulfills $\int_\Omega u(x,t) \, \mathrm{d}x = 0$ for $t \in [0,T]$ a.e.

(a) *u solves*

$$\partial_t u - \nabla \cdot (\kappa \nabla u) = 0 \text{ in } \Omega\times]0,T[, \tag{2.14}$$

$$\nabla \cdot (\kappa \nabla u) = 0 \text{ in } Q\times]0,T[, \tag{2.15}$$

$$[\kappa \partial_\nu u]_{\partial\Omega} = \psi, \tag{2.16}$$

$$[u]_{\partial\Omega} = f, \tag{2.17}$$

$$\kappa \partial_\nu u|_{\partial B} = g, \tag{2.18}$$

$$u(x,0) = 0 \ \text{in } \Omega. \tag{2.19}$$

(b) $u|_\Omega \in W$, $u(x,0) = 0$ *in* Ω, *and* $\tilde{u} := u - u_f$ *solves*

$$\begin{aligned} &\int_0^T \langle (\tilde{u}|_\Omega)', v|_\Omega \rangle \,\mathrm{d}t + \int_0^T \int_B \kappa \nabla \tilde{u} \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_0^T \langle g, v|_{\partial B} \rangle \,\mathrm{d}t - \int_0^T \langle \psi, v|_{\partial\Omega} \rangle \,\mathrm{d}t - \int_0^T \int_Q \kappa \nabla u_f \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{2.20}$$

for all $v \in H^{1,0}(B)$.

(c) $\tilde{u} := u - u_f$ *solves*

$$\begin{aligned} &\int_0^T \int_B \kappa \nabla \tilde{u} \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle (v|_\Omega)', \tilde{u}|_\Omega \rangle \,\mathrm{d}t \\ &= \int_0^T \langle g, v|_{\partial B} \rangle \,\mathrm{d}t - \int_0^T \langle \psi, v|_{\partial\Omega} \rangle \,\mathrm{d}t - \int_0^T \int_Q \kappa \nabla u_f \nabla v \,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

for all $v \in H^{1,0}(B)$ *with* $v|_\Omega \in W$ *and* $v(x,T) = 0$ *on* Ω.

Proof. We showed the unique solvability of the equations in (a) and the properties of the solution in Lemmas 2.3 and 2.5. Thus it remains only to prove the equivalence of (a), (b), and (c).

(a) $\Rightarrow$ (b). Note that $\tilde{u} \in H^{1,0}(B)$, $\kappa\partial_\nu u^-|_{\partial\Omega} \in H^{-\frac{1}{2},0}(\partial\Omega)$, and, by Lemma 2.3, $\tilde{u}|_\Omega = u|_\Omega \in W$.

It suffices to show (2.20) for $v \in \mathcal{D}(]0,T[\times \overline{B})$. Equations (2.14) and (2.15) imply that

$$\begin{aligned} 0 &= \int_0^T \langle (\tilde{u}|_\Omega)', v|_\Omega \rangle \,\mathrm{d}t - \int_0^T \langle \nabla \cdot (\kappa \nabla u|_\Omega), v|_\Omega \rangle \,\mathrm{d}t \\ &= \int_0^T \langle (\tilde{u}|_\Omega)', v|_\Omega \rangle \,\mathrm{d}t - \int_0^T \langle \kappa \,\partial_\nu u^-|_{\partial\Omega}, v|_{\partial\Omega} \rangle \,\mathrm{d}t + \int_0^T \int_\Omega \kappa \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

and

$$\begin{aligned} 0 &= \int_0^T \langle \nabla \cdot (\kappa \nabla u|_Q), v|_Q \rangle \,\mathrm{d}t \\ &= -\int_0^T \langle \kappa \,\partial_\nu u^+|_{\partial\Omega}, v|_{\partial\Omega} \rangle \,\mathrm{d}t + \int_0^T \langle \kappa \,\partial_\nu u|_{\partial B}, v|_{\partial B} \rangle \,\mathrm{d}t \\ &\quad - \int_0^T \int_Q \kappa \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Subtracting these two equations and using (2.16) and (2.18) give

$$\begin{aligned} 0 &= \int_0^T \langle (\tilde{u}|_\Omega)', v|_\Omega \rangle \,\mathrm{d}t + \int_0^T \langle \psi, v|_{\partial\Omega} \rangle \,\mathrm{d}t - \int_0^T \langle g, v|_{\partial B} \rangle \,\mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \kappa \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_Q \kappa \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Now (2.20) follows from

$$\int_0^T \int_\Omega \kappa \nabla u \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_Q \kappa \nabla u \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^T \int_B \kappa \nabla \tilde{u} \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_Q \kappa \nabla u_f \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}t.$$

(b) $\Rightarrow$ (c). This part of the proof follows from Green's formula on W.

(c) $\Rightarrow$ (a). This part of the proof was shown in the proof of Lemma 2.5. ∎

2.2. Boundary measurements and a reference problem. We assume that the inclusion not only has a higher heat capacity but also has a higher conductivity κ than the background. For simplicity we fix $\kappa = 1$ on Q and therefore require that $\kappa|_\Omega - 1 \in L^\infty_+(\Omega)$.

We introduce the measurement operator

$$\Lambda_1 : \; g \mapsto u_1|_{\partial B}, \text{ where } u_1 \text{ solves (2.1) with } \partial_\nu u_1|_{\partial B} = g, \; u_1|_\Omega = 0 \text{ at } t = 0.$$

Using the results from section 2.1 we know that Λ_1 is a continuous linear operator from $H_\diamond^{-\frac{1}{2},0}(\partial B)$ to $H^{\frac{1}{2},0}(\partial B)$.

To locate the inclusion Ω we compare Λ_1 with boundary measurements of a domain without inclusions, i.e., with the measurement operator

$$\Lambda_0 : \; g \mapsto u_0|_{\partial B}, \text{ where } \Delta u_0 = 0 \text{ on } B \times]0, T[\text{ and } \partial_\nu u_0|_{\partial B} = g.$$

The Lax–Milgram theorem shows that u_0 is uniquely determined up to addition of a spatially constant function $u(x,t) = c(t) \in L^2(0, T, \mathbb{R})$ and that Λ_0 is a continuous linear operator from $H_\diamond^{-\frac{1}{2},0}(\partial B)$ to $H_\diamond^{\frac{1}{2},0}(\partial B) := L^2(0, T, H_\diamond^{\frac{1}{2}}(\partial B))$, where the quotient space $H_\diamond^{\frac{1}{2}}(\partial B) := H^{\frac{1}{2}}(\partial B)/\mathbb{R}$ can be identified with the dual space of $H_\diamond^{-\frac{1}{2}}(\partial B)$ and $H_\diamond^{\frac{1}{2},0}(\partial B)$ with the dual space of $H_\diamond^{-\frac{1}{2},0}(\partial B)$.

Analogously we define quotient spaces on B, Q, and $\partial\Omega$ and note that in the case that $\partial\Omega$ is disconnected the quotient space $H_\diamond^{\frac{1}{2}}(\partial\Omega)$ is still obtained by factoring out the one-dimensional space of functions that are constant on $\partial\Omega$, and not the multidimensional space of functions that are constant on each connected component.

Mathematically the elements of the quotient spaces $H_\diamond^{r,0}$, $r \geq 0$, are equivalence classes; i.e., all functions that differ only by a spatially constant function are called equivalent and combined into one class. For the sake of readability we write an equivalence class as a function and keep in mind that it is a representant of its class. We also note that the space $H_\diamond^{-\frac{1}{2},0}$, which we defined earlier, is not a quotient space.

Without changing notation we use the canonical epimorphism to restrict Λ_1 to the spaces of the reference problem. Thus we will investigate the inverse problem of locating the inclusion Ω from knowledge of

$$\Lambda_0, \, \Lambda_1 : \; H_\diamond^{-\frac{1}{2},0}(\partial B) \to H_\diamond^{\frac{1}{2},0}(\partial B).$$

3. The inverse problem. We use the factorization method to reconstruct Ω from the boundary measurements. To this end we show that the difference of the measurement operators $\Lambda_0 - \Lambda_1$ can be factorized into the product

$$\Lambda_0 - \Lambda_1 = L(F_0 - F_1)L' \tag{3.1}$$

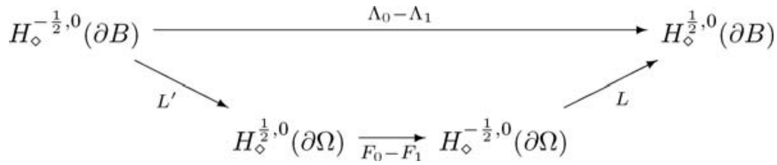

FIG. 3.1. *Factorization of* $\Lambda_0 - \Lambda_1$.

(cf. Figure 3.1), where the operator L corresponds to virtual measurements on the complement Q of the inclusion, and its range contains all information about Q and thus about the location of Ω.

Unlike previously known applications of the factorization method, the explicit time-dependence of the problem prevents us from calculating $\mathcal{R}(L)$ from the boundary measurements, but using a new approach we can show that the knowledge of $\Lambda_0 - \Lambda_1$ still suffices to determine Ω.

3.1. Factorization of the boundary measurements. We define a virtual measurement operator that corresponds to inducing a heat flux on the inclusion's boundary

$$L:\ H_\diamond^{-\frac{1}{2},0}(\partial\Omega) \to H_\diamond^{\frac{1}{2},0}(\partial B), \quad L\psi := v|_{\partial B},$$

where $v \in H_\diamond^{1,0}(Q)$ solves

$$\Delta v = 0 \text{ in } Q\times]0,T[, \quad \partial_\nu v = \begin{cases} -\psi & \text{on } \partial\Omega, \\ 0 & \text{on } \partial B. \end{cases} \tag{3.2}$$

We also need the two auxiliary operators

$$\begin{aligned} F_0:\ H^{\frac{1}{2},0}(\partial\Omega) \to H^{-\frac{1}{2},0}(\partial\Omega), \quad F_0\phi := \partial_\nu v_0^+|_{\partial\Omega}, \\ F_1:\ H^{\frac{1}{2},0}(\partial\Omega) \to H^{-\frac{1}{2},0}(\partial\Omega), \quad F_1\phi := \partial_\nu v_1^+|_{\partial\Omega}, \end{aligned}$$

where $v_0, v_1 \in H^{1,0}(B \setminus \partial\Omega)$ solve

$$\begin{array}{rclrcl} \Delta v_0 &=& 0 \text{ in } (Q\cup\Omega)\times]0,T[, & [\partial_\nu v_0]_{\partial\Omega} &=& 0, \\ \partial_\nu v_0|_{\partial B} &=& 0, & [v_0]_{\partial\Omega} &=& \phi, \end{array} \tag{3.3}$$

and

$$\begin{array}{rclrcl} \Delta v_1 &=& 0 \text{ in } Q\times]0,T[, & [\kappa\partial_\nu v_1]_{\partial\Omega} &=& 0, \\ \partial_t v_1 - \nabla\cdot(\kappa\,\nabla v_1) &=& 0 \text{ in } \Omega\times]0,T[, & [v_1]_{\partial\Omega} &=& \phi, \\ v_1(x,0) &=& 0 \text{ in } \Omega, & \partial_\nu v_1|_{\partial B} &=& 0. \end{array} \tag{3.4}$$

Note that F_0 is well defined even though (3.3) determines v_0 only up to addition of a spatially constant function. Since the ranges of F_0 and F_1 are contained in $H_\diamond^{-\frac{1}{2},0}(\partial\Omega)$ and their kernels contain $L^2(0,T,\mathbb{R})$, we will consider them as operators from

$$H_\diamond^{\frac{1}{2},0}(\partial\Omega) \text{ into } H_\diamond^{-\frac{1}{2},0}(\partial\Omega).$$

THEOREM 3.1. *The difference of the boundary measurements can be factorized into*

$$\Lambda_0 - \Lambda_1 = L(F_0 - F_1)L'.$$

The operators L and L' are injective.

Proof. For given $g \in H_\diamond^{-\frac{1}{2},0}(\partial B)$ let $w \in H_\diamond^{1,0}(Q)$ solve

$$\Delta w = 0 \text{ in } Q\times]0,T[, \text{ with } \partial_\nu w = \begin{cases} 0 & \text{on } \partial\Omega, \\ g & \text{on } \partial B. \end{cases}$$

Let $\psi \in H_\diamond^{-\frac{1}{2},0}(\partial\Omega)$ and $v \in H_\diamond^{1,0}(Q)$ be the solution of (3.2) in the definition of $L\psi$. Then

$$\begin{aligned} \langle \psi, L'g\rangle = \langle g, L\psi\rangle = \langle \partial_\nu w|_{\partial B}, v|_{\partial B}\rangle &= \int_0^T \int_Q \nabla w \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}t \\ = \left\langle -\partial_\nu v^+|_{\partial\Omega}, w^+|_{\partial\Omega}\right\rangle &= \left\langle \psi, w^+|_{\partial\Omega}\right\rangle, \end{aligned}$$

and thus $L'g = w^+|_{\partial\Omega}$.

Now let $v_0, v_1 \in H^{1,0}(B \setminus \partial\Omega)$ be the solutions of (3.3), resp., (3.4), from the definition of $F_0 w^+|_{\partial\Omega}$, resp., $F_1 w^+|_{\partial\Omega}$. We define $u_0, u_1 \in H^{1,0}(B \setminus \partial\Omega)$ by setting $u_i|_\Omega := -v_i|_\Omega$ and $u_i|_Q := w - v_i|_Q$, $i = 0, 1$. Then $u_0, u_1 \in H^{1,0}(B)$ and solve the equations in the definitions of $\Lambda_0 g$ and $\Lambda_1 g$. Thus

$$(\Lambda_0 - \Lambda_1)g = (u_0 - u_1)|_{\partial B} = -(v_0 - v_1)|_{\partial B}.$$

Since $\Delta(v_1 - v_0) = 0$ in $Q\times]0,T[$ and $\partial_\nu(v_1 - v_0)|_{\partial B} = 0$ we also have

$$L(\partial_\nu(v_0^+ - v_1^+)|_{\partial\Omega}) = -(v_0 - v_1)|_{\partial B},$$

and thus

$$(\Lambda_0 - \Lambda_1)g = L(\partial_\nu(v_0^+ - v_1^+)|_{\partial\Omega}) = L(F_0 - F_1)w^+|_{\partial\Omega} = L(F_0 - F_1)L'g.$$

To show injectivity of L' let $L'g = 0$ with some $g \in H_\diamond^{-\frac{1}{2},0}(\partial B)$. Then we obtain from the above characterization of L' a solution $w \in H^{1,0}(Q)$ of

$$\Delta w = 0 \quad \text{in } Q\times]0,T[, \qquad w^+|_{\partial\Omega} = 0, \quad \text{and} \quad \partial_\nu w = \begin{cases} 0 & \text{on } \partial\Omega, \\ g & \text{on } \partial B. \end{cases}$$

We set w to zero on $\Omega\times]0,T[$ and denote this continuation by $\tilde{w} \in H^{1,0}(B \setminus \partial\Omega)$. Then we have

$$\Delta\tilde{w} = 0 \quad \text{in } (B \setminus \partial\Omega)\times]0,T[, \qquad [\tilde{w}]_{\partial\Omega} = 0, \qquad [\kappa\partial_\nu\tilde{w}]_{\partial\Omega} = 0,$$

and thus $\tilde{w} \in H^{1,0}(B)$ and $\Delta\tilde{w} = 0$ in $B\times]0,T[$. Hence $\tilde{w}(\cdot, t)$ is analytic for $t \in]0,T[$ a.e. Since $\tilde{w}$ disappears on Ω and B is connected, we obtain that $w = \tilde{w} = 0$ in Q so that $g = 0$. Thus L' is injective.

The injectivity of L follows from the same arguments, when the function from the definition of L is set to zero in $(\mathbb{R}^n \setminus \overline{B})\times]0,T[$. Since Q is connected, $\mathbb{R}^n \setminus \overline{\Omega}$ is also connected. □

The injectivity of L and L' yields that they have dense ranges. The operator $F_0 - F_1$ satisfies a coerciveness condition; to show this we introduce the operators λ_1 and λ that correspond to measurements on the inclusion, resp., on its complement.

$$\begin{aligned} \lambda_1 : \ & H_\diamond^{-\frac{1}{2},0}(\partial\Omega) \to H^{\frac{1}{2},0}(\partial\Omega), \quad \lambda_1\psi := u_1^-|_{\partial\Omega}, \\ \lambda : \ & H_\diamond^{-\frac{1}{2},0}(\partial\Omega) \to H_\diamond^{\frac{1}{2},0}(\partial\Omega), \quad \lambda\psi := u^+|_{\partial\Omega}, \end{aligned}$$

where $u_1 \in W$ solves

$$\partial_t u_1 - \nabla\cdot(\kappa\,\nabla u_1) = 0 \text{ in } \Omega\times]0,T[, \quad \kappa\partial_\nu u_1^-|_{\partial\Omega} = \psi, \quad u_1(x,0) = 0 \text{ on } \Omega, \tag{3.5}$$

and $u \in H^{1,0}_\diamond(Q)$ solves

$$\Delta u = 0 \text{ in } Q\times]0,T[, \quad \partial_\nu u = \begin{cases} -\psi & \text{on } \partial\Omega, \\ 0 & \text{on } \partial B. \end{cases}$$

The unique solvability of (3.5) is shown in [8, Cor. 3.17] for general $\psi \in H^{-\frac{1}{2},-\frac{1}{4}}(\partial\Omega)$. In our case it can also be proven analogously to Lemmas 2.3 and 2.5.

Again we use the canonical epimorphism to restrict λ_1 to the same spaces as λ; i.e., from now on we consider it as an operator

$$\lambda_1 : \; H_\diamond^{-\frac{1}{2},0}(\partial\Omega) \to H_\diamond^{\frac{1}{2},0}(\partial\Omega).$$

LEMMA 3.2. (a) *For every* $\psi \in H_\diamond^{-\frac{1}{2},0}(\partial\Omega)$ *we have the identity*

$$\langle \psi, \lambda_1\psi\rangle = \int_0^T \langle (u_1|_\Omega)', u_1|_\Omega\rangle \,\mathrm{d}t + \int_0^T \int_\Omega \kappa|\nabla u_1|^2 \,\mathrm{d}x\,\mathrm{d}t,$$

where $u_1 \in W$ *is the solution of* (3.5) *in the definition of* λ_1.

(b) λ_1 *is coercive with respect to* $H^{-\frac{1}{2},-\frac{1}{4}}(\partial\Omega)$*; i.e., there exists* $c > 0$ *such that*

$$\langle \psi, \lambda_1\psi\rangle \geq c\,\|\psi\|^2_{H^{-\frac{1}{2},-\frac{1}{4}}(\partial\Omega)} \qquad \textit{for all } \psi \in H_\diamond^{-\frac{1}{2},0}(\partial\Omega). \tag{3.6}$$

Proof. By setting it to zero on Q, every solution $u_1 \in W$ of (3.5) can be extended to a solution of (2.14), (2.15), and (2.19) in Theorem 2.6(a), with

$$[\kappa\partial_\nu u_1]_{\partial\Omega} = -\psi, \qquad [u_1]_{\partial\Omega} = -\lambda_1\psi, \quad \text{ and } \quad \kappa\partial_\nu u_1|_{\partial B} = 0.$$

It follows that

$$\int_\Omega u_1(x,t)\,\mathrm{d}x = 0 \quad \text{ for } t \in [0,T] \text{ a.e.,} \tag{3.7}$$

and with $u_f \in H^{1,0}(B \setminus \partial\Omega)$ such that $u_f|_{\partial B} = 0$, $u_f|_{\partial\Omega} = -\lambda_1\psi$, and $u_f|_\Omega = 0$ we obtain from the variational formulation for $\tilde{u} := u_1 - u_f$ in Theorem 2.6(b)

$$\int_0^T \langle (\tilde{u}|_\Omega)', \tilde{u}|_\Omega\rangle \,\mathrm{d}t + \int_0^T \int_B \kappa|\nabla\tilde{u}|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle \psi, \tilde{u}|_{\partial\Omega}\rangle \,\mathrm{d}t - \int_0^T \int_Q \kappa\nabla u_f\cdot\nabla\tilde{u}\,\mathrm{d}x\,\mathrm{d}t.$$

Using $\tilde{u}|_{\partial\Omega} = \lambda_1\psi$, $u_f|_\Omega = 0$, and $u_1|_Q = 0$ we conclude that

$$\langle \psi, \lambda_1\psi\rangle = \int_0^T \langle (u_1|_\Omega)', u_1|_\Omega\rangle \,\mathrm{d}t + \int_0^T \int_\Omega \kappa|\nabla u_1|^2 \,\mathrm{d}x\,\mathrm{d}t,$$

and thus (a) holds.

Because of (3.7) Poincaré's inequality yields the existence of a $c' > 0$ such that

$$\langle \psi, \lambda_1\psi\rangle \geq c'\,\|u_1\|^2_{H^{1,0}(\Omega)}\,,$$

and so assertion (b) follows from the continuity of the Neumann trace in Theorem 2.1(c). □

LEMMA 3.3. *There exists $c' > 0$ such that*

$$\langle (F_0 - F_1)\phi, \phi\rangle \geq c' \, \|F_1\phi\|^2_{H^{-\frac{1}{2},-\frac{1}{4}}(\partial\Omega)},$$

and F_1 is bijective with $F_1^{-1} = -\lambda - \lambda_1$.

Proof. For given $\phi \in H^{\frac{1}{2},0}(\partial\Omega)$ let $v_0, v_1 \in H^{1,0}(B \setminus \partial\Omega)$ be the solutions of (3.3) and (3.4) in the definition of F_0 and F_1, and let $v_\phi \in H^{1,0}(B \setminus \partial\Omega)$ be such that $v_\phi^+|_{\partial\Omega} = \phi$, $v_\phi|_{\partial B} = 0$, and $v_\phi|_\Omega = 0$.

Then $\tilde{v}_i := v_i - v_\phi$, $i = 0, 1$, solve

$$\int_0^T \int_B \nabla \tilde{v}_0 \cdot \nabla w \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_Q \nabla v_\phi \cdot \nabla w \,\mathrm{d}x\,\mathrm{d}t,$$

$$\int_0^T \langle (\tilde{v}_1|_\Omega)', w|_\Omega\rangle \,\mathrm{d}t + \int_0^T \int_B \kappa \nabla \tilde{v}_1 \cdot \nabla w \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_Q \nabla v_\phi \cdot \nabla w \,\mathrm{d}x\,\mathrm{d}t$$

for all $w \in H^{1,0}(B)$ (cf. Theorem 2.6 for the second equation). From the Lax–Milgram theorem it follows that for $t \in]0, T[$ a.e. $\tilde{v}_0(\cdot, t)$ minimizes the functional

$$w \mapsto \frac{1}{2} \int_B |\nabla w(x)|^2 \,\mathrm{d}x + \int_Q \nabla v_\phi(x,t) \cdot \nabla w(x) \,\mathrm{d}x$$

in $H^1(B)$ so that

$$\begin{aligned}
&\int_0^T \int_B |\nabla \tilde{v}_0|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&= -2\left(-\frac{1}{2}\int_0^T \int_B |\nabla \tilde{v}_0|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_Q \nabla v_\phi \cdot \nabla \tilde{v}_0 \,\mathrm{d}x\,\mathrm{d}t\right) \\
&\geq -2\left(\frac{1}{2}\int_0^T \int_B |\nabla \tilde{v}_1|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_Q \nabla v_\phi \cdot \nabla \tilde{v}_1 \,\mathrm{d}x\,\mathrm{d}t\right) \\
&= -\int_0^T \int_B |\nabla \tilde{v}_1|^2 \,\mathrm{d}x\,\mathrm{d}t + 2\int_0^T \int_B \kappa |\nabla \tilde{v}_1|^2 \,\mathrm{d}x\,\mathrm{d}t + 2\int_0^T \langle (\tilde{v}_1|_\Omega)', \tilde{v}_1|_\Omega\rangle \,\mathrm{d}t
\end{aligned}$$

and thus

$$\begin{aligned}
&\langle (F_0 - F_1)\phi, \phi\rangle \\
&= \langle \partial_\nu v_0^+, \phi\rangle - \langle \partial_\nu v_1^+, \phi\rangle = \int_0^T \int_Q \nabla v_1 \cdot \nabla v_\phi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_Q \nabla v_0 \cdot \nabla v_\phi \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_Q \nabla v_\phi \cdot \nabla \tilde{v}_1 \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_Q \nabla v_\phi \cdot \nabla \tilde{v}_0 \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_B |\nabla \tilde{v}_0|^2 \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_B \kappa |\nabla \tilde{v}_1|^2 \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle (\tilde{v}_1|_\Omega)', \tilde{v}_1|_\Omega\rangle \,\mathrm{d}t \\
&\geq \int_0^T \int_\Omega (\kappa - 1)|\nabla \tilde{v}_1|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle (\tilde{v}_1|_\Omega)', \tilde{v}_1|_\Omega\rangle \,\mathrm{d}t.
\end{aligned}$$

Using $\kappa|_\Omega - 1 \in L^\infty_+(\Omega)$, $\int_0^T \langle (\tilde v_1|_\Omega)', \tilde v_1|_\Omega \rangle \,\mathrm{d}t \geq 0$, $\tilde v_1|_\Omega = v_1|_\Omega$, and Lemma 3.2(a) we conclude that there exists $c_\kappa > 0$ such that

$$\begin{aligned}\langle (F_0 - F_1)\phi, \phi \rangle &\geq c_\kappa \left(\int_0^T \int_\Omega \kappa |\nabla \tilde v_1|^2 + \int_0^T \langle (\tilde v_1|_\Omega)', \tilde v_1|_\Omega \rangle \,\mathrm{d}t \right) \\ &= c_\kappa \left\langle \left(\kappa \partial_\nu \tilde v_1^- |_{\partial\Omega} \right), \lambda_1 \left(\kappa \partial_\nu \tilde v_1^- |_{\partial\Omega} \right) \right\rangle = c_\kappa \left\langle \left(\partial_\nu v_1^+ |_{\partial\Omega} \right), \lambda_1 \left(\partial_\nu v_1^+ |_{\partial\Omega} \right) \right\rangle \\ &= c_\kappa \langle F_1 \phi, \lambda_1 F_1 \phi \rangle ,\end{aligned}$$

and so the first assertion follows from Lemma 3.2(b). To show surjectivity of F_1 let $\psi \in H_\diamond^{-\frac{1}{2},0}(\partial\Omega)$ and denote by $u \in H^{1,0}(Q)$, $u_1 \in W$ the functions from the definition of $\lambda\psi$ and $\lambda_1\psi$.

Define $v_1 \in H^{1,0}(B \setminus \partial\Omega)$ by setting $v_1 := -u$ on Q and $v_1 := u_1$ on Ω. Then v_1 solves the equations in the definition of F_1 with $[v_1]_{\partial\Omega} = (-\lambda - \lambda_1)\psi$ (up to a spatially constant function) and thus $F_1(-\lambda - \lambda_1)\psi = \partial_\nu u_1^+|_{\partial\Omega} = \psi$.

It remains to show injectivity of F_1. To this end let $F_1\phi = 0$ and $v_1 \in H^{1,0}(B \setminus \partial\Omega)$ be the function from the definition of F_1. Then v_1 solves the Laplace equation on Q and the heat equation on Ω each with zero Neumann boundary values. Thus it vanishes on Ω and is spatially constant on Q, which implies that $\phi \in L^2(0, T, \mathbb{R})$. □

3.2. Range characterization. Lemma 3.3 implies that the symmetric part of $F_0 - F_1$ is positive and thus also the symmetric part of $\Lambda_0 - \Lambda_1$ is positive. Identifying Hilbert spaces with their duals, these operators have positive square roots, and their ranges can be related. The key to provide this relation is the following lemma that has been used by Brühl to extend the factorization method to the case of nonconstant conductivities in EIT [4, Satz 4.9]. We state it in the form in which it is called the "14th important property of Banach spaces" in [3] and give an elementary proof for the sake of completeness.

LEMMA 3.4. *Let X,Y be two Banach spaces, and let $A \in \mathcal{L}(X;Y)$ and $x' \in X'$. Then*

$$x' \in \mathcal{R}(A') \text{ if and only if } \exists C > 0 \;:\; |\langle x', x \rangle| \leq C \|Ax\| \text{ for all } x \in X.$$

Proof. If $x' \in \mathcal{R}(A')$ then there exists $y' \in Y'$ such that $x' = A'y'$. Thus

$$|\langle x', x \rangle| = |\langle y', Ax \rangle| \leq \|y'\| \, \|Ax\| \text{ for all } x \in X,$$

and the assertion holds with $C = \|y'\|$.

Now let $x' \in X'$ such that there exists $C > 0$ with $|\langle x', x \rangle| \leq C \|Ax\|$ for all $x \in X$. Define

$$f(z) := \langle x', x \rangle \text{ for every } z = Ax \in \mathcal{R}(A).$$

Then f is a well-defined, continuous, linear functional, with $\|f(z)\| \leq C \|z\|$. Using the Hahn–Banach theorem there exists $y' \in Y'$ with $y'|_{\mathcal{R}(A)} = f$. For all $x \in X$ we have

$$\langle A'y', x \rangle = \langle y', Ax \rangle = f(A(x)) = \langle x', x \rangle$$

and thus $x' = A'y' \in \mathcal{R}(A')$. □

We will make use of the following simple corollary.

COROLLARY 3.5. *Let H_i, $i = 1, 2$, be Hilbert spaces with norms $\|\cdot\|_i$, X be a third Hilbert space, and $A_i \in \mathcal{L}(X, H_i)$.*

If $\|A_1x\|_1 \leq \|A_2x\|_2$ *for all* $x \in X$, *then* $\mathcal{R}(A_1^*) \subseteq \mathcal{R}(A_2^*)$.

Proof. Since $A_i'\iota_{H_1} = \iota_X A_i^*$, $i = 1, 2$, $y \in \mathcal{R}(A_1^*)$ implies $\iota_X y \in \mathcal{R}(A_1')$. Using Lemma 3.4 there exists $C > 0$ such that

$$|\langle \iota_X y, x\rangle| \leq C\,\|A_1x\|_1 \leq C\,\|A_2x\|_2 \text{ for all } x \in X,$$

and thus $\iota_X y \in \mathcal{R}(A_2')$, which implies $y \in \mathcal{R}(A_2^*)$. ∎

Note that in particular $A_1^*A_1 = A_2^*A_2$ implies $\mathcal{R}(A_1^*) = \mathcal{R}(A_2^*)$ (cf. [11]). Following the argument in [12] we can use Corollary 3.5 to characterize the range of the virtual measurement operator L by reformulating the symmetric part of (3.1) using adjoint operators.

We set

$$\begin{aligned} \Lambda &:= \Lambda_0 - \tfrac{1}{2}(\Lambda_1 + \Lambda_1'), &\qquad \tilde{\Lambda} &= \Lambda\iota_{H_\diamond^{1/2,0}(\partial B)},\\ F &:= F_0 - \tfrac{1}{2}(F_1 + F_1'), &\qquad \tilde{F} &= \iota^{-1}_{H_\diamond^{1/2,0}(\partial\Omega)}F. \end{aligned}$$

Lemma 3.6. $\tilde{\Lambda}$ *and* $\tilde{F}$ *are self-adjoint and positive operators and their square roots satisfy*

$$\mathcal{R}(\tilde{\Lambda}^{1/2}) = \mathcal{R}(L\iota_{H_\diamond^{1/2,0}(\partial\Omega)}\tilde{F}^{1/2}).$$

Proof. By construction $\tilde{\Lambda}$ and $\tilde{F}$ are self-adjoint and positive. From Theorem 3.1 it follows that

$$\begin{aligned} \tilde{\Lambda}^{1/2}\tilde{\Lambda}^{1/2} = \tilde{\Lambda} &= L\iota_{H_\diamond^{1/2,0}(\partial\Omega)}\tilde{F}L'\iota_{H_\diamond^{1/2,0}(\partial B)}\\ &= \left(L\iota_{H_\diamond^{1/2,0}(\partial\Omega)}\right)\tilde{F}\left(L\iota_{H_\diamond^{1/2,0}(\partial\Omega)}\right)^*\\ &= \left(L\iota_{H_\diamond^{1/2,0}(\partial\Omega)}\right)\tilde{F}^{1/2}\tilde{F}^{1/2}\left(L\iota_{H_\diamond^{1/2,0}(\partial\Omega)}\right)^*. \end{aligned}$$

The assertion now follows from Corollary 3.5. ∎

If F were coercive with respect to the space $H_\diamond^{-\frac{1}{2},0}(\partial\Omega)$, we would obtain surjectivity of $\tilde{F}^{1/2}$ and thus the range characterization $\mathcal{R}(\tilde{\Lambda}^{1/2}) = \mathcal{R}(L)$ that was used in previous applications of the factorization method. In our situation we have only the weaker coercivity condition from Lemma 3.3. The next theorem shows that this weaker condition is still enough to guarantee that $\mathcal{R}(\tilde{F}^{1/2})$ contains all functions of a certain time regularity, which turns out to be sufficient for the method to work.

Theorem 3.7.

$$\mathcal{R}(\tilde{\Lambda}^{1/2}) \subseteq \mathcal{R}(L) = L\left(H_\diamond^{-\frac{1}{2},0}(\partial\Omega)\right), \tag{3.8}$$

$$\mathcal{R}(\tilde{\Lambda}^{1/2}) \supseteq L\left(H^{\frac{1}{4}}(0,T,H_\diamond^{-\frac{1}{2}}(\partial\Omega))\right). \tag{3.9}$$

Proof. Equation (3.8) immediately follows from Lemma 3.6.

Denote by $j\colon H_\diamond^{-\frac{1}{2},0}(\partial\Omega) \hookrightarrow H^{-\frac{1}{2},-\frac{1}{4}}(\partial\Omega)$ the imbedding operator. Using Lemma 3.3 we have for all $\phi \in H_\diamond^{\frac{1}{2},0}(\partial\Omega)$

$$\left\|\tilde{F}^{1/2}\phi\right\|^2_{H_\diamond^{\frac{1}{2},0}(\partial\Omega)} = (\tilde{F}\phi,\phi)_{H_\diamond^{\frac{1}{2},0}(\partial\Omega)} \geq c'\,\|jF_1\phi\|^2_{H^{-\frac{1}{2},-\frac{1}{4}}(\partial\Omega)}.$$

Since $F_1^* j^* = \iota^{-1}_{H_\diamond^{1/2,0}(\partial\Omega)} F_1' j' \iota_{H^{-1/2,-1/4}(\partial\Omega)}$ we obtain from Corollary 3.5

$$\mathcal{R}\left(\tilde{F}^{1/2}\right) \supseteq \mathcal{R}(F_1^* j^*) = \mathcal{R}\left(\iota^{-1}_{H_\diamond^{1/2,0}(\partial\Omega)} F_1' j'\right)$$

and from Lemma 3.6

$$\mathcal{R}(\tilde{\Lambda}^{1/2}) = \mathcal{R}(L \iota_{H_\diamond^{1/2,0}(\partial\Omega)} \tilde{F}^{1/2}) \supseteq \mathcal{R}(L F_1' j').$$

Using Lemma 3.4 it is easily seen that

$$\mathcal{R}(j') = H_\diamond^{\frac{1}{2},\frac{1}{4}}(\partial\Omega) := (H^{\frac{1}{2},\frac{1}{4}}(\partial\Omega) + L^2(0,T,\mathbb{R}))/L^2(0,T,\mathbb{R}) \subset H_\diamond^{\frac{1}{2},0}(\partial\Omega).$$

(Note that by this definition $H_\diamond^{\frac{1}{2},\frac{1}{4}}(\partial\Omega)$ is isomorphic to $H^{\frac{1}{2},\frac{1}{4}}(\partial\Omega)/H^{\frac{1}{4}}(0,T,\mathbb{R})$.)

Using Lemma 3.3 we have $(F_1')^{-1} = -\lambda' - \lambda_1'$. Since $\lambda = \lambda'$ and $\mathcal{R}(\lambda_1') \subseteq H_\diamond^{\frac{1}{2},\frac{1}{4}}(\partial\Omega)$ (cf. [8]) it remains only to show that

$$\lambda\left(H^{\frac{1}{4}}(0,T,H_\diamond^{-\frac{1}{2}}(\partial\Omega))\right) \subseteq H_\diamond^{\frac{1}{2},\frac{1}{4}}(\partial\Omega). \tag{3.10}$$

To this end denote by $\overline{\lambda}:\ H_\diamond^{-\frac{1}{2}}(\partial\Omega) \to H_\diamond^{\frac{1}{2}}(\partial\Omega)$, $\overline{\psi} \mapsto \overline{u}^+|_{\partial\Omega}$, where $\overline{u} \in H_\diamond^1(Q)$ solves

$$\Delta \overline{u} = 0 \text{ in } Q, \quad \partial_\nu \overline{u} = \begin{cases} -\overline{\psi} & \text{on } \partial\Omega, \\ 0 & \text{on } \partial B. \end{cases}$$

Then for every $\psi \in H^1(0,T,H_\diamond^{-\frac{1}{2}}(\partial\Omega))$ and $\varphi \in \mathcal{D}(]0,T[)$

$$\begin{aligned} \int_0^T (-1)(\lambda\psi)\varphi' \,\mathrm{d}t &= \overline{\lambda}\left(\int_0^T (-1)\psi\varphi'(t)\,\mathrm{d}t\right) = \overline{\lambda}\left(\int_0^T \psi'\varphi(t)\,\mathrm{d}t\right) \\ &= \int_0^T (\lambda\psi')\varphi \,\mathrm{d}t \in H_\diamond^{\frac{1}{2}}(\partial\Omega). \end{aligned}$$

Thus $\lambda\psi \in H^1(0,T,H_\diamond^{\frac{1}{2}}(\partial\Omega))$ with $(\lambda\psi)' = \lambda(\psi')$, which shows that λ is a continuous operator not only from $L^2(0,T,H_\diamond^{-\frac{1}{2}}(\partial\Omega))$ to $L^2(0,T,H_\diamond^{\frac{1}{2}}(\partial\Omega))$ but also from $H^1(0,T,H_\diamond^{-\frac{1}{2}}(\partial\Omega))$ to $H^1(0,T,H_\diamond^{\frac{1}{2}}(\partial\Omega))$.

By interpolation (cf. [21]) λ is a continuous operator from

$$H^{\frac{1}{4}}(0,T,H_\diamond^{-\frac{1}{2}}(\partial\Omega)) \to H^{\frac{1}{4}}(0,T,H_\diamond^{\frac{1}{2}}(\partial\Omega)) \subset H_\diamond^{\frac{1}{2},\frac{1}{4}}(\partial\Omega).$$

Thus (3.10) holds and the assertion follows. ∎

3.3. Characterization of the inclusion. The composition of time integration and the (compact) imbedding $H_\diamond^{\frac{1}{2}}(\partial B) \hookrightarrow L_\diamond^2(\partial B) := L^2(\partial B)/\mathbb{R}$ defines the operator

$$I:\ H_\diamond^{\frac{1}{2},0}(\partial B) \to L_\diamond^2(\partial B), \quad u \mapsto \int_0^T u(\cdot,t)\,\mathrm{d}t.$$

Identifying $L_\diamond^2(\partial B)$ with its dual we have

$$I\tilde{\Lambda}I^* = I\Lambda I', \tag{3.11}$$

where $I' :\ L^2_\diamond(\partial B) \to H_\diamond^{-\frac{1}{2},0}(\partial B)$ is given by

$$I'v = w, \text{ with } w(\cdot,t) = v(\cdot) \text{ for } t \in [0,T] \text{ a.e.}$$

The operator $I\Lambda I'$ corresponds to measurements of applying temporal constant (and spatially square integrable) heat fluxes to a body and measuring time integrals of the resulting temperature on the boundary.

We use the same dipole functions as Brühl and Hanke used in [13] for the implementation of the factorization method in EIT. For a direction $d \in \mathbb{R}^n$, $|d| = 1$, and a point $z \in B$ let

$$D_{z,d}(x) := \frac{(z-x)\cdot d}{|z-x|^n}.$$

Then $D_{z,d}(x)$ is analytic and $\Delta D_{z,d}(x) = 0$ in $\mathbb{R}^n \setminus \{z\}$. Moreover, using a ball $B_\epsilon(z)$ centered at z with such small radius $\epsilon > 0$ such that $\overline{B_\epsilon(z)} \subset B$,

$$\int_{\partial B} \partial_\nu D_{z,d}(x)\,\mathrm{d}x = \int_{\partial B_\epsilon(z)} \partial_\nu D_{z,d}(x)\,\mathrm{d}x = 0,$$

so in particular $\partial_\nu D_{z,d} \in H_\diamond^{-\frac{1}{2}}(\partial B)$ and there exists $v_{z,d} \in H^1(B)$ that solves

$$\Delta v_{z,d} = 0 \text{ in } B \quad \text{ and } \quad \partial_\nu v_{z,d} = -\partial_\nu D_{z,d} \text{ on } \partial B.$$

Now $H_{z,d} := D_{z,d} + v_{z,d}$ is harmonic (and thus analytic) in $B \setminus \{z\}$ with $\partial_\nu H_{z,d}|_{\partial B} = 0$ but $H_{z,d} \notin L^2(B \setminus \{z\})$. The inclusion can now be characterized by the traces $h_{z,d} := H_{z,d}|_{\partial B} \in H_\diamond^{\frac{1}{2}}(\partial B)$ (again we use the same notation for the equivalence class of functions that are identical up to addition of constant functions as we used for the original function).

Theorem 3.8. *For every $d \in \mathbb{R}^n$, $|d| = 1$, and $z \in B$*

$$z \in \Omega \textit{ if and only if } h_{z,d} \in \mathcal{R}\left((I\Lambda I')^{1/2}\right).$$

Proof. From Corollary 3.5 and (3.11) it follows that $\mathcal{R}\left((I\Lambda I')^{1/2}\right) = \mathcal{R}(I\tilde{\Lambda}^{1/2})$ and consequently from Theorem 3.7 we obtain

$$\mathcal{R}\left((I\Lambda I')^{1/2}\right) \subseteq IL\left(H_\diamond^{-\frac{1}{2},0}(\partial\Omega)\right),$$
$$\mathcal{R}\left((I\Lambda I')^{1/2}\right) \supseteq IL\left(H^{\frac{1}{4}}(0,T,H_\diamond^{-1/2}(\partial\Omega))\right).$$

First let $z \in \Omega$; then we define $w \in H_\diamond^{1,0}(Q)$ by $w(x,t) := H_{z,d}(x)/T$ for $t \in [0,T]$ a.e. Then $-\partial_\nu w^+|_{\partial\Omega} \in H^{\frac{1}{4}}(0,T,H_\diamond^{-1/2}(\partial\Omega))$ and w solves (3.2) in the definition of L, so

$$\begin{aligned} h_{z,d} &= Iw|_{\partial B} = IL(-\partial_\nu w^+|_{\partial\Omega}) \\ &\in IL\left(H^{\frac{1}{4}}(0,T,H_\diamond^{-1/2}(\partial\Omega))\right) \subseteq \mathcal{R}\left((I\Lambda I')^{1/2}\right). \end{aligned}$$

To show the converse let $h_{z,d} \in \mathcal{R}\left((I\Lambda I')^{1/2}\right) \subseteq IL(H_\diamond^{-\frac{1}{2},0}(\partial\Omega))$. Then $h_{z,d}$ coincides with the integral of the trace of a solution of the Laplace equation on Q with vanishing

Neumann boundary values. Taking the integral of that solution we have that $h_{z,d} = w|_{\partial B}$, with some

$$w \in H^1_\diamond(Q) \text{ that solves } \Delta w = 0 \text{ on } Q,\ \partial_\nu w = 0 \text{ on } \partial B.$$

As $H_{z,d}$ and w are both harmonic on $Q \setminus \{z\}$ with the same Cauchy data on ∂B, they coincide near ∂B and thus by analytic continuation on $Q \setminus \{z\}$. If $z \notin \Omega$ this leads to the contradiction that $w \in L^2_\diamond(Q \setminus \{z\})$ but $H_{z,d} \notin L^2_\diamond(Q \setminus \{z\})$. ☐

By construction $I\Lambda I'$ is a compact and self-adjoint operator and from the factorization and the positiveness of F it follows that it is positive. Since $I\Lambda I' g = 0$ implies that $\langle F L' I' g, L' I' g\rangle = 0$ and thus $L' I' g = 0$, we also obtain injectivity of $I\Lambda I'$ from the injectivity of L' and I'. Hence there exists an orthonormal basis $(v_k)_{k\in\mathbb{N}}$ of eigenfunctions with associated positive eigenvalues $(\lambda_k)_{k\in N}$. Following [13] we use this spectral decomposition to reformulate Theorem 3.8 with the Picard criterion.

Corollary 3.9. *For every $d \in \mathbb{R}^n$, $|d| = 1$, and $z \in B$*

$$z \in \Omega \text{ if and only if } \sum_{k\in\mathbb{N}} \frac{1}{\lambda_k} \left(\int_{\partial B} h_{z,d} v_k \, \mathrm{d}x \right)^2 < \infty.$$

We remark that the results of this subsection remain valid with identical proofs when I is replaced by

$$I_S :\ H^{\frac{1}{2},0}_\diamond(\partial B) \to L^2_\diamond(S), \quad u \mapsto \int_0^T u|_S(\cdot, t)\, \mathrm{d}t,$$

where S is a relatively open subset of the boundary ∂B. Thus Ω is uniquely determined by $I_S \Lambda I'_S$, i.e., by measurements of applying (temporal constant) heat fluxes on a part of the boundary and measuring (time integrals of) the resulting temperature on the same part; cf., e.g., [14, 24, 25] for corresponding results in impedance tomography and the effect of partial boundary data on numerical reconstructions.

4. Numerics.

4.1. The direct problem.

In this section we show how the direct problem can be solved numerically with a coupling of a finite element method and a boundary element method similar to [9]. We start by reformulating the direct problem.

4.1.1. Reformulation of the direct problem.

Recall that λ was defined by

$$\lambda :\ H^{-\frac{1}{2},0}_\diamond(\partial\Omega) \to H^{\frac{1}{2},0}_\diamond(\partial\Omega), \quad \lambda\psi := \eta^+|_{\partial\Omega}\,,$$

where $\eta \in H^{1,0}_\diamond(Q)$ solves

$$\Delta\eta = 0 \text{ in } Q, \quad \partial_\nu \eta = \begin{cases} -\psi & \text{on } \partial\Omega\,, \\ 0 & \text{on } \partial B\,. \end{cases}$$

We use the same notation for the time-independent Neumann–Dirichlet operator

$$\lambda :\ H^{-\frac{1}{2}}_\diamond(\partial\Omega) \to H^{\frac{1}{2}}_\diamond(\partial\Omega).$$

Note that λ is linear, continuous, and coercive, i.e., $\langle \psi, \lambda\psi\rangle \geq c\|\psi\|^2_{H^{-\frac{1}{2}}_\diamond(\partial\Omega)}$.

For the rest of this section we assume that $g \in H_\diamond^{-\frac{1}{2},0}(\partial B)$ and $\xi = \xi(g) \in H_\diamond^{1,0}(Q)$ solves

$$\Delta \xi = 0 \text{ in } Q, \quad \partial_\nu \xi = \begin{cases} 0 & \text{on } \partial\Omega\,, \\ g & \text{on } \partial B\,. \end{cases}$$

THEOREM 4.1. *If $u \in H^{1,0}(B)$ solves* (2.5)–(2.8), (2.11), *and* (2.12), *then $v := u|_\Omega$ and $\phi := -\kappa \partial_\nu u^-|_{\partial\Omega}$ satisfy*

$$\partial_t v - \nabla \cdot (\kappa \nabla v) = 0 \text{ in } \Omega \times]0, T[, \tag{4.1}$$

$$v^-|_{\partial\Omega} - \lambda\phi = \xi^+|_{\partial\Omega} \text{ in } H_\diamond^{\frac{1}{2},0}(\partial\Omega), \tag{4.2}$$

$$v(x, 0) = 0 \text{ in } \Omega\,. \tag{4.3}$$

On the other hand, if $(v, \phi) \in H^{1,0}(\Omega) \times H_\diamond^{-\frac{1}{2},0}(\partial\Omega)$ solves (4.1)–(4.3) *and*

$$\kappa \partial_\nu v^-|_{\partial\Omega} = -\phi, \tag{4.4}$$

then there exists $u \in H^{1,0}(B)$ that solves (2.5)–(2.8), (2.11), (2.12), *and $v = u|_\Omega$. Moreover $u|_Q \in H^{1,0}(Q)$ is the representant of $\xi + \eta \in H_\diamond^{1,0}(Q)$ with $\int_{\partial\Omega} u^+|_{\partial\Omega}\,\mathrm{d}s = \int_{\partial\Omega} u^-|_{\partial\Omega}\,\mathrm{d}s$, where η is as in the definition of $\lambda\phi$.*

Proof. The proof immediately follows from the definitions of ξ and λ. □

THEOREM 4.2. *The following problems are equivalent:*

(a) $(u, \phi) \in H^{1,0}(\Omega) \times H_\diamond^{-\frac{1}{2},0}(\partial\Omega)$ *solves* (4.1)–(4.4).

(b) $(u, \phi) \in W \times H_\diamond^{-\frac{1}{2},0}(\partial\Omega)$, *$u(x, 0) = 0$ in Ω, and (u, ϕ) solves*

$$\begin{aligned} &\int_0^T \langle u', v\rangle\,\mathrm{d}t + \int_0^T \int_\Omega \kappa \nabla u \cdot \nabla v\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle \phi, v^-|_{\partial\Omega}\rangle\,\mathrm{d}t \\ &- \int_0^T \langle \tilde\psi, \lambda\phi\rangle\,\mathrm{d}t + \int_0^T \langle \tilde\psi, u^-|_{\partial\Omega}\rangle\,\mathrm{d}t = \int_0^T \langle \tilde\psi, \xi^+|_{\partial\Omega}\rangle\,\mathrm{d}t \end{aligned} \tag{4.5}$$

for all $v \in H^{1,0}(\Omega)$ and for all $\tilde\psi \in H_\diamond^{-\frac{1}{2},0}(\partial\Omega)$.

(c) $(u, \phi) \in W \times H_\diamond^{-\frac{1}{2},0}(\partial\Omega)$, *$u(x, 0) = 0$ in Ω, and (u, ϕ) solves*

$$\begin{aligned} \langle u'(t), v\rangle + \int_\Omega \kappa \nabla u(t) \cdot \nabla v\,\mathrm{d}x + \langle \phi(t), v^-|_{\partial\Omega}\rangle - \langle \tilde\psi, \lambda\phi(t)\rangle + \langle \tilde\psi, u(t)^-|_{\partial\Omega}\rangle \\ = \langle \tilde\psi, \xi(t)^+|_{\partial\Omega}\rangle \end{aligned} \tag{4.6}$$

for $t \in [0, T]$ a.e. and for all $v \in H^1(\Omega)$ and for all $\tilde\psi \in H_\diamond^{-\frac{1}{2}}(\partial\Omega)$.

Proof. (a)⇔(b) can be shown analogously to the proof of Theorem 2.6.

To show (b)⇔(c), note that (4.6) is fulfilled for $t \in [0, T]$ a.e. if and only if it is fulfilled in the sense of $L^2([0, T])$. The equivalence then follows from the fact that $L^2([0, T]) \otimes H^1(\Omega)$, resp., $L^2([0, T]) \otimes H_\diamond^{-\frac{1}{2}}(\partial\Omega)$, are dense in $H^{1,0}(\Omega)$, resp., $H_\diamond^{-\frac{1}{2},0}(\partial\Omega)$. □

4.1.2. Implementation and convergence analysis of the reformulated problem. Let $\{H_h,\, h > 0\}$ and $\{B_h,\, h > 0\}$ be families of finite dimensional subspaces of $H^1(\Omega)$ and $H_\diamond^{-\frac{1}{2}}(\partial\Omega)$, respectively. Accordingly the family of L^2-projections $P_h : H^1(\Omega) \to H_h$ is defined by $\int_\Omega P_h v w_h\,\mathrm{d}x = \int_\Omega v w_h\,\mathrm{d}x$ for all $w_h \in H_h$. We assume that P_h satisfies the following estimate: There exists a constant $\gamma > 0$, independent of h, such that

$$\sup_{v\in H^1(\Omega)} \frac{\|P_h v\|_{H^1(\Omega)}}{\|v\|_{H^1(\Omega)}} \le \gamma \;\text{ for all } h > 0\,. \tag{4.7}$$

For example, let $\mathcal{T}$ be a regular triangulation of Ω with generic mesh spacing h and H_h be a space of piecewise linear polynomials on $\mathcal{T}$. Then following [23] the operator P_h fulfills (4.7).

We consider the following Galerkin scheme.

Find $u_h : [0,T] \to H_h$, $\phi_h : [0,T] \to B_h$ such that

$$\begin{aligned} \langle u_h', v_h\rangle + \int_\Omega \kappa\nabla u_h\cdot\nabla v_h\,\mathrm{d}x + \langle \phi_h, v_h^-|_{\partial\Omega}\rangle & \\ - \langle\psi_h,\lambda\phi_h\rangle + \langle\psi_h, u_h^-|_{\partial\Omega}\rangle \quad &= \quad \langle\psi_h,\xi^+|_{\partial\Omega}\rangle \end{aligned} \tag{4.8}$$

for all $(v_h,\psi_h) \in H_h\times B_h$, $t\in[0,T]$ a.e., and $u_h(0) = 0$.

LEMMA 4.3. *For every $h > 0$ the Galerkin scheme* (4.8) *has a unique solution in $H_h^T \times B_h^T$, where*

$$\begin{aligned} H_h^T &:= \left\{u\in L^2(0,T,H_h) : u'\in L^2(0,T,H_h), u(x,0)=0\right\} \subset W,\\ B_h^T &:= L^2(0,T,B_h) \subset H_\diamond^{-\frac{1}{2},0}(\partial\Omega). \end{aligned}$$

Proof. Let $(w_k)_{k=1}^{n_h}$ be a basis of H_h which is orthonormal with respect to the $L^2(\Omega)$ scalar product and $(\psi_j)_{j=1}^{m_h}$ be a basis of B_h. Moreover, if we write $u_h(x,t) = \sum_{k=1}^{n_h}\alpha_k(t)w_k(x)$ and $\phi_h(x,t) = \sum_{j=1}^{m_h}\beta_j(t)\psi_j(x)$, then (4.8) is equivalent to

$$\partial_t\alpha(t) + K\alpha(t) + B\beta(t) = 0\,,\quad \alpha(0) = 0\,, \tag{4.9}$$

and

$$D\beta(t) - B^T\alpha(t) = d(t), \tag{4.10}$$

where $\alpha = (\alpha_1,\ldots,\alpha_{n_h})^T$ and $\beta = (\beta_1,\ldots,\beta_{m_h})^T$. Since λ is linear and coercive we can solve (4.10) for β in terms of α and substitute into (4.9) to obtain a system of ODEs for α. According to standard existence theory for ODEs, there exists a unique absolutely continuous solution α. ∎

LEMMA 4.4. *Assume that $(w_h,\zeta_h)\in H_h^T\times B_h^T$, $\zeta\in H_\diamond^{-\frac{1}{2},0}(\partial\Omega)$, $w\in W$ with $w(x,0) = 0$ in Ω. Moreover assume that the following equation is fulfilled:*

$$\begin{aligned} &\langle w_h', v_h\rangle + \int_\Omega \kappa\nabla w_h\cdot\nabla v_h\,\mathrm{d}x + \langle\zeta_h, v_h^-|_{\partial\Omega}\rangle - \langle\psi_h,\lambda\zeta_h\rangle + \langle\psi_h, w_h^-|_{\partial\Omega}\rangle \\ &= \langle w', v_h\rangle + \int_\Omega \kappa\nabla w\cdot\nabla v_h\,\mathrm{d}x + \langle\zeta, v_h^-|_{\partial\Omega}\rangle - \langle\psi_h,\lambda\zeta\rangle + \langle\psi_h, w^-|_{\partial\Omega}\rangle \end{aligned} \tag{4.11}$$

for $t\in[0,T]$ a.e. and for all $(v_h,\psi_h)\in H_h\times B_h$. Then there exists $c > 0$ independent of h such that

$$\|w_h\|_W + \|\zeta_h\|_{H_\diamond^{-\frac{1}{2},0}(\partial\Omega)} \le c\left(\|w\|_W + \|\zeta\|_{H_\diamond^{-\frac{1}{2},0}(\partial\Omega)}\right).$$

Proof. 1. We use $(v_h, \psi_h) = (w_h, -\zeta_h)$ in (4.11) and obtain for $t \in [0,T]$ a.e.

$$\begin{aligned}
&\langle w_h', w_h\rangle + \int_\Omega \kappa |\nabla w_h|^2 \,\mathrm{d}x + \langle \zeta_h, \lambda\zeta_h\rangle \\
&= \langle w', w_h\rangle + \int_\Omega \kappa\nabla w\cdot\nabla w_h \,\mathrm{d}x + \langle \zeta, w_h^-|_{\partial\Omega}\rangle + \langle \zeta_h, \lambda\zeta\rangle - \langle \zeta_h, w^-|_{\partial\Omega}\rangle \\
&\le c_1 \left(\|w'\|_{H^1(\Omega)'} + \|w\|_{H^1(\Omega)} + \|\zeta\|_{H_\diamond^{-\frac12}(\partial\Omega)} \right) \|w_h\|_{H^1(\Omega)} \\
&\quad + c_2 \left(\|\zeta\|_{H_\diamond^{-\frac12}(\partial\Omega)} + \|w\|_{H^1(\Omega)} \right) \|\zeta_h\|_{H_\diamond^{-\frac12}(\partial\Omega)},
\end{aligned}$$

where c_i, $i = 1, \ldots, 8$, are not depending on h. Note that $\|\zeta\|_{H^{-\frac12}(\partial\Omega)} = \|\zeta\|_{H_\diamond^{-\frac12}(\partial\Omega)}$ and $\|\zeta_h\|_{H^{-\frac12}(\partial\Omega)} = \|\zeta_h\|_{H_\diamond^{-\frac12}(\partial\Omega)}$. Now we integrate the left- and the right-hand sides of this inequality from 0 to t and get for $t \in [0,T]$ a.e.

$$\begin{aligned}
&\frac12 \|w_h(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \kappa|\nabla w_h|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^t \langle \zeta_h, \lambda\zeta_h\rangle \,\mathrm{d}t \\
&\le c_3 \left(\|w\|_W + \|\zeta\|_{H_\diamond^{-\frac12,0}(\partial\Omega)} \right) \left(\|w_h\|_{H^{1,0}(\Omega)} + \|\zeta_h\|_{H_\diamond^{-\frac12,0}(\partial\Omega)} \right). \qquad (4.12)
\end{aligned}$$

Again integrating both sides of this inequality from 0 to T yields

$$\begin{aligned}
&\|w_h\|^2_{L^2(0,T,L^2(\Omega))} \\
&\le c_4 \left(\|w\|_W + \|\zeta\|_{H_\diamond^{-\frac12,0}(\partial\Omega)} \right) \left(\|w_h\|_{H^{1,0}(\Omega)} + \|\zeta_h\|_{H_\diamond^{-\frac12,0}(\partial\Omega)} \right). \qquad (4.13)
\end{aligned}$$

2. Since λ is coercive, using (4.12) and (4.13) we get

$$\begin{aligned}
&\|w_h\|^2_{H^{1,0}(\Omega)} + \|\zeta_h\|^2_{H_\diamond^{-\frac12,0}(\partial\Omega)} \\
&\le c_5 \left(\|w\|_W + \|\zeta\|_{H_\diamond^{-\frac12,0}(\partial\Omega)} \right) \left(\|w_h\|_{H^{1,0}(\Omega)} + \|\zeta_h\|_{H_\diamond^{-\frac12,0}(\partial\Omega)} \right).
\end{aligned}$$

Therefore, we have

$$\|w_h\|_{H^{1,0}(\Omega)} + \|\zeta_h\|_{H_\diamond^{-\frac12,0}(\partial\Omega)} \le c_6 \left(\|w\|_W + \|\zeta\|_{H_\diamond^{-\frac12,0}(\partial\Omega)} \right). \qquad (4.14)$$

3. Since $w_h' \in H_h$ for $t \in [0,T]$ a.e., using the L^2-projection P_h we have for $t \in [0,T]$ a.e.

$$\|w_h'\|_{H^1(\Omega)'} = \sup_{w\in H^1(\Omega)} \frac{\langle w_h', w\rangle}{\|w\|_{H^1(\Omega)}} = \sup_{w\in H^1(\Omega)} \frac{\langle w_h', P_h w\rangle}{\|w\|_{H^1(\Omega)}}. \qquad (4.15)$$

Now we use $(v_h, \psi_h) = (P_h w, 0)$ in (4.11) and obtain for $t \in [0,T]$ a.e.

$$\begin{aligned}
\langle w_h', P_h w\rangle &= -\int_\Omega \kappa\nabla w_h\cdot\nabla P_h w \,\mathrm{d}x - \langle \zeta_h, P_h w^-|_{\partial\Omega}\rangle + \langle w', P_h w\rangle \\
&\quad + \int_\Omega \kappa\nabla w\cdot\nabla P_h w \,\mathrm{d}x + \langle \zeta, P_h w^-|_{\partial\Omega}\rangle \\
&\le c_7 \|P_h w\|_{H^1(\Omega)} \left(\|w_h\|_{H^1(\Omega)} + \|\zeta_h\|_{H_\diamond^{-\frac12}(\partial\Omega)} \right. \\
&\quad \left. + \|w\|_W + \|\zeta\|_{H_\diamond^{-\frac12}(\partial\Omega)} \right).
\end{aligned}$$

Squaring and then integrating (4.15) from 0 to T and combining it with the inequality above, (4.7), and (4.14), we obtain

$$\|w_h'\|_{L^2(0,T,H^1(\Omega)')} \le c_8 \left(\|w\|_W + \|\zeta\|_{H_\diamond^{-\frac{1}{2},0}(\partial\Omega)} \right) . \qquad \square$$

In particular, Lemma 4.4 holds for (u_h, ϕ_h) and (u, ϕ).

We proof a variant of Céa's lemma for this time-dependent problem.

THEOREM 4.5. *Assume that* (u, ϕ) *and* (u_h, ϕ_h) *are solutions of* (4.5) *with* $u(x, 0) = 0$ *and of the Galerkin scheme, respectively. Then there exists* $c > 0$ *such that*

$$\begin{aligned} &\|u - u_h\|_W + \|\phi - \phi_h\|_{H_\diamond^{-\frac{1}{2},0}(\partial\Omega)} \\ &\qquad \le c \inf \left\{ \|u - z_h\|_W + \|\phi - \chi_h\|_{H_\diamond^{-\frac{1}{2},0}(\partial\Omega)} : z_h \in H_h^T,\, \chi_h \in B_h^T \right\} . \end{aligned}$$

Proof. (u, ϕ) and (u_h, ϕ_h) obviously satisfy (4.11).

Let $(z_h, \chi_h) \in H_h^T \times B_h^T$. We set $(e_1, e_2) := (u_h, \phi_h) - (z_h, \chi_h)$ and $(\epsilon_1, \epsilon_2) := (u, \phi) - (z_h, \chi_h)$. Then (4.11) yields

$$\begin{aligned} &\langle e_1', v_h\rangle + \int_\Omega \kappa \nabla e_1 \cdot \nabla v_h \,\mathrm{d}x + \langle e_2, v_h^-|_{\partial\Omega}\rangle - \langle \psi_h, \lambda e_2\rangle + \langle \psi_h, e_1^-|_{\partial\Omega}\rangle \\ &= \langle \epsilon_1', v_h\rangle + \int_\Omega \kappa \nabla \epsilon_1 \cdot \nabla v_h \,\mathrm{d}x + \langle \epsilon_2, v_h^-|_{\partial\Omega}\rangle - \langle \psi_h, \lambda \epsilon_2\rangle + \langle \psi_h, \epsilon_1^-|_{\partial\Omega}\rangle \end{aligned}$$

for all $(v_h, \psi_h) \in H_h \times B_h$ and for $t \in [0, T]$ a.e. Lemma 4.4 shows that

$$\|e_1\|_W + \|e_2\|_{H_\diamond^{-\frac{1}{2},0}(\partial\Omega)} \le c_1 \left(\|\epsilon_1\|_W + \|\epsilon_2\|_{H_\diamond^{-\frac{1}{2},0}(\partial\Omega)} \right) .$$

Hence

$$\begin{aligned} &\|(u_h, \phi_h) - (u, \phi)\|_{W \times H_\diamond^{-\frac{1}{2},0}(\partial\Omega)} \\ &\qquad \le c_2 \inf \left\{ \|(u, \phi) - (z_h, \chi_h)\|_{W \times H_\diamond^{-\frac{1}{2},0}(\partial\Omega)} : z_h \in H_h^T,\, \chi_h \in B_h^T \right\}, \end{aligned}$$

which is the desired estimate. $\square$

For our numerical examples we choose the same subspaces as in [9] and [23]; i.e., H_h consists of continuous functions, which are piecewise linear on a finite element grid, and B_h consists of piecewise constant functions. Equations (4.9) and (4.10) are solved numerically by a Crank–Nicolson method; i.e., we solve in each time-step the linear system of equations

$$\begin{bmatrix} I + \frac{\Delta t}{2} K & \frac{\Delta t}{2} B \\ -\frac{1}{2} B^T & \frac{1}{2} D \end{bmatrix} \begin{bmatrix} \alpha(t + \Delta t) \\ \beta(t + \Delta t) \end{bmatrix} = \begin{bmatrix} I - \frac{\Delta t}{2} K & -\frac{\Delta t}{2} B \\ \frac{1}{2} B^T & -\frac{1}{2} D \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ d(t) \end{bmatrix} ,$$

with $\alpha(0) = 0$ and $\beta(0) = D^{-1} d(0)$.

For the calculation of ξ and λ we use a boundary element method.

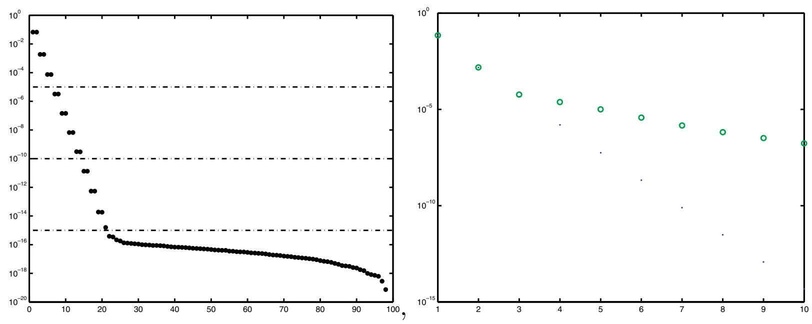

FIG. 4.1. *Left: The exact eigenvalues $\tilde{\lambda}_k$ from A. Right: Exact ($\cdot$) and perturbed ($\circ$) averaged eigenvalues.*

4.2. Implementation of the inverse problem. In this subsection we demonstrate how the factorization method can be used to solve the inverse problem, i.e., to locate the inclusion Ω from the knowledge of $I\Lambda I'$. We assume that we are given a finite dimensional approximation of $I\Lambda I'$ and thus a matrix $A \in \mathbb{R}^{m\times m}$. Let $(v_k)_{k\in\mathbb{N}}$, resp., $(\tilde{v}_k)_{k=1}^m$, be the eigenfunctions of $I\Lambda I'$, resp., A, with associated eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$, resp., $(\tilde{\lambda}_k)_{k\in\mathbb{N}}$. Since $I\Lambda I'$ is self-adjoint and positive, the matrix A is symmetric and positive, too.

According to Corollary 3.9 a point $z \in B$ belongs to the inclusion Ω if and only if the infinite series

$$\sum_{k\in\mathbb{N}} \frac{(h_{z,d}, v_k)^2_{L^2(\partial B)}}{\lambda_k}$$

converges. For the numerical realization we have to decide about the convergence of this series from the knowledge of the finite sum

$$\sum_{k=1}^{m} \frac{(h_{z,d}, \tilde{v}_k)^2_{L^2(\partial B)}}{\tilde{\lambda}_k} .$$

For that we carry forward the ideas from [4]. Numerical examples show that the numerator and the denominator of the above series decay more or less exponentially and that every two eigenvalues have approximately the same value; cf. the left picture of Figure 4.1. Motivated by the examples and the method from [4], we compare the slopes of the least squares fitting straight lines of $h_1(k) = \log\left(\sqrt{\tilde{\lambda}_{2k-1}\tilde{\lambda}_{2k}}\right)$ and of

$$h_2(k) = \log\left(\frac{1}{2}\left((h_{z,d}, \tilde{v}_{2k-1})^2_{L^2(\partial B)} + (h_{z,d}, \tilde{v}_{2k})^2_{L^2(\partial B)}\right)\right), \quad k = 1, \dots, r .$$

We mark a sampling point z as inside the inclusion if h_1 decays slower than h_2. On the right side of Figure 4.2 the algorithm is demonstrated for two test points. If we apply this method to a large number of points, the black area on the left side of Figure 4.2 illustrates the reconstruction of the inclusion (dashed curve).

The number of the eigenvalues and Fourier coefficients which are used in the reconstruction procedure depends on the quality of the data. If A is known up to a perturbation of $\delta > 0$ (with respect to the spectral norm), then we trust in those

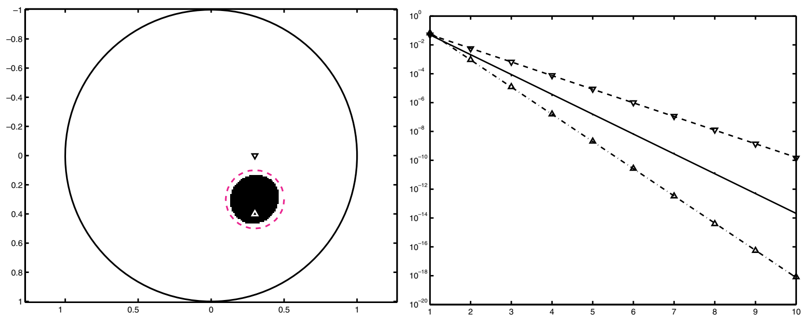

FIG. 4.2. *Least squares fitting straight lines of $h_2(k)$ for a point inside ($\triangle$) and outside ($\bigtriangledown$) the inclusion compared with the least squares fitting straight line of $h_1(k)$ ($\cdot$).*

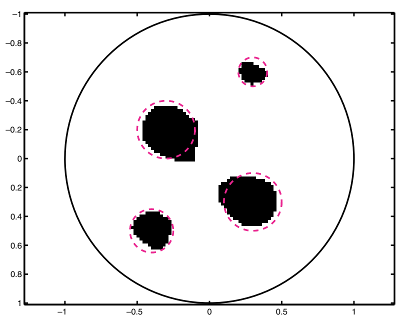

FIG. 4.3. *Reconstruction of four inclusions (dashed curves).*

eigenvalues which are larger than δ. On the left side of Figure 4.1, δ corresponds to the computational accuracy. The right side of Figure 4.1 shows the effect of a relative noise of 0.1% on the eigenvalues, and thus $\delta = 0.1\% \cdot \tilde{\lambda}_1$. The first three averaged pairs of the perturbed eigenvalues have nearly the exact values and they show the same exponential decay rates.

4.3. Numerical examples. To test this reconstruction algorithm we simulate the direct problem to produce the data. For this purpose we calculate the Dirichlet boundary data $f_k = I\Lambda I' g_k$, where $(g_k)_{k=1}^m$ are orthogonal input patterns. In the first examples this data was used for inversion. In the final example this data was perturbed with noise.

We restrict our attention to the case where $\kappa(x) = 2$ for $x \in \Omega$ and B is the unit disc in $\mathbb{R}^2$. For this case the function $h_{z,d}$ is known explicitly:

$$h_{z,d}(x) = \frac{1}{\pi} \frac{(z-x) \cdot d}{|z-x|^2} \,.$$

First we aim to reconstruct a single circle in the interior of B. The result is shown in the left picture of Figure 4.2. The location of Ω is detected but the size is underestimated.

In the second example four inclusions of different size should be located. In Figure 4.3 we demonstrate the possibility of the method to reconstruct nonconnected inclusions. The position and the different size of each are detected.

Our next example is to detect a nonconvex moon-like inclusion; cf. Figure 4.4. The top left picture shows the reconstruction with exact data. The shape of the moon

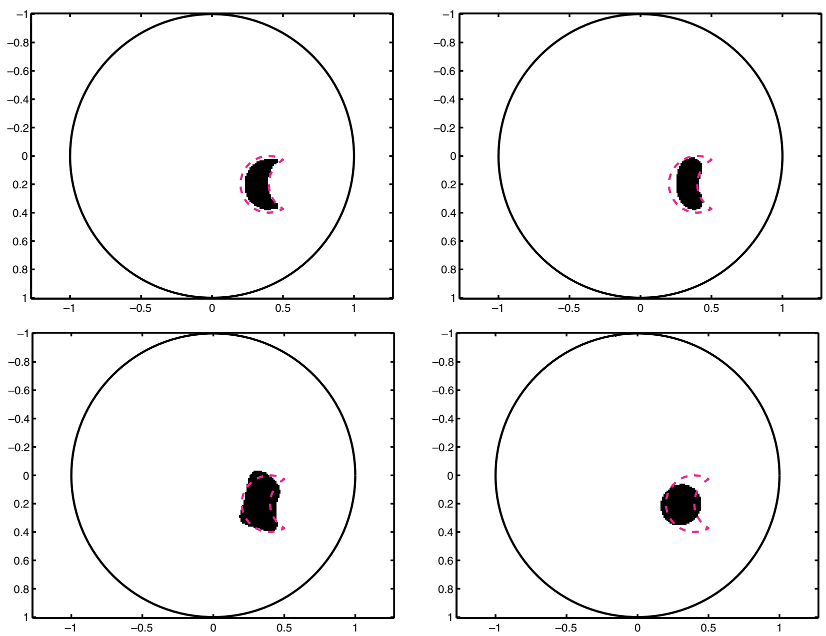

FIG. 4.4. *Reconstruction of a nonconvex inclusion (dashed curve). Top left: With exact data, and with perturbed data. Top right:* 0.05% *noise. Bottom left:* 0.1% *noise. Bottom right:* 1% *noise.*

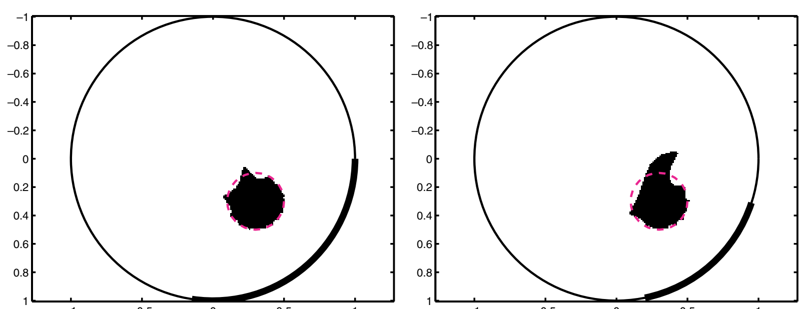

FIG. 4.5. *Reconstruction of an inclusion (dashed curve) by using partial boundary measurements (bold boundary).*

is recovered but the size is underestimated. Next we show the influence of noise on the reconstructions. By adding 0.05%, 0.1%, resp., 1%, noise the position of the inclusion is found, but the quality decreases with increasing noise level; cf. the top right and bottom pictures in Figure 4.4.

The last example shows the reconstruction of a single circle by partial boundary measurements (cf. our remark at the end of section 3). The location of the inclusion is detected and the shape next to the measuring boundary is recovered; see Figure 4.5.

Acknowledgments. We thank Prof. Martin Hanke-Bourgeois and Prof. Olaf Steinbach for stimulating discussion and constructive remarks.

REFERENCES

[1] H. Ammari, R. Griesmaier, and M. Hanke, *Identification of small inhomogeneities: asymptotic factorization*, Math. Comp., to appear.

[2] G. Bal, *Reconstructions in impedance and optical tomography with singular interfaces*, Inverse Problems, 21 (2005), pp. 113–131.

[3] N. Bourbaki, *Topological Vector Spaces, Chapters* 1–5, Elem. Math. (Berlin), Springer-Verlag, Berlin, 1987.

[4] M. Brühl *Gebietserkennung in der elektrischen Impedanztomographie*, Dissertation, Universität Karlsruhe, Karlsruhe, Germany, 1999.

[5] M. Brühl and M. Hanke, *Numerical implementation of two noniterative methods for locating inclusions by impedance tomography*, Inverse Problems, 16 (2000), pp. 1029–1042.

[6] M. Brühl, *Explicit characterization of inclusions in electrical impedance tomography*, SIAM J. Math. Anal., 32 (2001), pp. 1327–1341.

[7] M. Brühl, M. Hanke, and M. S. Vogelius, *A direct impedance tomography algorithm for locating small inhomogeneities*, Numer. Math., 93 (2003), pp. 635–654.

[8] M. Costabel, *Boundary integral operators for the heat equation*, Integral Equations Operator Theory, 13 (1990), pp. 488–552.

[9] M. Costabel, V. J. Ervin, and E. P. Stephan, *Symmetric coupling of finite elements and boundary elements for a parabolic-elliptic interface problem*, Quart. Appl. Math., 48 (1990), pp. 265–279.

[10] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology. Volume* 5: *Evolution Problems* I, Springer-Verlag, Berlin, 1992.

[11] H. W. Engl, M. Hanke, and A. Neubauer, *Regularization of Inverse Problems*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[12] B. Gebauer, *The factorization method for real elliptic problems*, Z. Anal. Anwend., 25 (2006), pp. 81–102.

[13] M. Hanke and M. Brühl, *Recent progress in electrical impedance tomography*, Inverse Problems, 19 (2003), pp. S65–S90.

[14] M. Hanke and B. Schappel, *The Factorization Method for Electrical Impedance Tomography in the Half Space*, submitted.

[15] N. Hyvönen, *Characterizing inclusions in optical tomography*, Inverse Problems, 20 (2004), pp. 737–751.

[16] A. Kirsch, *Characterization of the shape of a scattering obstacle using the spectral data of the far field operator*, Inverse Problems, 14 (1998), pp. 1489–1512.

[17] A. Kirsch, *The factorization method for a class of inverse elliptic problems*, Math. Nachr., 278 (2005), pp. 258–277.

[18] A. Kirsch, *The factorization method for Maxwell's equations*, Inverse Problems, 20 (2004), pp. S117–134.

[19] R. Kress, *A factorisation method for an inverse Neumann problem for harmonic vector fields*, Georgian Math. J., 10 (2003), pp. 549–560.

[20] J.-L. Lions, *Equations Differentielles Operationnelles et Problèmes aux Limites*, Springer-Verlag, Berlin, 1961.

[21] J.-L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications* I, Springer-Verlag, Berlin, 1972.

[22] J.-L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications* II, Springer-Verlag, Berlin, 1972.

[23] R. C. MacCamy and M. Suri, *A time-dependent interface problem for two-dimensional eddy currents*, Quart. Appl. Math., 44 (1987), pp. 675–690.

[24] B. Schappel *Electrical impedance tomography in the half space: Locating obstacles by electrostatic measurements on the boundary*, in Proceedings of the 3rd World Congress on Industrial Process Tomography, Banff, VCIPT, 2003, pp. 788–793.

[25] B. Schappel *Die Faktorisierungsmethode für die elektrische Impedanztomographie im Halbraum*, Dissertation, Joh. Gutenberg-Universität Mainz, Mainz, Germany, 2005. Available online at http://nbn-resolving.de/urn/resolver.pl?urn=urn:nbn:de:hebis:77-7427.