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Monotony based inclusion detection in EIT for realistic electrode models

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Abstract. In electrical impedance tomography (EIT), there is a monotony relation between the conductivity of an imaging subject and the EIT measurements. Such monotony relations can be used for inclusion detection methods. However, up to now, monotony-based methods were only known to reconstruct an upper or lower bound of inclusions.

In a recent preprint, the authors showed that, for continuous boundary data, monotony based methods are in fact capable of reconstructing the exact shape of inclusions. In this work, we discuss how to extend our results to the practically relevant complete electrode model setting and study the possibility of rigorous resolution guarantees.

1. Introduction

The aim of electrical impedance tomography is to image the conductivity inside a conducting subject by boundary current-voltage measurements. In this work we focus on detecting the shape of conducting anomalies (aka inclusions) within a practically relevant setting. To obtain current-voltage measurements we consider the following setting, cf. figure 1.

We are given a conducting object with a set of separated electrodes attached to its boundary. A prescribed current pattern is applied to the electrodes and yields measurement data of the resulting potential on each electrode (up to a real constant representing the ground level). The boundary of the object is isolated, currents can only flow trough the electrodes. The electrodes are highly conducting, such that the potentials can be assumed to be constant on each electrode. Furthermore we assume that the contact impedance of a thin contact layer between each electrode and the object is known.

Mathematically, for this setting, the current-voltage measurements can be modeled by the complete electrode model (CEM) as follows. Let $\Omega \in \mathbb{R}^n$ be a bounded domain with smooth boundaries, $\sigma \in L^\infty_+(\Omega)$ (where $L^\infty_+(\Omega)$ denotes the subspace of $L^\infty(\Omega)$-functions with positive essential infima), the electrodes are identified with their corresponding boundary parts $e_1, e_2, \ldots, e_L \subseteq \partial\Omega$ and the contact impedances are real values $z_1, z_2, \ldots, z_L \in \mathbb{R}$. Then, for a prescribed current pattern $I = (I_l)_{l=1}^L \in \mathbb{R}^L_\diamond$ there exists a unique pair $(u, U) \in H^1_\diamond(\Omega) \times \mathbb{R}^L_\diamond$ (where the subscript $\diamond$ denotes the subspace of $H^1(\Omega)$-functions resp. $\mathbb{R}^L$-elements with vanishing integral mean resp. sum of components), such that

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega, \tag{1}$$

$$u|_{\partial\Omega} + z_l \sigma \partial_\nu u|_{\partial\Omega} = U_l \quad \text{on } e_l, \quad 1 \le l \le L, \tag{2}$$

$$\sigma \partial_\nu u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega \setminus \bigcup_{l=1}^{L} e_l, \tag{3}$$

$$\int_{e_l} \sigma \partial_\nu u|_{\partial\Omega} \mathrm{d}S = I_l, \tag{4}$$

where $\sigma \partial_\nu u|_{\partial\Omega}$ denotes the Neumann boundary values and $u|_{\partial\Omega} \in L^2_\diamond(\partial\Omega)$ is the trace of u. By this formulation of the CEM, we can identify the current-voltage measurements for an object with conductivity σ, with the linear symmetric mapping $R(\sigma) : \mathbb{R}^L_\diamond \to \mathbb{R}^L_\diamond, I \mapsto U$, cf. e.g. [1].

2. Monotony based methods

2.1. The monotony relation

Theorem 1. *Let $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$ and (v, V) be the solution of the CEM for the conductivity σ_2. Then*

$$\int_\Omega (\sigma_1 - \sigma_2)|\nabla v|^2 \mathrm{d}x \ge \langle I, (R(\sigma_2) - R(\sigma_1))\, I \rangle \ge \int_\Omega \frac{\sigma_2}{\sigma_1}(\sigma_1 - \sigma_2)|\nabla v|^2 \mathrm{d}x, \tag{5}$$

where $\langle \cdot, \cdot \rangle : \mathbb{R}^L \times \mathbb{R}^L \to \mathbb{R}^L$ denotes the euclidean inner product.

Note that there is an analogue monotony results for the continuous boundary model that seem to go back to Ikehata, Kang, Seo and Sheen [2, 3].

Proof. It is known, that $(u, U) \in H^1_\diamond(\Omega) \times \mathbb{R}^L_\diamond$ is the solution of the CEM for the conductivity σ and the prescribed current pattern $I \in \mathbb{R}^L_\diamond$ if and only if the following variational formulation

$$B_\sigma((u, U), (v, V)) := \int_\Omega \sigma \nabla u \cdot \nabla v \mathrm{d}x + \sum_{l=1}^{L} \frac{1}{z_l} \int_{e_l} (u|_{\partial\Omega} - U_l)(v|_{\partial\Omega} - V_l) \mathrm{d}S = \sum_{l=1}^{L} I_l V_l = \langle I, V \rangle$$

holds for all $(v, V) \in H^1_\diamond(\Omega) \times \mathbb{R}^L_\diamond$, cf. e.g. [1].

With this equivalent formulation of the CEM-System we obtain the first inequality. Let (u, U) resp. (v, V) be the solution of the CEM for the conductivity σ_1 resp. σ_2.

$$\begin{aligned}
0 &\le \int_\Omega \sigma_1 |\nabla(u - v)|^2 \mathrm{d}x + \sum_{l=1}^{L} \frac{1}{z_l} \int_{e_i} \left((u|_{\partial\Omega} - U_i) - (v|_{\partial\Omega} - V_i) \right)^2 \mathrm{d}S \\
&= B_{\sigma_1}((u, U), (u, U)) + B_{\sigma_2}((v, V), (v, V)) - 2B_{\sigma_1}((u, U), (v, V)) + \int_\Omega (\sigma_1 - \sigma_2)|\nabla v|^2 \mathrm{d}x \\
&= \langle I, (R(\sigma_1) - R(\sigma_2)) I \rangle + \int_\Omega (\sigma_1 - \sigma_2)|\nabla v|^2 \mathrm{d}x.
\end{aligned}$$

Thus, we obtain the first inequality.

By interchanging (u, U) and (v, V) as well as σ_1 and σ_2, the equation above yields:

$$\begin{aligned}
&\langle I, (R(\sigma_2) - R(\sigma_1)) I \rangle \\
&= \int_\Omega (\sigma_1 - \sigma_2)|\nabla u|^2 \mathrm{d}x + \int_\Omega \sigma_2 |\nabla(v - u)|^2 \mathrm{d}x + \sum_{l=1}^{L} \frac{1}{z_l} \int_{e_i} \left((v|_{\partial\Omega} - V_i) - (u|_{\partial\Omega} - U_i) \right)^2 \mathrm{d}S \\
&= \int_\Omega \sigma_1 \left| \nabla u - \frac{\sigma_2}{\sigma_1} \nabla v \right|^2 + \frac{\sigma_2}{\sigma_1}(\sigma_1 - \sigma_2)|\nabla v|^2 \mathrm{d}x + \sum_{l=1}^{L} \frac{1}{z_l} \int_{e_i} \left((v|_{\partial\Omega} - V_i) - (u|_{\partial\Omega} - U_i) \right)^2 \mathrm{d}S.
\end{aligned}$$

Since σ_1 and z_l are positive the second inequality follows. □

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A simple consequence is

$$\sigma_1 \leq \sigma_2 \quad \Longrightarrow \quad R(\sigma_1) \geq R(\sigma_2), \tag{6}$$

where the first inequality is meant in the pointwise (a.e.) on Ω sense, and the second one is to be understood in the sense that $R(\sigma_1) - R(\sigma_2)$ has only non-negative eigenvalues.

2.2. A simple monotony based shape reconstruction method

We start by presenting the main idea on a simple case. Let the conductivity σ of the conducting object Ω be given by $\sigma = 1 + \chi_D \in L^\infty_+(\Omega)$, where $D \subseteq \Omega$ is an open set. Then (6) yields that

$$D_{\mathrm{R}} := \bigcup_{\epsilon>0,\ x\in\Omega,\ R(1+\chi_{B_\epsilon(x)})\geq R(\sigma)} B_\epsilon(x)$$

is an upper bound of the inclusion D.

Such monotony based reconstruction methods were numerically studied by Tamburrino and Rubinacci [4]. In the recent preprint [5] (see also [6]), the authors showed that, for continuous boundary data, monotony methods are actually capable of reconstructing the exact shape D (if D has connected complement) and that they allow a fast linearized but still exact implementation.

3. Resolution guaranties for monotony based methods

In the realistic CEM-model with finitely many electrodes, we can no longer expect the monotony based method to reconstruct the exact shape of an inclusion. However, we will now present a new approach that allows us to guarantee certain detectability properties (or resolution guarantees) for the CEM-model, even with imprecisely known backgrounds and measurement noise.

For a resolution given by a partition $(\omega_i)_{i=1}^N$, $\dot{\bigcup}_{i=1}^n \omega_i = \Omega$ $(w_i \neq \emptyset)$, we will show how to test whether a set of electrodes $e_1, e_2, \ldots, e_L$ is suitable to ensure some reconstruction guaranties or not.

Similar as in section 2.2, let us consider the case where the conductivity σ is given by $\sigma = \sigma_0 + \chi_D\kappa \in L^\infty_+(\Omega)$ with $\sigma_0, \kappa \in L^\infty(\Omega)$, $\kappa \geq c$ (a.e.) on a set $D \subseteq \Omega$ and $|\sigma_0 - 1| \leq \epsilon$ for some small constants $c > 2\epsilon > 0$.

3.1. A basic resolution guarantee

For an inclusion detection method that yields a reconstruction D_R to D, we say that it fulfills a resolution guarantee with respect to a partition $(\omega_i)_{i=1}^N$ if the following holds.

If a piece ω_i of the resolution partition is completely overlapped by an inclusion D, the method will mark this piece as a part of the reconstruction D_R. On the other hand, if there is no inclusion, no piece of the resolution partition will be marked. Mathematically speaking, this means that

(i) $\omega_i \subseteq D$ implies $\omega_i \subseteq D_R$ for $i \in \{1, 2, \ldots, N\}$, and
(ii) $D = \emptyset$ implies $D_R = \emptyset$.

3.2. Verification of resolution guarantee

Theorem 2. *The inclusion detection method*

$$D_R := \bigcup_{i\in\{1,2,\ldots,N\},\ R(1-\epsilon+c\chi_{\omega_i})\geq R(\sigma)} \omega_i \tag{7}$$

fulfills the resolution guarantee of 3.1 if and only if

$$R(1 - \epsilon + c\chi_{\omega_i}) \not\geq R(1 + \epsilon), \ \textit{for all } i \in \{1, 2, \ldots, N\}. \tag{8}$$

Proof. By the monotony relation (6), assertion (i) is always satisfied, since $\omega_i \subseteq D$ implies $1 - \epsilon + c\chi_{\omega_i} \leq \sigma$ for all $i \in \{1, 2, \ldots, N\}$. Hence, only assertion (ii) needs to be verified.

First, we assume that $D = \emptyset$. Thus, $\sigma = \sigma_0 \leq 1 + \epsilon$ and by the monotony relation (6) we obtain that $R(\sigma) \geq R(1+\epsilon)$. Hence, $R(1 - \epsilon + c\chi_{\omega_i}) \ngeq R(1+\epsilon)$ for all $i \in \{1, 2, \ldots, N\}$ implies $R(1 - \epsilon + c\chi_{\omega_i}) \ngeq R(\sigma)$ for all $i \in \{1, 2, \ldots, N\}$.

For the converse, we assume that $R(1-\epsilon+c\chi_{\omega_i}) \geq R(1+\epsilon)$ for at least one $i \in \{1, 2, \ldots, N\}$ and the conductivity be given by $\sigma = 1 + \epsilon$. For this case $D_R \supseteq \omega_i \neq \emptyset$.

□

Theorem 2 offers a criterion to check, if the resolution and the electrode setting fit together, such that the reconstruction method, given by (7), fulfills the resolution guarantee or not.

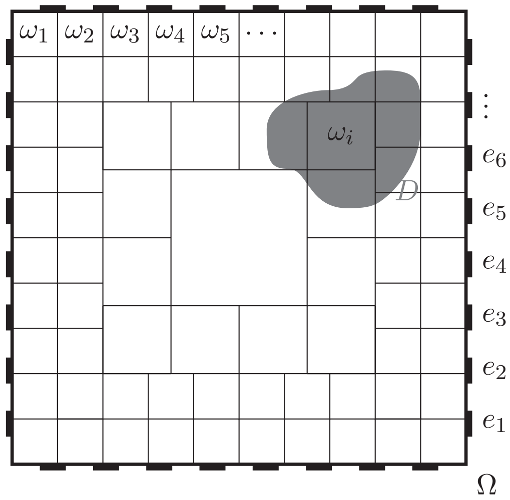

Figure 1. CEM-setting with a potentially resolution $(\omega_i)_{i=1}^N$ and sample inclusion D.

3.3. Presence of measurement errors

We consider noisy measurement data $R^\delta(\sigma)$ with $\|R(\sigma) - R^\delta(\sigma)\| = \delta$. If the reconstruction method with exact data fulfills the resolution guarantee, the test criteria (8) also yield a bound $\Delta > 0$ for the measurement error δ, such that the resolution guarantee is still valid for measurement with noise up to Δ. This bound depend on the smallest eigenvalue of all the matrices $R(1 - \epsilon + c\chi_{\omega_i}) - R(1+\epsilon)$ from the test criterion.

References

[1] Somersalo E, Cheney M and Isaacson D 1992 *SIAM J. Appl. Math.* **52** 1023–1040
[2] Ikehata M 1998 *J. Inverse Ill-Posed Probl.* **6** 127–140
[3] Kang H, Seo J K and Sheen D 1997 *SIAM J. Math. Anal.* **28** 1389–1405
[4] Tamburrino A and Rubinacci G 2002 *Inverse Problems* **18** 1809–1829
[5] Harrach B and Ullrich M submitted for publication, online available at `http://www.mathematik.uni-wuerzburg.de/~harrach/publikationen.html`
[6] Harrach B and Ullrich M 2010 *AIP Conf. Proc.* **1281** 1975–1978