Optimisation Problems in Non-Life Insurance

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The Problem

Denote by \( Y_n \) the result of an insurance company in year \( n \). We suppose \( Y_n \in \mathbb{N} \) and the variables \( \{ Y_n \} \) are iid. Without any control the surplus of the insurer becomes

\[
X_n^0 = x + \sum_{k=1}^{n} Y_k .
\]

The insurer pays a dividend \( U_n \) at the end of year \( n \). Then the pre-dividend process becomes

\[
X_n^U = x + \sum_{k=1}^{n} (Y_k - U_{k-1}) .
\]

An admissible dividend fulfils \( 0 \leq U_n \leq X_n^U \).
Let $\tau^U = \inf\{n : X_n^U < 0\}$ be the time of ruin. The value of a dividend strategy is

$$V^U(x) = \mathbb{E}\left[\sum_{k=0}^{\tau^U-1} U_k r^k\right]$$

for some discounting variable $r \in (0, 1)$.

The goal is to maximise the value

$$V(x) = \sup_{U} V^U(x),$$

and — if it exists — to find an optimal strategy.
Bellman’s Equation

Suppose we know already the dividend $U_0 = u$ at time zero. This is the same as starting with initial capital $x - u$ and not paying a dividend at time zero. At time 1 one has the capital $x - u + Y_1$. The optimal dividend value at time 1 is $V(x - u + Y_1)$. The value at time zero is $rV(x - u + Y_1)$. Thus the value at time zero is

$$u + rE[V(x - u + Y_1)].$$

We have the maximal value, thus

$$V(x) = \sup_{0 \leq u \leq x} u + rE[V(x - u + Y_1)].$$
Some Properties

The dividend strategy \( U'_0 = x - \lfloor x - U_0 \rfloor \) and
\( U'_n = \lfloor X_n \rfloor - \lfloor X_n - U_n \rfloor \) leads to the process \( X'_n = \lfloor X_n \rfloor \).
Ruin occurs at the same time, but dividends are paid out earlier. Thus \( V^U(x) \leq V^{U'}(x) \). If \( \{X^n_U : n \geq 1\} \neq \{X^n_{U'} : n \geq 1\} \), then \( V^U(x) < V^{U'}(x) \).

Moreover, \( V(x) = V(\lfloor x \rfloor) + (x - \lfloor x \rfloor) \). Thus it is enough to consider integer values only. In particular, there exists \( u(x) \), such that

\[
V(x) = \sup_{0 \leq u \leq x} u + r \mathbb{E}[V(x-u+Y_1)] = u(x) + r \mathbb{E}[V(x-u(x)+Y_1)].
\]
A Lower Bound

Use the following strategy. Let $U_0 = x$ and $U_n = Y_n^+$. Then $X_n = Y_n$ and ruin occurs at the first time where $Y_n < 0$. The value of this strategy is

$$x + \frac{r \mathbb{E}[Y^+]}{1 - p_+ r} = V^U(x) \leq V(x),$$

where $p_+ = \mathbb{P}[Y_k \geq 0]$. 
An Upper Bound

Suppose we use the dividend strategy \( U_0 = x \) and \( U_n = Y_n^+ \), but we do not stop at ruin. The aggregate dividend payment of this strategy is an upper bound for the aggregate dividend payment of any admissible strategy. Hence any admissible strategy must have a smaller value. Thus we get the upper bound

\[
V(x) \leq x + \frac{r\mathbb{E}[Y^+]}{1 - r}.
\]
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Consider the Strategy $U_n = u(X_n)$. Then the process

$$\left\{ V(X_n) r^n + \sum_{k=0}^{n-1} r^k U_k \right\}$$

is a martingale.

In particular,

$$V(x) = \lim_{n \to \infty} \mathbb{E}\left[ V(X_{\tau \wedge n}) r^{\tau \wedge n} + \sum_{k=0}^{(\tau \wedge n)-1} r^k U_k \right] = \mathbb{E}\left[ \sum_{k=0}^{\tau-1} r^k U_k \right].$$
The same argument shows that if $U_n$ is not a maximiser of

$$u + r\mathbb{E}[V(X_n - u + Y_{n+1})]$$

then $\{U_n\}$ is not an optimal strategy. If there are several maximisers $u$ we choose $u(x)$ the largest value. We can characterise the strategy in the following way

$$u(x) = \sup\{n : V(x) = V(x - n) + n\}.$$ 

Thus $u(x) = 0$ if $V(x) > V(x - 1) + 1$. 

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Suppose \( u(x) = 0 \). Then

\[
x + \frac{r \mathbb{E}[Y^+]}{1 - p + r} \leq V(x) = r \sum_{j=-x}^{\infty} p_j V(x + j)
\]

\[
\leq r \sum_{j=-x}^{\infty} p_j \left( x + j + \frac{r \mathbb{E}[Y^+]}{1 - r} \right)
\]

\[
\leq rx + r \left( \mathbb{E}[Y^+] + \frac{r \mathbb{E}[Y^+]}{1 - r} \right)
\]

\[
= rx + \frac{r \mathbb{E}[Y^+]}{1 - r}.
\]
The Optimal Strategy

Solving for \( x \) gives

\[
x \leq \frac{r^2 \mathbb{E}[Y^+](1 - p_+)}{(1 - r)^2(1 - p_+ r)}.
\]

Thus there is a barrier \( x_0 = \sup\{x : u(x) = 0\} \). The post-dividend process fulfils \( X_n - U_n \leq x_0 \), thus ruin will occur almost surely.
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De Finetti’s Example

Let $\mathbb{P}[Y = 1] = 1 - \mathbb{P}[Y = -1] = p$. Consider first the case where all capital is paid out as dividend. Plugging in the function $V(x) = x + \frac{pr}{(1 - pr)}$ into Bellman’s equation

$$V(x) = \max\{r(pV(x + 1) + (1 - p)V(x - 1)), V(x - 1) + 1\}$$

yields that this is the solution whenever

$$1 \geq r \left( p + \sqrt{p(1 - p)} \right).$$
De Finetti’s Example: Equations

Let \( x_1 = \sup \{ n : u(n) = 0 \} \) and suppose \( x_1 > 0 \) and let \( n \leq x_1 \). Then we have the equations

\[
V(n + 1) = \frac{1}{pr} V(n) - \frac{1 - p}{p} V(n - 1),
\]

if \( n \geq 1 \),

\[
V(1) = \frac{1}{pr} V(0),
\]

and

\[
V(x_1 + 1) = V(x_1) + 1.
\]
The solution is

\[ V(n) = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2} V(0). \]

where

\[ \lambda_{1/2} = \frac{1 \pm \sqrt{1 - 4p(1 - p)r^2}}{2pr} = \frac{r^{-1} \pm \sqrt{r^{-2} - 1 + (2p - 1)^2}}{2p}. \]
de Finetti’s Example: Solution

The equation

\[ V(x_1 + 1) = V(x_1) + 1 \]

yields \( V(0) \). Maximising \( V(0) \) gives the solution on \([0, x_1 + 1]\). It turns out that this solution solves Bellman’s equation, thus the optimal strategy is a barrier strategy.
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The Classical Risk Model

\[ X_t^1 = x + ct - \sum_{i=1}^{N_t} Y_i \]

- **\( x \):** initial capital
- **\( c \):** premium rate
- \( \{N_t\} \): Poisson process with rate \( \lambda \)
- \( \{Y_i\} \): iid, independent of \( \{N_t\} \)
- \( 0 = T_0 < T_1 < T_2 < \cdots \): claim times
- \( G(y) \): distribution of \( Y_i \), \( G(0) = 0 \)
- \( \mu_n = \mathbb{E}[Y_i^n], \quad \mu = \mu_1 \)
Proportional Reinsurance

The insurer can buy proportional reinsurance. Choosing retention level $b$ the premium rate $c - c(b)$ has to be paid. For a strategy $\{b_t\}$ the surplus process becomes

$$X_t^b = x + \int_0^t c(b_s) \, ds - \sum_{k=1}^{N_t} b_{T_k} - Y_k.$$

We let $\tau^b = \inf\{t : X_t^b < 0\}$ denote the time of ruin and by $\psi^b(x) = \mathbb{P}[\tau^b < \infty]$ the ruin probability. We define the value function

$$\delta(x) = 1 - \inf_b \psi^b(x).$$
Proportional Reinsurance

We assume:

- $G(x)$ is continuous.
- $c(b)$ continuous.
- $c(b)$ increasing.
- $c(1) = c$.
- $c(0) < 0$. 
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The Hamilton-Jacobi-Bellman Equation

If the function $\delta(x)$ is ‘nice’ it fulfils the Hamilton-Jacobi-Bellman equation:

$$\sup_{b \in [0,1]} c(b)\delta'(x) + \lambda \left[ \int_0^{x/b} \delta(x - by) \, dG(y) - \delta(x) \right] = 0.$$ 

Reformulation yields

$$\delta'(x) = \lambda \inf_{b \in (0,1]} \frac{\delta(x) - \int_0^{x/b} \delta(x - by) \, dG(y)}{(c(b))^+}.$$ 

Continuity gives that there is a value $b(x)$ where the sup (inf) is taken.
Denote by $\delta_1(x)$ the survival probability without reinsurance. Then $f_0(x) = c \delta_1(x) / (c - \lambda \mu)$ fulfills

$$f_0'(x) = \lambda \frac{f_0(x) - \int_0^x f_0(x - y) \, dG(y)}{c(1)} , \quad f_0(0) = 1 .$$

Define recursively

$$f_{n+1}'(x) = \lambda \inf_{b \in (0,1]} \frac{f_n(x) - \int_0^{x/b} f_n(x - by) \, dG(y)}{(c(b))^+} , \quad f_{n+1}(0) = 1 .$$

Then $f_n(x)$ is decreasing in $n$ and converges to some function $f(x)$ fulfilling the Hamilton-Jacobi-Bellman equation. Thus there is a solution to the Hamilton-Jacobi-Bellman equation.
Let \( f(x) \) be an increasing solution to the Hamilton-Jacobi-Bellman equation. Then the process

\[
f(X_t) - \int_0^t \left\{ c(b_s)f'(X_s) + \lambda \left[ \int_0^{X_s/b_s} f(X_s - b_s y) \, dG(y) - f(X_s) \right] \right\} \, ds
\]

is a martingale.

Because \( f(x) \) solves the Hamilton-Jacobi-Bellman equation the process \( \{f(X_t)\} \) is a supermartingale.
We get 

\[ f(x) = f(X_0^b) \geq \mathbb{E}[f(X_t^b)] \rightarrow f(\infty)\delta^b(x). \]

Thus \( f(x) \) is bounded.

Let \( b_t^* = b(X_t^*) \). Then \( f(X_s^*) \) is a martingale and

\[ f(x) = \mathbb{E}[f(X_t^*)] \rightarrow f(\infty)\delta^*(x) = f(\infty)\delta(x). \]

Thus \( \delta(x) \) really solves the Hamilton-Jacobi-Bellman equation and \( \{b_t^*\} \) is an optimal strategy.
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Exponentially Distributed Claim Sizes

In the following two examples we choose \( \lambda = \mu = 1 \) and \( c(b) = 1.7b - 0.2 \).

**Example**

Exponentially distributed claim sizes:

\[
G(y) = 1 - e^y.
\]

\[
\delta_1(x) = 1 - 2e^{-x/3}/3. \text{ From Waters (1983)}
\]

\[
b^R = \left(1 - \frac{1.5}{1.7}\right)\left(1 + \frac{1}{\sqrt{1.7}}\right) = 0.504847.
\]

\( \delta^R(x) \) known explicitly.
Exponentially Distributed Claim Sizes: $\delta(u)$
Exponentially Distributed Claim Sizes: $b^*(u)$
Example

Pareto distributed claim sizes:

\[ G(y) = 1 - (1 + y)^{-2}. \]

Asymptotically optimal strategy?

\[ \psi^b(x) \approx \frac{1}{0.7b - 0.2} \frac{b}{1 + x/b} = \frac{1}{(0.7 - 0.2/b)(1 + x/b)}. \]

Minimised for \( b = 0.4x/(0.7x - 0.2) \).

\( b^R = 4/7 \).
Pareto Distributed Claim Sizes: $\delta(u)$
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Pareto Distributed Claim Sizes: $b^*(u)$
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Optimal Dividends in a Diffusion Approximation

Consider the Diffusion Approximation

\[ X_t^0 = x + \sigma W_t + mt \]

where \( \{ W_t \} \) is a standard Brownian motion. 
A dividend strategy is an increasing process \( \{ D_t \} \) with \( D_0 \geq 0 \). 
The surplus becomes \( X_t^D = X_t^0 - D_t \). 
The value of a strategy is

\[ V^D(x) = \mathbb{E} \left[ \int_0^\tau - e^{-\delta t} \, dD_t \right] . \]

We let \( V(x) = \sup_D V^D(x) \).
The value function then fulfils

$$\max\{1 - V'(x), \frac{1}{2}\sigma^2 V''(x) + mV'(x) - \delta V(x)\} = 0.$$ 

We know that $V(0) = 0$.

**Optimal Strategy:** Do not pay dividend if $V'(x) > 1$. At a point where $V'(x) = 1$ reflect the process.
The Value Function

Intuition: Barrier strategy close to zero.
Solve $\frac{1}{2}\sigma^2 f''(x) + mf'(x) - \delta f(x) = 0$ with $f(0) = 0$.

$$f(x) = e^{\theta_1 x} - e^{\theta_2 x}, \quad V(x) = Cf(x).$$

Barrier in $a$ yields $V'(a) = 1$. Solution is maximal if

$$C = \frac{1}{f'(a)}$$

is maximal, i.e. $f''(a) = 0$. Note that $V(a) = m/\delta$.

The optimal solution is

$$V(x) = \begin{cases} f(x)/f'(a), & \text{if } x \leq a, \\ f(a) + x - a, & \text{otherwise.} \end{cases}$$
By Itô’s formula

\[ 0 = V(X_\tau) e^{-\delta \tau} = V(X_0) + \int_0^\tau \sigma V'(X_s) e^{-\delta s} \, dW_s \]
\[ + \int_0^\tau \left[ \frac{1}{2} \sigma^2 V''(X_s) + m V'(X_s) - \delta V(X_s) \right] e^{-\delta s} \, ds \]
\[ - \int_0^\tau e^{-\delta s} V'(X_s) \, dD_s \]
\[ \leq V(X_0) + \int_0^\tau \sigma V'(X_s) e^{-\delta s} \, dW_s - \int_0^\tau e^{-\delta s} \, dD_s . \]

Taking expected values yields the result. Using the optimal strategy equality holds.
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Consider the classical model

\[ X_t^0 = x + ct - \sum_{k=1}^{N_t} Y_k. \]

With dividend payments we have \( X_t^D = X_t^0 - D_t. \)
Here we get the equation

\[
\max\left\{ 1 - V'(x), \right. \\
\left. cV'(x) - \delta V(x) + \lambda \left[ \int_0^x V(x - y) \, dG(y) - V(x) \right] \right\} = 0.
\]
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The Optimal Strategy

It turns out: The solution is not unique and not necessarily differentiable. But the jumps of $V'(x)$ are all upwards with $V'(x-) = 1$. The correct solution is the minimal solution.

The optimal strategy is the following:
No dividend if $V'(x-) > 1$.
Dividends at rate $c$ if $x$ is an upper boundary point of the set where no dividend is paid.
A dividend of size $\sup\{y : V(x) = V(x - y) + y\}$.
Dividends in the Classical Model

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We now consider the process $X^D_t = X^0_t - D_t + A_t$ where $A_t$ is the minimal increasing process such that $X^D_t \geq 0$ for all $t$. That is, the process is reflected in 0 and $A_t$ is the cumulative investments until time $t$. The value of a strategy $\{D_t\}$ is

$$V^D(x) = \mathbb{E}\left[ \int_0^\infty e^{-\delta t} \, dD_t - \theta \int_0^\infty e^{-\delta t} \, dA_t \right].$$

The value function is $V(x) = \sup_D V^D(x)$

If $\theta < 1$ then $V(x) = \infty$.

If $\theta = 1$ then the optimal process fulfils $X^*_t = 0$.

We assume $\theta > 1$. 
Concavity of $V(x)$

Let $x \neq y$ and $\alpha \in (0, 1)$, $\beta = 1 - \alpha$. Let $D^v_t$ and $A^v_t$ be a strategy for initial capital $v$. We let $z = \alpha x + \beta y$, $D_t = \alpha D^x_t + \beta D^y_t$, $B_t = \alpha A^x_t + \beta A^y_t$.

For initial capital $z$ we get the process

$$X^z_t = \alpha X^x_t + \beta X^y_t \geq 0.$$

We conclude that $A^z_t \leq B^z_t$.

Thus $V(z) \geq V^z(z) \geq \alpha V^x(x) + \beta V^y(y)$. Taking the supremum we find

$$V(\alpha x + \beta y) \geq \alpha V(x) + \beta V(y).$$
The Diffusion Approximation

The value function fulfills

$$\max\{1 - V'(x), \frac{1}{2}\sigma^2 V''(x) + mV'(x) - \delta V(x)\} = 0.$$ 

Moreover, $V'(0) \leq \theta$. The strategy becomes a barrier strategy at a barrier $a$ with $V'(a) = 1$.

The solution is of the form

$$V(x) = C_1 e^{\theta_1 x} + C_2 e^{\theta_2 x}.$$ 

The constants are determined such that $V'(0) = \theta$, $V'(a) = 1$ and $V(0)$ becomes maximal.
The value function fulfills

\[
\max \left\{ 1 - V'(x), cV'(x) - \delta V(x) \right\} + \lambda \left[ \int_0^x V(x - y) \, dG(y) + \int_x^\infty (V(0) - \theta(y - x)) \, dG(y) - V(x) \right] = 0.
\]

The optimal strategy is a barrier strategy with a barrier at \(a\).

Problem: \(1 \leq V'(0) < \theta\). Thus no condition at zero.
Calculate the value of a barrier strategy, and maximise \(V(0)\).
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Optimal Investment

Let $Z_t = \exp\{(r - \frac{1}{2}\sigma_I^2)t + \sigma_I W_t^I\}$ be the value of some risky asset or some portfolio. The insurer invests the amount $A_t$ into the risky asset. Then the surplus fulfils

$$dX_t^A = dX_t^0 + rA_t \, dt + \sigma_I A_t \, dW_t^I.$$

We want to minimise the ruin probability.

The corresponding HJB equation becomes

$$\sup_{A \in \mathbb{R}} \mathcal{A} \delta(x) + rA \delta'(x) + \frac{1}{2} \sigma_I^2 A^2 \delta''(x) = 0.$$ 

Here $\mathcal{A}$ is the generator of the uncontrolled process $\{X_t^0\}$. 

Let \( \{ Y_t \} \) be another independent surplus process. The insurer can buy part of the risk. Then the surplus fulfils

\[
X_t^b = X_t^0 + \int_0^t b_s dY_s ,
\]

where \( b_t \in [0, 1] \). We again minimise the ruin probability. The corresponding HJB equation is

\[
\sup_{b\in[0,1]} \Lambda_X \delta(x) + bc_Y \delta'(x) + \lambda_Y \left[ \int_0^{x/b} \delta(x-by) dG_Y(y) - \delta(x) \right] = 0 .
\]
Consider a diffusion approximation with reinsurance. We only allow 'cheap' reinsurance. We want to maximise

\[ \mathbb{E} \left[ \int_0^T e^{-\delta s} X_s^b \, ds \right] . \]

The HJB equation is

\[ x + \sup_b mbV'(x) + \frac{1}{2} \sigma^2 b^2 V''(x) - \delta V(x) = 0. \]

It turns out that ruin will never occur (\(b(x) \to 0\) as \(x \to 0\)).
Maximising the Surplus

For a classical model the HJB equation is

\[ x + \sup_{b} c(b)V'(x) - \delta V(x) + \lambda \left[ \int_{0}^{x/b} V(x - by) \, dG(y) - V(x) \right]. \]

Usually, there is a positive probability for ruin.
Consider the process with reinsurance \( \{X^b_t\} \). Suppose the insurer has to reinvest a possible deficit. The surplus becomes \( Y_t^b = X_t^b + A_t^b \). The value of the strategy is

\[
V^b(x) = \mathbb{E}\left[ \int_0^\infty e^{-\delta t} \, dA_t \right].
\]

The value function is \( V(x) = \inf_b V^b(x) \).

\( \delta > 0 \): Future reinvestment preferred to present reinvestment.

\( \delta = 0 \): Indifferent between future and present reinvestment.

\( \delta < 0 \): Present reinvestment preferred to future reinvestment.
Minimal Reinvestment

\( V(x) \) fulfils for \( x \geq 0 \) the HJB equation

\[
\inf_b c(b) V'(x) + \lambda \left[ \int_0^\infty V(x - by) \, dG(y) - V(x) \right] - \delta V(x) = 0 ,
\]

with \( V(x) = V(0) - x \) for \( x < 0 \).

The solution we look for is the maximal decreasing solution.
References


References

