Illiquid financial market models and absence of arbitrage

Diplomarbeit

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von
Matthias Paul Jüttner

Die Arbeit wurde von
Herrn Prof. Dr. Achim Klenke und
Herrn Prof. Dr. Christoph Kühn betreut.
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Introduction

Banks borrow short and lend long. This gives rise to three obvious risks - credit risk on the lending, as well as long-term interest rate risk and liquidity risk from the mismatch between the term structure of the assets and liabilities. Besides we know two additional categories: market and operational risk. The international agreement Basel II sets out a regulatory framework for most of the risks but is almost silent on the liquidity. Within their trading books banks hold trading assets backed by regulatory capital calculated for example on a Value at Risk basis that assumes that the market risk from those assets could be mitigated within a short period of time. This in turn assumes that those assets are liquid, although this was probably not true for the bonds of the Argentine Republic in the year 2000 or for most of the securities during the Long-Term Capital Management\(^1\) crisis in 1998. The actual turbulences in the worldwide stock markets triggered by the development in the housing market of the United States, especially the decline in lending standards, is a more recent example for illiquidity. Therefore, after credit and market risk, liquidity risk is the next important issue faced by the financial sector - ”Next” in the sense that the study of liquidity is yet far less advanced.

First of all we have to clarify what is meant by liquidity. We are able to distinguish between market and funding liquidity. Market liquidity is the ease of trading an asset on the security’s market or more precisely is understood as the ability to rapidly exchange a non-cash asset into a cash-asset at minimal costs. Funding liquidity is the availability of funds, like the cash account for settling debts. These two types of liquidity are interdependent,

\(^1\)Long-Term Capital Management (LTCM) was a hedge fund founded in 1994. On its board of directors were Myron Scholes and Robert C. Merton, who shared the 1997 Nobel Memorial Prize in Economics. Initially enormously successful with annualized returns of over 40% in its first years, in 1998 it lost 4.6 billion dollars in less than four months and became the most prominent example of the risk potential in the hedge fund industry. The fund folded in early 2000.
as the limited funding liquidity leads to an immediate need to sell assets; for instance if depositors who provide the bank with short-term funding withdraw or do not roll over that funding. On the other hand, market liquidity is negatively correlated with costs. If relatively illiquid assets are held during a time of crisis, a decline in price will lead to a value reduction of the company holding those assets and to a decrease in the credit rating. The necessity of raising cash increases even more. The interdependency between market and funding liquidity will not be further discussed in this thesis, but can be found e.g. in Brunnermeier and Pedersen [BP]. We will discuss exclusively the subject area of market liquidity.

Everyone would agree that the Bund Future\(^2\) pit at the futures exchange Eurex in Frankfurt is more liquid than the regional markets for residential housing. But how much more? Although we have a rough idea of what liquidity is, we do not have a precise - perhaps in mathematical terms - definition of market liquidity. The classical theories of financial markets, in particular the famous papers of Merton (1969) or Black and Scholes [BS73], assume perfect elasticity for the supply and demand of traded assets so that orders of arbitrary size do not influence asset prices. This assumption can be justified in a polyopolistic market, where ”small traders”, whose trading volume is covered by market liquidity, act as price takers. These market models stand in the Walrasian tradition, where prices are cried, and agents register how much of each good they would like to offer or purchase. No transactions and no production take place at disequilibrium prices. Instead, prices are lowered for goods with positive prices and excess supply. In the general equilibrium the market price is clearing. Obviously the market microstructure is an important determinant of market liquidity. Those market models which contain liquidity effects remove the assumption of the polypolistic structure and divide the market participants into two groups, referred to as small agents and large traders, where a large trader is defined as any investor whose trades change prices. In contrast the small agents are price takers and have no influence on the market price.

Now we can decompose the liquidity effect - the influence by large traders - into two main features, price manipulation and transaction costs. In the context of price manipulations the obvious question at hand is, whether the large trader when trading strategically is able to generate profits at no risk, i.e., to create arbitrage opportunities. For instance a large

\(^2\)A Bund Future contract is an agreement to buy or sell a certain amount of federal bonds of Germany at a certain date in the future for a certain price.
trader can try to "corner the market". She takes a significant long future position and simultaneously buys up the underlying asset. As the expiry date approaches the small investors who are short in the future contract may realize that there is not enough supply to meet their demand. Therefore the large trader has a certain scope to push up the price for the underlying asset in his portfolio. Another possibility is to establish a trend in the price evolution (a bubble) and then to trade against it before it collapses, as for instance occured in July 2004, where investment bankers of the large bank Citigroup could create a trend in the price development of federal bonds by pushing up the number of deals. Then they sold the total stock position at the same time and created a sizeable profit. A detailed assessment on market manipulation, examples and sufficient conditions for or against it, can be found in Jarrow [J92]. In all these cases the large trader benefits from the illiquid market structure.

The other dimension of illiquidity is the difficulty to trade large volumes of the underlying asset in a short period of time while minimizing the corresponding cost premium. If a large trader wants to purchase a large number of shares (in relation to the total number of supplied shares) she will have to pay an inflated price. On the other hand, if the amount of assets she trades is relatively small, it will have a negligible impact on the price of the assets, because she is acting like a small agent in the market and we are back in a polypolistic market with price takers only. Here the liquidity effect is like that of a transaction cost. Therefore it is reasonable to assume the existence of a liquidity premium as an additional, independent factor next to the known premia of risk, equity and debt risk, taking into account the existence of market frictions.

As with the different viewpoints on liquidity effects, there exist different modelling approaches to capture the interdependency of the evolution of the asset price and the dynamic trading strategy of a large trader in the asset. We categorize these approaches into two groups: the feedback effect models and short-term price impact models. Within the models of the first group the stock price responds instantaneously to the amount of the stock held by a large trader. In Jarrow [J92], Bank and Baum [BB], Esser and Mönch [EM] and Frey and Stremme [FS] the market price stays at the new level after a transaction of the large trader. In a natural way the feedback models are suitable for studying market manipulation strategies like establishing a trend as described above, because the market price process of an asset depends on the strategy of a large trader. Without that dependency the Citigroup bankers would not have been able to establish a trend. By contrast,
in Çetin, Jarrow, Protter [CJP], Çetin, Rogers [CR], Rogers, Singh [RS] and others, a transaction of a large trader has only a short-term price impact and the asset price jumps back to its previous level. What becomes obvious is that the impact of liquidity modelled in this fashion can be interpreted as a transaction cost, but not one which is necessarily proportional to the amounts traded. Within this class of models, especially the transaction cost issue of illiquidity can be studied in depth. Recently Kühn [Kü] has developed a market model which unifies both approaches. As we have seen there are some parallels between the two types of liquidity effects, namely market manipulation and transaction costs, and the different modelling approaches, feedback effect and short-term price impact models, e.g., feedback models are assigned to model the aspect of market manipulation. We remark that there exist approaches in which both types of liquidity effects are included. For instance the feedback model approach of [BB] incorporates a transaction cost element as we see in chapter 2 of this thesis.

The following provides a short overview of the coming:

Chapter 1
We give a short introduction of some of the most basic or fundamental definitions and theorems used in stochastics and finance as well as those being used throughout this work.

Chapter 2
We present two modelling approaches. As an example for a feedback effect model we first give a short description of the model of Bank and Baum [BB]. Then we study in more detail the illiquid economy of Rogers and Singh [RS], which is a short-term price impact approach. We analyze the discrete-time model and show convergence to a continuous-time model, when the trading times approach to each other.

Chapter 3
The second part of this thesis is the study of arbitrage opportunities in illiquid economies. We study a modification of the Bank and Baum model. The central result is the absence of arbitrage within this modified model, where we have to assume that the price process
is described by a continuous semimartingale, in contrast to the chapter before where we work with a general semimartingale. The general case could not be solved in this context up to now and is left open for further studies.

Chapter 4
In the last chapter we draw some conclusions.
Chapter 1

Preliminaries from Finance and Stochastics

In this chapter we summarize some definitions and fundamental concepts from finance and stochastics. Moreover we introduce the underlying assumptions and notations we need. We follow [Pro], [Bj] and [DS94].

We consider a model of a security market with a finite time horizon $[0, T]$. The market consists of two assets: a risk-free asset with zero rate of return, called a bond or cash account, and a risky traded asset, called a stock. The asset price process of the stock is denoted by $S = (S_t)_{t \in [0, T]}$, the price process of the bond is denoted by $S^0 = (S^0_t)_{t \in [0, T]}$. We suppose that the price of the bond $S^0$ is constant, i.e., $S^0_t \equiv 1$ for all $t \in [0, T]$. This assumption does not restrict the generality of our model, because we may express the values of the stock in units of the bond. In other words $S^0 = (S^0_t)_{t \in [0, T]}$ is an $\mathbb{R}$-valued process modeling the discounted price process of one risky asset. The price process $S$ is assumed to be a semimartingale, based on and adapted to a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$.

At first we provide some definitions and results from stochastics.

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1We will analyze models with different price processes, modelled by semimartingales. In most of the cases we will not distinguish between the corresponding probability spaces.
1.1 Basic facts and definitions of stochastic processes

Definition 1.1.1 A filtered probability space \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) satisfies the usual conditions, if and only if:

(i) The \(\sigma\)-field \(\mathcal{F}\) is \(\mathbb{P}\)-complete, that is, if \(A \in \mathcal{F}, \mathbb{P}[A] = 0\) and \(B \subset A\), then \(B \in \mathcal{F}\).

(ii) \(\mathcal{F}_0\) contains all the \(\mathbb{P}\)-null sets of \(\mathcal{F}\).

(iii) The filtration \(\mathcal{F}\) is right-continuous, that is, for all \(t \geq 0\)
\[
\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u.
\]

Assumption 1.1.2 We assume that the filtered probability space \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\), where the price process \(S\) is based on and adapted to, satisfies the usual conditions.

Definition 1.1.3 A process \((H_t)_{t \geq 0}\) is called predictable, if \((\omega, t) \mapsto H_t(\omega)\) is measurable with respect to the predictable \(\sigma\)-algebra \(\mathcal{P}\) on \(\Omega \times \mathbb{R}_+\):
\[
\mathcal{P} := \sigma(H : H \text{ is a left-continuous, adapted process}).
\]

We let \(\mathcal{bP}\) denote the space of bounded processes that are predictably measurable.

Definition 1.1.4 (i) An adapted, càdlàg process \(X\) is a local martingale if there exists a sequence of stopping times \((\tau_n)_{n \in \mathbb{N}}\) increasing to \(\infty\) a.s such that \((X_{t \wedge \tau_n} \mathbb{1}_{\tau_n > 0})_{t \geq 0}\) is a uniformly integrable martingale for each \(n\).

(ii) A process \(X\) is said to be locally bounded if there exists a sequence of stopping times \((\tau_n)_{n \in \mathbb{N}}\) increasing to \(\infty\) a.s such that for each \(n\), \((X_{t \wedge \tau_n} \mathbb{1}_{\tau_n > 0})_{t \geq 0}\) is bounded.

Definition 1.1.5 A semimartingale \(X\) is an adapted and càdlàg process of the form \(X = X_0 + M + B\) with \(M_0 = B_0 = 0\) where \(M\) is a local martingale and \(B\) is an adapted process of finite variation (FV). If \(B\) is also predictable, we say that \(X\) is a special semimartingale. In this case the decomposition is unique (up to a null-set) and is called the canonical decomposition of \(X\).

Definition 1.1.6 Let \(X, Y\) be semimartingales. The quadratic variation process of \(X\), denoted \([X, X] = ([X, X]_t)_{t \in [0,T]}\), is defined by
\[
[X, X] = X^2 - 2 \int X_- \, dX
\]
with \( X_{0-} = 0 \).

The **quadratic covariation** of \( X \) and \( Y \) is defined by

\[
[X, Y] = XY - \int X_- dY - \int Y_- dX.
\]

**Definition 1.1.7** Let \( X \) be a special semimartingale with canonical decomposition \( X = M + B \), where \( M \) is a local martingale and \( B \) is a predictable finite variation process. The \( \mathcal{H}^2 \) norm of \( X \) is defined to be

\[
\|X\|_{\mathcal{H}^2} = \|[M, M]_{\infty}^{1/2}\|_{L^2} + \left\| \int_0^\infty |dB_s| \right\|_{L^2},
\]

where we write \( \int_0^\infty |dB_s| \) for the random variable which is the total variation of the paths of \( B \). The **space of semimartingales** \( \mathcal{H}^2 \) consists of all special semimartingales with finite \( \mathcal{H}^2 \) norm. \( \mathcal{H}^2_{loc} \) is its localized class, the so-called "locally square-integrable martingales".

**Definition 1.1.8** Let \( X \in \mathcal{H}^2 \) with canonical decomposition \( X = M + B \). We say \( H \in \mathcal{P} \) is \((\mathcal{H}^2, X)\) **integrable** if

\[
E\left[ \int_0^\infty H_s^2 d[M, M]_s \right] + E\left[ \left( \int_0^\infty |H_s| dB_s \right)^2 \right] < \infty.
\]

**Definition 1.1.9** Let \( X \) be a semimartingale and \( H \in \mathcal{P} \). The stochastic integral \( H \cdot X \) is said to exist if there exists a sequence of stopping times \( \tau^n \) increasing to \( \infty \) a.s. such that \( X_{\tau^n -} \in \mathcal{H}^2 \), for each \( n \geq 1 \), and such that \( H \) is \((\mathcal{H}^2, X_{\tau^n -})\) integrable for each \( n \). In this case we say \( H \) is \( X \) **integrable**, written \( H \in L(X) \), and we define the **stochastic integral** by that

\[
(H \cdot X)_t = (H \cdot X_{\tau^n -})_t \quad \text{on } [0, \tau^n),
\]

for each \( n \).

**Proposition 1.1.10** Let \( X \) be a semimartingale and let \( H \in \mathcal{P} \) be locally bounded. Then \( H \in L(X) \). That is, the stochastic integral \( H \cdot X \) exists.

**Proof.** See for example p. 166. in [Pro]. ■

**Proposition 1.1.11** Let \( X, Y \in \mathcal{H}^2_{loc} \). Then there exists a finite, predictable process \( A \in \mathcal{V} \) such that \( XY - A \) is a local martingale, where \( \mathcal{V} \) is the set of all real-valued processes that are càdlàg, adapted, with \( A_0 = 0 \), and whose each path \( t \mapsto A_t(\omega) \) has a finite variation
over each interval $[0, t]$. The process $A$ is called the *predictable* or conditional quadratic covariation (resp. variation) of $X, Y$ and is denoted by $\langle X, Y \rangle$. It is the compensator of $[X, Y]$.

**Proof.** See for example I.4.2. in [JS]. □

### 1.2 Financial modelling

In the following chapters we will clearly distinguish between concepts from discrete-time or continuous-time finance. This thesis deals primarily with continuous-time models. We recall that we consider security markets with a finite time horizon $[0, T]$. The risk-free asset has constant prices and $S$ describes the process modelling the discounted price process of one risky asset.

**Assumption 1.2.1** The price process $S$ is a semimartingale, based on and adapted to a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$. Moreover $S$ is locally bounded.

**Definition 1.2.2** A *trading strategy* $\theta = (\theta_t)_{t \in [0,T]}$ is a predictable and $\mathbb{R}$-valued process, i.e., $(\omega, t) \mapsto \theta_t(\omega)$ is $\mathbb{P}$–measurable. As already said, we assume a market with two assets. $\theta_t$ is the number of risky assets in the portfolio at time $t$. The number of risk-free bonds in the portfolio at time $t$ is denoted by $K_t$.

**Definition 1.2.3** The *portfolio value process* $V(\theta) = (V_t(\theta))_{t \in [0,T]}$ is given by

$$V_t(\theta) = K_t + \theta_t S_t \quad \text{for all} \quad t \in [0,T],$$

where $K_t$ is the cash account or equivalently the number of bonds in the portfolio, $\theta$ is a trading strategy and $S$ the price process.

**Definition 1.2.4** (i) A trading strategy $\theta \in L(S)$ is said to be *self-financing*, if for the accompanying value process $V(\theta)$

$$V_t(\theta) = v_0 + \int_0^t \theta_s dS_s = v_0 + \theta \cdot S \quad \text{for all} \quad t \in [0,T],$$

where $v_0 := V_0(\theta) = K_0 + \theta_0 S_0$ is the initial value of the portfolio.

(ii) A self-financing trading strategy $\theta$ is said to be *admissible*, if there exist a real number $a > 0$ such that $V_t(\theta) \geq -a$ for all $t \in [0,T]$.
Remark. (i) We see that the integrand $\theta$ only describes the variation of the stock position, because the bond price process is assumed to be constant:

$$V_t(\theta) = v_0 + \int_0^t \theta_s dS_s$$

$$\Leftrightarrow K_t + \theta_t S_t = K_0 + \theta_0 S_0 + \int_0^t \theta_s dS_s$$

$$\Leftrightarrow K_t - K_0 = \theta_0 S_0 - \theta_t S_t + \int_0^t \theta_s dS_s.$$ 

Therefore the process $(\theta_t)_{t \in [0,T]}$ (stock position) determines the variation of the bond position.

(ii) The term "admissibility" describes the realistic situation of a finite credit facility. The concept was developed by Harrison and Pliska (see [HP]). The condition serves to exclude strategies (like doubling-strategies), which enable the investor to generate riskless profits. In chapter three we will discuss a relaxation of this condition in more detail.

Definition 1.2.5 We say that the market is **complete**, if every bounded claim $H \in L^\infty(\Omega, \mathcal{F}_T, P)$ can be replicated by a portfolio $V(\theta)$, i.e., $H = V_T(\theta)$ a.s., where $\theta$ is a self-financing strategy.

Definition 1.2.6 (i) An **arbitrage possibility** on a financial market is an admissible self-financing strategy $\theta$ such that $V_0(\theta) = 0$, $P[V_T(\theta) \geq 0] = 1$ and $P[V_T(\theta) > 0] > 0$.

(ii) A sequence $(\theta^{(n)})_{n \in \mathbb{N}}$ of admissible trading strategies is called **free lunch with vanishing risk**, if there exist a nonnegative random variable $f$ with $P[f > 0] > 0$ and a null sequence $(\epsilon_n)_{n \in \mathbb{N}} \subset \mathbb{R} \setminus \{0\}, \epsilon_n \downarrow 0$, such that for the sequence of accompanying value processes $V(\theta^{(n)}): V_0(\theta^{(n)}) = 0$ and

$$f \leq V_T(\theta^{(n)}) + \epsilon_n, \ P\text{-a.s. for all } n \in \mathbb{N}.$$ 

Otherwise we say that the market satisfies the condition "no free lunch with vanishing risk" (NFLVR).

Definition 1.2.7 Let $S$ be a nonnegative semimartingale. A probability measure $Q \sim P$ on $(\Omega, \mathcal{F})$ is called an **equivalent martingale measure** (EMM) with respect to $S^0$ and the time interval $[0,T]$, if $S$ is a local $(Q, \mathcal{F})$-martingale. Let $\mathcal{M}(S)$ denote the set of equivalent martingale measures.
Proposition 1.2.8 If $S$ is a local martingale and if $\theta$ is an admissible integrand (strategy) for $S$, then $\theta \cdot S$ is a local martingale. Consequently, $\theta \cdot S$ is a supermartingale.

Proof. With Corollary 3.5 in [AS] we obtain the local martingale property. Since $\theta$ is admissible we have that the local martingale $\theta \cdot S$ is bounded from below and therefore a supermartingale.

Remark. (i) Proposition 1.2.8 can be rephrased as: Let $Q \in \mathcal{M}(S)$ and $\theta$ be an admissible strategy, then the portfolio value process $V(\theta)$ is a $Q$–supermartingale.
(ii) Moreover, we can interpret $\mathcal{M}(S)$ as the set of probability measures $Q \sim P$ such that for each admissible integrand $\theta$, the process $\theta \cdot S$ is a local $(Q, \mathcal{F})$-martingale.
(iii) In the general setting, where $S$ is an arbitrary semimartingale (possibly unbounded), we need a wider concept. We say that a semimartingale $S$ is a sigma-martingale if there is a strictly positive predictable process $\theta$ such that $\theta \in L(S)$ and such that $\theta \cdot S$ is a local martingale (see [DS98]). In our case, where $S$ is locally bounded, the notions coincide. The following first theorem of asset pricing holds also for the general case.

Now we are ready to present the fundamental theorems of asset pricing. For the first theorem we present a version of Delbaen and Schachermayer.

Theorem 1.2.9 (First fundamental theorem of asset pricing, [DS94],[DS98]) Let $S$ be a bounded real valued semimartingale. There is an equivalent martingale measure $Q \in \mathcal{M}(S)$ if and only if $S$ satisfies NFLVR.

Proof. We refer to Delbaen and Schachermayer [DS94] and for the general case see [DS98].

Since a security market containing arbitrage opportunities cannot be one in which an economic equilibrium exists, we shall assume the following no-arbitrage condition on $S$ to get consistent and viable models.

Assumption 1.2.10 The set $\mathcal{M}(S)$ is not empty.

Under this assumption we obtain the second fundamental theorem, which is the version of Harrison and Pliska:
Theorem 1.2.11 (Second fundamental theorem of asset pricing, [HP]) The market is complete if and only if the martingale measure is unique, i.e., $\mathcal{M}(S)$ is a singleton.

Proof. We refer to [HP].

In this thesis we consider market models with liquidity costs. Apart from this we assume no frictions like transaction costs or information asymmetries.
Chapter 2

Modelling Liquidity

In this chapter we give a short overview of different aspects of modelling liquidity. As we have already noted there exist different modelling approaches to capture the interdependency of the evolution of the asset price and the dynamic trading strategy of the large trader in the asset. We have categorized these approaches into two groups: the feedback effect models and short-term price impact models. We want to present two models, each belonging to one of the categories: the prominent model of Bank and Baum [BB] for an example of a feedback effect model and the short-term approach of Rogers and Singh [RS].

2.1 A feedback effect model

A feedback effect model is a model where the stock price responds instantaneously to the amount of the stock held by the large trader. The market price stays at a new level after a transaction of the large trader. If the large trader wants to purchase a certain number of assets she will have to pay an inflated price, because she is trading in a shallow market, where she has caused this structural inelasticity. But not only the large trader has to pay the higher price, but also any agent on the market. In this context we want to present the continuous-time model of [BB]. In chapter 3 we will modify this model and study the arbitrage opportunities within this economy. Other examples of feedback models can be found, e.g., in [J92], [EM] or [FS].

Bank and Baum assume that the price fluctuations of the risky asset, given that the
large investor\(^1\) holds a constant stake of \(\vartheta\) shares\(^2\) in this asset, are described by a family of continuous semimartingales \(S^\vartheta = (S^\vartheta_t)_{t \in [0,T]}, \vartheta \in \mathbb{R}\). Therefore the market price will no longer evolve independently of the trading strategies chosen by this big player. If the large investor chooses a time-varying strategy \(\theta = (\theta_t)_{t \in [0,T]}\), the resulting asset price evolution can be modeled as
\[
S^\theta_t = S(\theta_t, t) := S^0_t.
\]
Since we want to give an overview of the general ideas and results of Bank and Baum, we forego the quite demanding technical details. In the following Assumption 2.1.1 the term ”smooth” stands for a bundle of nice properties of the family of semimartingales. Definition 2.2 in [BB] gives the exact conditions, which \(S^\vartheta\) has to satisfy to be called smooth. This smoothness will be an decisive assumption for applying the Itô-Wentzell formula, which is an important tool in the paper of [BB].

**Assumption 2.1.1**

(i) The family of semimartingales \(S^\vartheta (\vartheta \in \mathbb{R})\) is smooth (compare Definition 2.2 in [BB]).

(ii) Asset prices are nondecreasing with respect to the large investor’s position \(\vartheta\), resp. \(\vartheta’\):
\[
S^\vartheta \leq S^{\vartheta’} \text{ for } \vartheta \leq \vartheta’.
\]

The second assumption is important to rule out trivial arbitrage possibilities. As already noted feedback models are suitable for studying market manipulation strategies, because the market price process of an asset depends on the strategy of a large investor. On the other hand, illiquidity causes transaction costs, if the large trader’s orders are only exercised after prices have adversely adjusted to them. The Bank/Baum-model incorporates both elements.

In order to investigate the question of existence of arbitrage opportunities, we present the wealth dynamics generated by self-financing strategies. Let \(\theta = (\theta_t)_{t \in [0,T]}\) be a semimartingale self-financing strategy and let \((K_t(\theta))_{t \in [0,T]}\) be the accompanying discounted holdings in the bank account. Then the wealth dynamics can be described by
\[
K_t(\theta) = K_0 - \int_0^t S(\theta_s, s) \, d\theta_s - [S(\theta, \cdot), \theta]_t. \tag{2.1.1}
\]

\(^1\)We use the terms ”large trader”, ”large investor” or later ”hedger” synonymously.

\(^2\)\(\vartheta\) describes the absolute number of assets the large trader holds. Another view could be that \(\vartheta\) represents the proportion to the total number of assets in the market. Then \((S^\vartheta_t)_{t \in [0,T]}\) describes the price dynamics if the large trader holds a stake of 40% of the total number of shares.
The last term implies that asset prices are affected by the large investor’s orders when these are actually exercised. Assume, for example, the order is of size \( \Delta \theta_t = \theta_t - \theta_{t-} > 0 \) (we set in general \( \theta_{t-} := \lim_{s \uparrow t} \theta_s \)). Then the investor’s bank account will be charged

\[
\Delta K_t(\theta) = -S(\theta_{t-}, t)\Delta \theta_t - \Delta [S(\theta_{t-}, \cdot), \theta_t]_t \\
= -S(\theta_{t-}, t)\Delta \theta_t - \Delta S(\theta_t, t)\Delta \theta_t \\
= -S(\theta_{t-}, t)\Delta \theta_t - (S(\theta_t, t) - S(\theta_{t-}, t))\Delta \theta_t \\
= -S(\theta_t, t)\Delta \theta_t
\]

and she has to pay \( S(\theta_t, t) \) for each of his \( \Delta \theta_t \) ordered shares. Therefore the large investor trades on the ”bad side”, i.e., the orders are only exercised after prices have adjusted (compare Assumption 2.1.1).

Now we present the concept of ”paper” wealth and ”real” wealth, which was introduced by Jarrow [J92]. Paper wealth is defined to be the value of the portfolio position when prices are evaluated using the current holdings. Real wealth is the value of the large trader’s position when relative prices are evaluated as if the stock holdings were liquidated. For a price taker, these values are identical, but for the large investor they are distinct.

**Definition 2.1.2**

(i) Let \( \theta \) be a self-financing strategy. The **book value** at time \( t \in [0, T] \) of the portfolio is defined as

\[
V_{t}^{b}(\theta) := K_t(\theta) + S(\theta_t, t)\theta_t. \tag{2.1.2}
\]

(ii) Analogously we can define the **block liquidation value** of the portfolio:

\[
V_{t}^{\text{block}}(\theta) := K_t(\theta) + S(0, t)\theta_t. \tag{2.1.3}
\]

Of course the difference between \( V_{t}^{\text{block}}(\theta) \) and \( V_{t}^{b}(\theta) \) can be quite substantial. Perhaps the investor is forced to liquidate her stock position immediately by a single block trade, therefore she would not be able to sell her shares at price \( S(\theta_t, t) \) but only at \( S(0, t) \). Consequently the investor should choose a liquidation strategy, that she does not sell her position en bloc, but in smaller packages.

**Definition 2.1.3**

(i) Let \( L(\rho, t) \) denote the **liquidation value** of \( \rho \in \mathbb{R} \) shares at time \( t \in [0, T] \), which is defined by

\[
L(\rho, t) := \int_{0}^{\rho} S(x, t)dx. \tag{2.1.3}
\]
(ii) The real value or realizable value of a portfolio at time \( t \in [0, T] \) is defined by

\[
V^\text{real}_t(\theta) = K^\theta_t + L(\theta_t, t).
\]  

(2.1.4)

Now we can study the explicit dynamics of the real value. For this Bank and Baum apply the Itô-Wentzell formula to \( L(\theta_t, t) \) and \( S(\theta_t, t) \).

**Proposition 2.1.4** Let \( \theta \) be a self-financing semimartingale strategy and let \((S^\theta)_{\theta \in \mathbb{R}}\) be a family of smooth semimartingales, then the dynamics of the real value process \((V^\text{real}_t(\theta))_{t \in [0, T]}\) are given by

\[
V^\text{real}_t(\theta) - V^\text{real}_0(\theta) = \int_0^t L(\theta_s, ds) - \frac{1}{2} \int_0^t S'(\theta_s, s)\, d[\theta, \theta]^c_s \\
- \sum_{0 \leq s \leq t} \int_{\theta_s^-}^{\theta_s^+} \{S(\theta_s, s) - S(x, s)\}\, dx,
\]  

(2.1.5)

where \( \int_0^t L(\theta_s, ds) \) denotes the stochastic integral\(^3\) of \( \theta \) with respect to the semimartingale kernel \( L(\theta, ds) \) and where all derivatives are taken with respect to \( \theta \).

**Proof.** See p.7 in [BB].

The first part, \( \int_0^t L(\theta_s, ds) \), accounts for profits or losses from exogenous stock price fluctuations. The second and third parts can be interpreted as transaction costs due to limited liquidity. As we see the transaction costs in the case of block orders are reflected in the jump term \( \sum_{0 \leq s \leq t} \int_{\theta_s^-}^{\theta_s^+} \{S(\theta_s, s) - S(x, s)\}\, dx \) and the costs which are produced by trading in a fluctuation manner are described by the quadratic variation term \( \frac{1}{2} \int_0^t S'(\theta_s, s)\, d[\theta, \theta]^c_s \).

Therefore it is possible to avoid liquidity costs by using trading strategies whose trajectories \((\theta_t)_{t \in [0, T]}\) are continuous and of finite variation, because due to a finite variation

\(^3\)Again we remind that we only want to give a rough overview of the paper. Bank and Baum use nonlinear stochastic integration theory to model the trading gains of a large trader. Kunita gives in Chapter 3.2 in [K] a more detailed account of how to construct stochastic integrals with respect to a semimartingale kernel \( S(\theta, ds) \). We only give the elementary definition for simple integrands \( \theta = \sum_{k=1}^n H^{k-1}1_{[T_{k-1}, T_k)} \), with \( n \in \mathbb{N}, 0 = T_0 \leq T_1 \ldots \leq T_n = T \), stopping times and \( H^{k-1} \in L^0(\Omega, \mathcal{F}_{T_{k-1}}, P), k = 1, \ldots, n \):

\[
t \mapsto \int_0^t S(\theta_s, ds) := \sum_{k=1}^n (S(H^{k-1}, t \wedge T_k) - S(H^{k-1}, t \wedge T_{k-1})), \quad t \in [0, T].
\]

In Proposition 2.1.4 we have to extend the integral to semimartingale processes \( \theta \).
(FV) process the quadratic variation is zero and due to continuity there are no jumps and therefore the last expression vanishes too.

Now we are ready to study market manipulation strategies, which enable the large investor to make riskless profits. She can move the market price, but only at certain costs. Therefore it is interesting under which conditions the liquidity costs suffice to rule out these strategies. At first Bank and Baum exclude any doubling-strategies by introducing the concept of an admissible strategy for the large investor. In addition they need the following assumption.

**Assumption 2.1.5** There exists a measure $Q^* \sim P$, which simultaneously is a local martingale measure for all our processes $S^\theta (\theta \in \mathbb{R})$.

Under Assumptions 2.1.1 and 2.1.5 Bank and Baum can show quite easily the absence of arbitrage within the presented illiquid market model (see Theorem 3.3 in [BB]).

**Remark.** (i) For showing this substantial result Bank and Baum have to use some quite restrictive assumptions. First of all they assume that the price process follows a continuous semimartingale. Moreover also the trading strategy is described by a semimartingale. In this context this is essential for applying the Itô-Wentzell formula to obtain the dynamics of the value process, but very often the trading strategy is nothing more than a predictable process. Assumption 2.1.5 includes the assumption of no arbitrage for small investors. By the First fundamental theorem of asset pricing 1.2.9 the small agents cannot make any riskless profits in periods where the large trader does not trade. Here we can choose $Q^*$ independently of $\theta_t \in \mathbb{R}$.

(ii) The Bank/Baum-model is very similar to the approach of [CJP]. Çetin, Jarrow and Protter develop a reaction function in form of a supply curve that depends on the order size of the large investor. But in contrast to the Bank/Baum-model the actions of the large investor have no lasting effect. Furthermore they show that it is also possible to avoid liquidity costs by using these ”tame” strategies, i.e., $\theta$ is a continuous and FV process.

(iii) Refering to the example in the Introduction of the investment bankers of the Citigroup, who could establish a trend and make a profit by selling the whole bond position en bloc, we realize that the Bank/Baum model does not cover this strategy, because the liquidity costs would be too high. The liquidation strategy should be the division into smaller packages. Perhaps the smart bankers could exploit information asymmetries, which are
not anticipated in the Bank/Baum-model.

2.2 A short-term price impact model

In this section we want to investigate the modelling framework from Rogers and Singh as an example for a short-term price impact model (see [RS] and [Si]). First we will present the discrete-time model in some detail, where we extend the setting to semimartingale price processes. Secondly we develop the continuous-time dynamics. Under certain assumptions we are able to show the convergence of the discrete-time into the continuous-time modelling framework. This part is quite technical, but the exact relationship between discrete-time and continuous world was omitted in the paper of Rogers and Singh.

2.2.1 Price dynamics in the presence of a large investor

We start with a simple discrete-time microeconomic viewpoint to derive the price process. In our model of an illiquid financial market the agents can choose to invest in equities or bonds; for simplicity of exposition, we shall assume a zero interest rate. We shall consider a single asset, whose price process is described by a nonnegative semimartingale \((S_t)_{t \in [0,T]}\). The price of the asset at time \(t\) is denoted by \(S_t\). The time axis \([0,T]\) is divided by the grid points \(0 = T_0 < T_1 < \ldots < T_k = T\) into equal intervals of length \(\Delta T\). The period \(l\) is denoted by \(\Delta T_l = (T_{l-1}, T_l]\). All orders arising during the period \((T_{l-1}, T_l]\) are executed at the endpoint \(T_{l-}\), i.e., the traders order the shares independently of the stock price in time point \(T\); for instance in \(T\) the price could make a jump. It would be possible to choose a strategy for investing in the jumps of the price process. The continuous-time trading strategy of the large investor is as agreed \(\theta = (\theta_t)_{t \in [0,T]}\), but the portfolio is only rebalanced at each \(T_{l-}\).

The market price of the asset is determined by equalisation of supply and demand in each period. We can imagine that in each period \(\lambda_a \Delta T\) shareholders enter the market to sell their shares and on the other side (the demand-side) \(\lambda_b \Delta T\) agents want to purchase shares. An additional agent also comes to the market, the so-called hedger or large trader, with the intention of buying or selling shares in period \(l\). She wants to buy or sell \(\Delta \theta_{T_l} := \theta_{T_l} - \theta_{T_{l-1}}\) shares, where \(\theta_{T_l}\) is the number of shares in the portfolio of the hedger at time \(T_l\). In contrast to the representative agents (the price takers) on the market the hedger has to
pay a higher price for the additional shares she wants to purchase, or she gets a lower price for the additional shares she wants to sell. This mark-up (or "mark-down") represents the liquidity costs in the model. The liquidity costs have no permanent price effect, because in the next period the equalisation process starts again and only the hedger will be punished for his excess demand or supply.

In the following let $p_l := \log(S_{t-})$, where we set in general $S_{t-} := \lim_{s \uparrow t} S_s$, the left limit at $t$. The logarithmic price the hedger has to pay is denoted by $\tilde{p}_l$. First we assume that the hedger only wants to purchase shares, i.e., $\theta_{T_l} \geq \theta_{T_{l-1}}$, therefore she has to pay a higher price than the price takers on the market.

The market clearing-condition for the illiquid market at time $T_l$ becomes:

$$
\lambda_a \Delta T f_s(\tilde{p}_l - p_l) = \lambda_b \Delta T f_d(\tilde{p}_l - p_l) + \Delta \theta_{T_l}, \tag{2.2.1}
$$

where the supply function $f_s : P \rightarrow Q$, $p \mapsto q_s$ is continuous and strictly increasing, the demand function $f_d : P \rightarrow Q$, $p \mapsto q_d$ is continuous too and strictly decreasing. $P$ is the set of all logarithmic prices and $Q$ the set of all quantities: $P = Q = \mathbb{R}$. The function $F : P \rightarrow Q$

$$
F(p) \mapsto \lambda_a f_s(p) - \lambda_b f_d(p)
$$

describes the difference of the quantities, which are supplied and demanded at price $p$ by the small agents. It is clear that $F$ is continuous and strictly increasing.

If there is no hedger on the market and the market has achieved its equilibrium with its equilibrium price $p^*$, supply and demand will be equal and therefore the difference will be zero:

$$
F(p^*) = 0 \iff \lambda_a f_s(p^*) = \lambda_b f_d(p^*).
$$

Otherwise, if the large investor is on the market and wants to buy a certain amount of shares, the difference will be equal to the quantity which the large investor wants to purchase in the equilibrium situation. With the market clearing condition (2.2.1) we get:

$$
\lambda_a \Delta T f_s(\tilde{p}_l - p_l) - \lambda_b \Delta T f_d(\tilde{p}_l - p_l) = \Delta \theta_{T_l} = \theta_{T_l} - \theta_{T_{l-1}}
$$

$$
\iff \lambda_a f_s(\tilde{p}_l - p_l) - \lambda_b f_d(\tilde{p}_l - p_l) = \frac{\theta_{T_l} - \theta_{T_{l-1}}}{T_l - T_{l-1}}
$$

$$
\iff F(\tilde{p}_l - p_l) = \frac{\theta_{T_l} - \theta_{T_{l-1}}}{T_l - T_{l-1}}.
$$
We see that the difference of the prices \( \tilde{p}_l - p_l \) is - after application of \( F \) - the demand of the hedger relative to the length \( \Delta T \) of the time interval. Or more economically, the higher price the hedger has to pay is necessary to achieve a market equilibrium.

We define the inverse function of \( F \) to achieve the equilibrium price on the market by a given quantity of shares:

\[
\psi := F^{-1} : Q \rightarrow P.
\]  

(2.2.2)

As \( F \) is a continuous and strictly increasing function, the inverse function \( \psi \) is also continuous and strictly increasing. Therefore we get the price \( \tilde{p}_l \) per share, which the hedger has to pay in period \( l \):

\[
\tilde{p}_l - p_l = \psi \left( \frac{\theta_{T_i} - \theta_{T_{i-1}}}{T_i - T_{i-1}} \right)
\]

\[
\Leftrightarrow \tilde{p}_l = \psi \left( \frac{\Delta \theta_{T_i}}{\Delta T_i} \right) + p_l.
\]

After application of the exponential function we obtain the “original” price process

\[
\tilde{S}_{T_{i-}} := \exp \left( \psi \left( \frac{\Delta \theta_{T_i}}{\Delta T_i} \right) \right) S_{T_{i-}}.
\]

(2.2.3)

We can assume that \( \psi(0) = 0 \), because if the hedger as described above is not on the market, there will not exist any excess demand (\( \Delta \theta_{T_i} = 0 \)). Consequently the prices will be equal. Because \( \psi \) is strictly increasing, \( \frac{\Delta \theta_{T_i}}{\Delta T_i} > 0 \) and therefore \( \exp(\psi(\frac{\Delta \theta_{T_i}}{\Delta T_i})) > 1 \), the price the hedger has to pay is higher. The difference of the prices \( \tilde{p}_l - p_l \) gets larger, if on the one hand the hedger expands his demand, on the other hand the length of the time interval becomes shorter. Both effects describe trading in a shallow market with the consequence of inflated prices.

Now let us have a quick look at the situation where the hedger wants to sell her shares (\( \theta_{T_i} < \theta_{T_{i-}} \)), here interpreted as a negative demand. Therefore she will get a lower price compared to the representative agents:

\[
\lambda_a \Delta T f_a(\tilde{p}_l - p_l) = \lambda_b \Delta T f_d(\tilde{p}_l - p_l) + \Delta \theta_{T_i}
\]

\[
\Leftrightarrow \tilde{S}_{T_{i-}} = \exp \left( \psi \left( \frac{\Delta \theta_{T_i}}{\Delta T_i} \right) \right) S_{T_{i-}},
\]

\[<1\]

4All time intervals are equidistant; therefore we do not distinguish between the length in general \( \Delta T \) or a specific time interval \( \Delta T_i \).
because the argument of $\psi$ is negative.

**Remark.** (i) The function $\psi$ describes the liquidity costs the hedger has to pay for the excess demand. Since $\psi$ does not depend on time $t$, this connection is constant in time. Therefore the preferences of the small agents and so the supply and demand functions are known and constant during $[0, T]$. A more realistic approach could be the extension of the model with a family of functions $\psi^t, t \in [0, T]$. Then it would be possible that the preferences (expectations, risk aversion) and therefore the supply and demand curves of the agents vary during the time. The price process of the hedger could be modelled by $(\tilde{S}_T^i)_{T \in [0, T]}$ with

$$\tilde{S}_{T_l}^i = \exp \left( \psi^t \left( \frac{\Delta \theta_{T_l}}{\Delta T_l} \right) \right) S_{T_l}^-.$$

(ii) The quotient $\frac{\theta_{T_l} - \theta_{T_{l-1}}}{T_l - T_{l-1}}$ in equation (2.2.3) can be written as

$$\frac{\theta_{T_l} - \theta_{T_{l-1}}}{\lambda((T_{l-1}, T_l])},$$

where $\lambda$ is the Lebesgue-measure. We extend the model by introducing a measure $\mu$, which describes the market activity. In contrast to the Lebesgue-measure $\lambda$ the measure $\mu$ has not to weight all equidistant time intervals equally.

Now we are ready to introduce some analogous concepts to the Bank and Baum model for this illiquid economy. Of course we are especially interested in the situation of the large trader, because she has to pay a higher price. Therefore the concepts are defined in terms of the large investor. We begin with the formulation of the discrete-time case.

**Definition 2.2.1** A trading strategy $\theta : \Omega \times [0, T] \longrightarrow \mathbb{R}_+, (\omega, t) \mapsto \theta_t(\omega)$ is **self-financing** if and only if it fulfills the discrete-time self-financing condition:

$$K_{T_l} - K_{T_{l-1}} = - (\theta_{T_l} - \theta_{T_{l-1}}) \exp \left( \psi \left( \frac{\theta_{T_l} - \theta_{T_{l-1}}}{\mu((T_{l-1}, T_l])} \right) \right) S_{T_l}^- = - (\theta_{T_l} - \theta_{T_{l-1}}) \tilde{S}_{T_l}^-,$$

where $K_{T_l}$ denotes the number of riskless bonds in the portfolio of the hedger at time $T_l$. The measure $\mu$ is positive, finite, that is $\mu([0, T]) < \infty$, and $\mu(\{t\}) = 0$ for $t \in \mathcal{B}([0, T])$. 
Remark. The discrete-time model is defined for the grid points $0 = T_0 < T_1 < \ldots < T_k = T$, i.e., the stock position is constant between these trading points. Therefore the cash account $(K^n_t)_{t \in (T^n_{k-1}, T^n_k)}$ is constant.

Again we are able to introduce some interpretations of the portfolio value of the large trader:

**Definition 2.2.2** (i) Let $\theta$ be a self-financing strategy. The **book value** at time $T_i \in [0, T]$ of the portfolio is defined as

$$V^b_{T_i}(\theta) := K_{T_i}(\theta) + \theta_{T_i} S_{T_i}.$$ 

(ii) The **book value** development of two periods is defined inductively:

$$V^b_{T_i} - V^b_{T_{i-1}} = \theta_{T_{i-1}}(S_{T_i} - S_{T_{i-1}}) + \left[ \theta_{T_i} S_{T_i} - \theta_{T_i} \exp \left( \psi \left( \frac{\Delta \theta_{T_i}}{\mu(\Delta T_i)} \right) \right) S_{T_{i-1}} \right].$$

(iii) Furthermore the **block liquidation value** of the portfolio is defined as:

$$V^{\text{block}}_{T_i}(\theta) := K_{T_i}(\theta) + \theta_{T_i} \exp \left( \psi \left( - \frac{\theta_{T_i}}{\mu(\Delta T_i)} \right) \right) S_{T_{i-1}}.$$

Remark. As already remarked in Section 2.1, the book value or paper wealth is defined to be the value of the portfolio position when prices are evaluated using the current holdings. In Definition 2.2.2 (i) we see that the stock position $\theta_{T_i}$ is evaluated at the current price $S_{T_i}$. The book value development gives us information about the structure of the value difference of two periods. The first term describes the change in the asset price. The second term anticipates the liquidity costs of our model. The large trader has to pay a higher price for the stock, but the stocks are credited to the portfolio by the market price.

### 2.2.2 Continuous-time dynamics

In this section we will derive the continuous-time dynamics of the Rogers/Singh-model. Most of the concepts seem to be quite clear and obvious, especially if we keep the discrete-time dynamics in mind. Therefore Rogers and Singh take these dynamics for granted and do not give any exposition regarding the nature of the relation between the two models.
We close this gap and show the convergence of the discrete-time model to the continuous-time one letting $\Delta T \downarrow 0$. For this we have to make some assumptions regarding the price process $S$ and the trading strategy $\theta$.

**Proposition 2.2.3** Let $S$ be a price process satisfying Assumption 1.2.1, i.e., $S$ is a locally bounded semimartingale based on and adapted to a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$. Then $S$ is a special semimartingale and the price process can be decomposed as

$$S = S^M + S^B,$$

where $S^M_0 = S^B_0 = 0$, $S^M$ is a local $L^2$-integrable martingale and $S^B$ is a locally bounded predictable finite variation process.

**Proof.** Since $S$ is locally bounded, the process is locally square integrable. With Theorem 37 on p. 132 in [Pro] we obtain that $S$ is a special semimartingale and therefore we get its canonical decompososition: $S = S^M + S^B$. $S^M$ is a local martingale and $S^B$ is the predictable finite variation part. Both components have also càdlàg paths. So $S^B$ is predictable and possesses càdlàg paths and therefore is also locally bounded (see p.121 in [DM]) and therefore $S^M$ too. In particular $S^M$ is square integrable too. 

**Definition 2.2.4** (i) We denote by $\mathcal{V}$ the set of all real-valued processes $X$ that are adapted, càdlàg, with $X_0 = 0$, and whose each path $t \mapsto X_t(\omega)$ has a finite variation on each interval $[0,t]$. $\text{Var}(X)$ denotes the variation process of $X$.

(ii) Let $X \in \mathcal{V}$. For each $\omega \in \Omega$, the path $t \mapsto X_t(\omega)$ is the distribution function of a signed measure on $\mathbb{R}_+$ if and only if $\text{Var}(X)_\infty(\omega) < \infty$. We denote this measure by $dX_t(\omega)$.

(ii) We write $dX \ll dY$, where $X, Y \in \mathcal{V}$, if the measure $dX_t(\omega)$ is absolutely continuous with respect to the measure $dY_t(\omega)$ for almost all $\omega \in \Omega$.

**Assumption 2.2.5** The trading strategy $\theta = (\theta_t)_{t \in [0,T]}$ is a predictable real-valued process with FV, therefore in particular $\theta \in \mathcal{V}$. Moreover $d\theta$ is absolutely continuous w.r.t. some measure $\mu$, where $\mu$ is positive, finite, that is $\mu([0,T]) < \infty$, and $\mu(\{t\}) = 0$ for $t \in \mathcal{B}([0,T])$. 

Proposition 2.2.6 Let $\theta$ be a trading strategy fulfilling Assumption 2.2.5. Then there exists a predictable density $h$, i.e.,
\[
\theta_t(\omega) = \int_0^t h_s(\omega) \mu(ds) < \infty
\]
(2.2.6)
and the paths $t \mapsto \theta_t$ are Borel-measurable.

Proof. We show at first the existence of the predictable density, then we give two alternative proofs for the measurability of the paths of $\theta$.

First we note that by the theorem of Radon-Nikodym the existence of the density $h$ is guaranteed. Next we show the predictability of the density. By assumption, $\theta \in \mathcal{V}$. According to the construction in equation (2.2.6) $d\theta \ll \mu$. Then there exists an optional process $h$, such that $\theta = h \cdot \mu$ up to a null-set. Moreover $\theta$ is assumed to be predictable, consequently one may choose $h$ to be predictable (Proposition 3.13, p. 30 in [JS]).

Variant I: By assumption, the process $(\theta_t)_{t \in [0,T]}$ is predictable, i.e. $(\omega, t) \mapsto \theta_t(\omega)$ is measurable with respect to the predictable $\sigma$-algebra $\mathcal{P}$ on $\Omega \times [0, T]$, moreover it is measurable with respect to the product $\sigma$-algebra $\mathcal{F}_T \otimes \mathcal{B}([0, T])$, because the predictable $\sigma$-algebra is also generated by the following set: $A \times \{0\}$ where $A \in \mathcal{F}_0$, and $A \times (s, t]$, where $s < t, A \in F_s$ (see e.g. p. 16 in [JS]); and therefore $\mathcal{P} \subset \mathcal{F}_T \otimes \mathcal{B}([0, T])$. Then the mapping $\theta(\cdot, t) : (\Omega, \mathcal{F}_T) \rightarrow (\mathbb{R}_+, \mathcal{B}([0, T]))$ is $\mathcal{F}_T$-measurable and $\theta(\omega, \cdot) : ([0, T], \mathcal{B}([0, T])) \rightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ is $\mathcal{B}([0, T])$-measurable (see e.g. p. 162 in [Els]).

Variant II: Of course this is not a specific property of predictable processes as the alternative proof shows. We consider the positive part $h^+$ of the density $h$. Therefore there exist simple $(h^{(n)})_{n \in \mathbb{N}}$ with $h^{(n)} \uparrow h^+$, where $h^{(n)} = \sum_{i=1}^{N^{(n)}} \alpha_{i,n} 1_{A_{i,n}}$, where $\alpha_1, \ldots, \alpha_{N^{(n)}} \in \mathbb{R}$ and $A_1, \ldots, A_{N^{(n)}} \in \mathcal{A}$ with $\bigcup_{i=1}^{N^{(n)}} A_i = \Xi = [0, T]$. We can write by the theorem of monotone convergence:
\[
\int_0^t h^+_s \mu(ds) = \int_{\Xi \cap [0,t]} h^+_s \mu(ds) = \lim_{n \rightarrow \infty} \int_{\Xi \cap [0,t]} h^{(n)}_s \mu(ds) = \lim_{n \rightarrow \infty} \sum_{i=1}^{N^{(n)}} \alpha_{i,n} \mu(A_{i,n} \cap [0, t])
\]
As the limit and linear combinations of a measurable function is also measurable, we have to show that the function $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}_+, t \mapsto \mu(A_{i,n} \cap [0, t])$ is measurable. Since right-continuous functions are Borel measurable, we show the right-continuity of $\xi$. Let $(t_m)_{m \in \mathbb{N}}$
be a sequence with $t_m \downarrow t$. Then we have with the continuity from above of the measure $\mu$

$$\lim_{m \to \infty} \mu(A_{i,n} \cap [0, t_m]) = \mu\left( \lim_{m \to \infty} A_{i,n} \cap [0, t_m] \right) = \mu\left( \bigcap_{m=1}^{\infty} A_{i,n} \cap [0, t_m] \right) = \mu(A_{i,n} \cap \bigcap_{m=1}^{\infty} [0, t_m]) = \mu(A_{i,n} \cap [0, t]).$$

With the decomposition $h = h^+ - h^-$ the assertion follows.

The next two definitions serve to make clear in which sense the models converge precisely.

**Definition 2.2.7** Let $\sigma$ denote a finite sequence of finite stopping times:

$$0 = T_0 \leq T_1 \leq \cdots \leq T_k = T.$$

The sequence $\sigma$ is called a **random partition**. A sequence of random partitions $(\sigma_n)_{n \in \mathbb{N}}$ with $\sigma_n = (T^n_0, \ldots, T^n_{k_n})$, where $k_n \in \mathbb{N}$, is said to tend to the identity if

(i) $\lim_{n \to \infty} \sup_{k} T^n_k = T$ a.s.; and

(ii) $\|\sigma_n\| = \sup_{k=1,\ldots,k_n} |T^n_{k-1} - T^n_k|$ converges to 0 a.s.

**Definition 2.2.8** A sequence of processes $(K^n)_{n \geq 1}$ converges to a process $K$ uniformly on compacts in probability (ucp) if, for each $t > 0$, $\sup_{0 \leq s \leq t} |K^n_s - K_s|$ converges to 0 in probability.

Now we present the continuous-time dynamics. Up to now we have the definitions of a self-financing strategy 2.2.1 and of different portfolio values 2.2.2 for the discrete environment in mind. Intuitively we develop now the continuous-time dynamics.

**Definition 2.2.9** A strategy $(K_t, \theta_t)_{t \in [0, T]}$ is **self-financing** with initial capital $K_0$ if and only if

$$K_t - K_0 = -\int_0^t h_s S_s \exp(\psi(h_s)) \mu(ds), \quad (2.2.7)$$

where $K_t$ is the cash account at time $t$ of the large trader. $h$ is the density function introduced in Proposition 2.2.6.
Remark. The expression \( S_s - \exp(\psi(h)) \) is the continuous-time analogon of the discrete-time price process of the large trader. Recall that \( \psi \) (compare (2.2.2)) is a continuous and strictly increasing function. By applying \( \psi \) we achieve the equilibrium price on the market by a given quantity of shares. At this point we accept this intuitive setting. In Theorem 2.2.13 this definition will be justified.

**Definition 2.2.10** Let \( \theta \) be a self-financing strategy.

(i) The **book value** at time \( t \in [0, T] \) of the portfolio is defined as

\[
V^b_t(\theta) := K_t(\theta) + S_t \theta_t = v_0 + \int_0^t \theta_s dS_s - \int_0^t h_s(\exp(\psi(h_s))S_s - S_s) \mu(ds),
\]

where \( v_0 := K_0 + \theta_0 S_0 \) is the initial value of the portfolio.

(ii) Furthermore, \( C_t(\theta) \) denotes the **liquidity costs** with respect to a trading strategy \( \theta \), which are generated up to time \( t \in [0, T] \):

\[
C_t(\theta) := \int_0^t h_s(\exp(\psi(h_s))S_s - S_s) \mu(ds).
\]

Before we study the convergence of the models, we examine the connection between the predictable density \( h \) and the discrete-time stock purchasing rate \( \theta_{TK-1} \triangleq \int_{T_{k-1}}^{T_k} \theta_s \mu(ds) \). In Definition 2.2.9 the density \( h \) has already played the formal role of the quotient.

**Proposition 2.2.11** If Assumption 2.2.5 holds, if \( \theta \) is of the form (2.2.6) and if \( h \in L^1(\mu) \), then

\[
\sum_{k=1}^k \int_{T_{k-1}^n}^{T_k^n} \left| h_s - \frac{\theta_{T_k^n} - \theta_{T_{k-1}^n}}{\mu(T_{k-1}^n, T_k^n)} \right| \mu(ds) \xrightarrow{n \to \infty} 0 \quad \mu-a.s.
\]

**Proof.** In order to show (2.2.9) we split the proof into two parts. For \( t \in [0, T] \) define \( k := k(t, n) \) such that \( T_{k-1}^n < t \leq T_k^n \). We show the pointwise convergence, that is

\[
\left| h_t - \frac{\int_{T_{k-1}^n}^{T_k^n} h_s \mu(ds)}{\mu(T_{k-1}^n, T_k^n)} \right| \xrightarrow{n \to \infty} 0 \quad \mu-a.s.
\]

where we have used that \( \theta_{T_k^n} = \int_{T_{k-1}^n}^{T_k^n} h_s \mu(ds) \). Secondly we show the uniform integrability of the limiting sequence and therefore we get the required \( L^1 \)-convergence. We can assume
2.2 A short-term price impact model

w.l.o.g. that $\mu$ is a probability measure, otherwise consider $\mu_{[0,T]}$.

Before, we consider the increasing sequence of $\sigma$-algebras $(\mathcal{G}_n)_{n \in \mathbb{N}}$ with

$$\mathcal{G}_n = \sigma \left( (T^n_{k-1}, T^n_k] : m \leq n, k = 1, \ldots \right) \quad \text{and} \quad \sigma( \bigcup_{n \in \mathbb{N}} \mathcal{G}_n) = \mathcal{B}([0,T]).$$

For fixed $\omega \in \Omega$ the function $h : [0,T] \to \mathbb{R}$ is a Borel-measurable function. We remark that we can write:

$$E_{\mu}[h|\mathcal{G}_n](t) = \int_{T^n_{k(t)-1}}^{T^n_{k(t)}} h_s \mu(ds) / \mu((T^n_{k(t)-1}, T^n_k)),$$

where we clearly distinguish between the dependence on $t$ ($k$ or $k(t)$).

(i) As described above, for fixed $\omega \in \Omega$ the quotient in (2.2.10) can be written as $E_{\mu}[h|\mathcal{G}_n]$. Since $(\mathcal{G}_n)_{n \in \mathbb{N}}$ is an increasing sequence of $\sigma$-algebras we have $E_{\mu}[h|\mathcal{G}_n] \rightarrow h$, $\mu$-a.s., as $n \to \infty$ (see e.g. Corollary II.2.4, p. 62 in [RY]), which concludes the proof of the pointwise convergence.

(ii) Since $h \in L^1(\mu)$, $h$ is uniformly integrable. Therefore there exists a strictly increasing and convex function $\kappa : [0,\infty) \to [0,\infty)$ with $\lim_{x \to \infty} \frac{\kappa(x)}{x} = \infty$ and $\int \kappa(|h|)d\mu < \infty$. Moreover the function $x \mapsto \kappa(|x|)$ is convex and by Jensen’s inequality, we have:

$$E_{\mu}[\kappa(|E_{\mu}[h|\mathcal{G}_n]|)] \leq E_{\mu}[E_{\mu}[\kappa(|h|) |\mathcal{G}_n]] = E_{\mu}[\kappa(|h|)] < \infty.$$

Consequently, $(E_{\mu}[h|\mathcal{G}_n])_{n \in \mathbb{N}}$ is uniformly integrable.

With (i) and (ii) we obtain $L^1$-convergence, which is the assertion (2.2.9).

In preparation of the following theorem, we present the following lemma.

**Lemma 2.2.12** Let $\mu$ and $\nu$ be $\sigma$-finite signed measures on $(\Omega, \mathcal{F})$ with $\nu \sim \mu$. Moreover, let $\mathcal{G} \subset \mathcal{F}$ be a sub-$\sigma$-algebra. Then

$$\frac{d\nu|_{\mathcal{G}}}{d\mu|_{\mathcal{G}}} = E_{\mu}[h|\mathcal{G}] \quad \text{and} \quad \frac{d\mu|_{\mathcal{G}}}{d\nu|_{\mathcal{G}}} = E_{\nu}[1_h|\mathcal{G}],$$
where \( h := \frac{d\nu}{d\mu} \) is the density, i.e., \( \nu(A) = \int_A h \, d\mu \) for all \( A \in \mathcal{F} \), respectively \( \frac{d\mu}{d\nu} = \frac{1}{h} \).

**Proof.** Let \( B \in \mathcal{G} \). Then

\[
\nu(B) = \int_B h \, d\mu = \int_B \mathbb{E}_\mu[h|\mathcal{G}] \, d\mu.
\]

And analogously we have for \( B \in \mathcal{G} \):

\[
\mu(B) = \int_B \frac{1}{h} \, d\nu = \int_B \mathbb{E}_\nu[\frac{1}{h}|\mathcal{G}] \, d\nu.
\]

\[\blacksquare\]

**Remark.** In the following Theorem 2.2.13 we use a sequence of random partitions \( \sigma_n = (T^n_0, \ldots, T^n_{k_n}) \) to derive the continuous-time dynamics. We use the following notations for the introduced models:

1. \( (K^n_t)_{t \in [0,T]} \) describes the cash account of the discrete-time model. We remark that the discrete-time model is defined for the grid points \( 0 = T^n_0 < T^n_1 < \ldots < T^n_{k_n} = T \). The cash account is by construction a right-continuous step function, that is \( (K^n_t)_{t \in [T^n_{k_n-1}, T^n_k]} \) is constant. The discrete-time self-financing condition (compare Definition 2.2.1) is then defined as:

\[
K^n_{T^n_l} - K^n_{T^n_{l-1}} = - (\theta_{T^n_l} - \theta_{T^n_{l-1}}) \exp\left( \psi\left( \frac{\theta_{T^n_l} - \theta_{T^n_{l-1}}}{\mu((T^n_l - T^n_{l-1}))} \right) \right) S^n_{T^n_l}.
\]

2. \( (K_t)_{t \in [0,T]} \) describes the continuous-time model. Recall the self-financing condition (compare Definition 2.2.9):

\[
K_t - K_0 = - \int_0^t h_s S_s^- \exp(\psi(h_s))\mu(ds).
\]

\[\blacklozenge\]
Theorem 2.2.13 Let $\theta$ be a strategy of the form (2.2.6) satisfying
\[
\int_0^T (|h_s| \exp(\psi(h_s)))^2 \mu(ds) < C \in \mathbb{R}_+ \quad \mathbf{P} - \text{a.s.,}
\]  
(2.2.11)
where the function $\psi$ meets the following shape property: $\psi''(x) \geq -(\psi'(x))^2$. Let $(\sigma_n)_{n \in \mathbb{N}} = (T^n_0, \ldots, T^n_{K_n})_{n \in \mathbb{N}}$ be a sequence of random partitions tending to identity. Then, the sequence of processes $(K^n_t)_{t \in [0,T]}$ from Definition 2.2.1 converges for $n \to \infty$ to the process $K = (K_t)_{t \in [0,T]}$ from Definition 2.2.9 uniformly on compacts in probability.

\textbf{Proof.} We want to show the ucp convergence, i.e., $\mathbf{P}[\sup_{0 \leq s \leq t} |K^n_s - K_s| > \epsilon] \xrightarrow{n \to \infty} 0$.

We divide the proof into three steps. By the triangle inequality we obtain:
\[
\sup_{t \in [0,T]} |K^n_t - K_t| = \sup_{t \in [0,T]} |K_t - K^n_{T^n_{k-1}} + K^n_{T^n_{k-1}} - K^n_{T^n_{k-1}} + K^n_{T^n_{k-1}} - K^n_t| \\
\leq \sup_{t \in [0,T]} \left\{|K^n_{T^n_{k-1}} - K^n_t| + |K_t - K^n_{T^n_{k-1}}| + |K^n_{T^n_{k-1}} - K^n_{T^n_{k-1}}|\right\}. 
\]  
(2.2.12)

We show convergence in probability for each of the summands in equation (2.2.12) as $n$ tends to zero.

As already remarked we use the notation $(K^n_t)_{t \in [0,T]}$ for the discrete-time model (Definition 2.2.1) and $(K_t)_{t \in [0,T]}$ for the continuous-time model (Definition 2.2.9). As we see, step 1 deals with the difference of the cash accounts $|K^n_{T^n_{k-1}} - K^n_t|$ between an arbitrary time point $t$ and a grid point $T^n_{k-1}$ of the discrete-time model. Analogously, we study this difference in step 2 but within the continuous-time model: $|K_t - K^n_{T^n_{k-1}}|$. Naturally step 3 is the most demanding part of the proof, because we show the convergence of the discrete and continuous time models.

\textbf{Step 1:} We show pointwise convergence of $|K^n_{T^n_{k-1}} - K^n_t| \to 0$ as $n \to 0$, which implies convergence in probability. Let $t \in [0,T]$ and $\omega \in \Omega$ arbitrary but fixed. Take $k := k(t,n)$ such that $T^n_{k-1} \leq t < T^n_k$. Then $|K^n_{T^n_{k-1}} - K^n_t| \xrightarrow{n \to \infty} 0$, because $K^n$ is by construction a right continuous step function; $\theta$ and the semimartingale $S$ have càdlàg paths and $\exp \circ \psi$ is continuous.

\textbf{Step 2:} We consider now the continuous-time model and show pointwise convergence of $|K_t - K^n_{T^n_{k-1}}| \to 0$ as $n \to 0$, which implies convergence in probability. Again let $t \in [0,T]$ and $\omega \in \Omega$ arbitrary but fixed. Define $k := k(t,n)$ such that $T^n_{k-1} \leq t < T^n_k$. 


Then we have with Definition 2.2.9:

\[
|K_t - K_{T_{n-1}}| = |K_t - K_0 - (K_{T_{n-1}} - K_0)|
= \left| - \int_0^t h_s S_{s-} \exp(\psi(h_s)) \mu(ds) - \left( - \int_0^{T_{n-1}} h_s S_{s-} \exp(\psi(h_s)) \mu(ds) \right) \right|
= \left| - \int_{T_{n-1}}^t h_s S_{s-} \exp(\psi(h_s)) \mu(ds) \right|.
\]

As the semimartingale \( S \) has càdlàg paths, it is bounded along each trajectory in the interval \( (T_{n-1}, t] \). We get the following upper bound:

\[
\left| - \int_{T_{n-1}}^t h_s S_{s-} \exp(\psi(h_s)) \mu(ds) \right| \leq \sup_{s \in (T_{n-1}, t]} |S_{s-}| \int_{T_{n-1}}^t |h_s| \exp(\psi(h_s)) \mu(ds).
\]

By assumption, \( \int_0^T f_s \mu(ds) < \infty \). Therefore \( f \) is uniformly integrable. Moreover by Assumption 2.2.5, \( \mu([0, T]) < \infty \). Then for any \( \epsilon > 0 \) there exists a \( \delta(\epsilon) > 0 \) such that \( \int_A |f| d\mu \leq \epsilon \) for any measurable set \( A \subset [0, T] \) with \( \mu(A) < \delta(\epsilon) \). Letting \( A_n := (T_{n-1}, t] \), we infer \( \int_{T_{n-1}}^t |f| d\mu \rightarrow_{n \to \infty} 0 \), because \( \mu(A_n) \rightarrow_{n \to \infty} 0 \) (remember \( \mu(\{t\}) = 0 \)). As the semimartingale \( S \) is bounded along each trajectory in \( (T_{n-1}, t] \), we obtain \( |K_t - K_{T_{n-1}}| \to 0 \) for \( n \to \infty \).

**Step 3:** In the last step we show the convergence of the discrete-time and the continuous-time model at one grid point \( T_k \), that is \( |K_{T_{k-1}} - K_{T_k}^n| \to 0 \) as \( n \to 0 \). We take the supremum over all possible grid points. We have:

\[
\sup_{k=1, \ldots, k_n} |K_{T_k}^n - K_{T_k}^n| = \sup_{k=1, \ldots, k_n} \left| (K_{T_k}^n - K_{T_k}^n) - (K_{T_k}^n - K_{T_k}^n) \right|
= \sup_{k=1, \ldots, k_n} \left| \sum_{l=1}^k (K_{T_l}^n - K_{T_{l-1}}^n) - (K_{T_l}^n - K_{T_{l-1}}^n) \right|
\leq \sum_{l=1}^k \left| (K_{T_l}^n - K_{T_{l-1}}^n) - (K_{T_l}^n - K_{T_{l-1}}^n) \right|,
\]

(2.2.13)

where the first equation is clear, because \( K_{T_0}^n = K_{T_0}^n = 0 \) (\( T_0^0 = 0 \) for all \( n \)). Then we extend the expression by a telescope sum and obtain with the triangle inequality equation (2.2.13). We remark that the differences of the cash accounts correspond to the self-financing condition of each model. Therefore we just plug in the formulas from the
Definitions 2.2.1 and 2.2.9 and use in equation (2.2.14) the representation of θ with density h (compare Proposition 2.2.6):

\[
\begin{align*}
&\sum_{l=1}^{k_n} \left| - \int_{T_{l-1}^n}^{T_l^n} h_s S_{s-} \exp(\psi(h_s)) \, \mu(ds) + \left[ (\theta_{T_l^n} - \theta_{T_{l-1}^n}) \exp \left( \frac{\theta_{T_l^n} - \theta_{T_{l-1}^n}}{\mu((T_{l-1}^n, T_l^n))} \right) \right] S_{T_l^n-} \right| \\
= &\sum_{l=1}^{k_n} \int_{T_{l-1}^n}^{T_l^n} h_s \exp \left( \frac{\theta_{T_l^n} - \theta_{T_{l-1}^n}}{\mu((T_{l-1}^n, T_l^n))} \right) S_{T_l^n-} - \exp(\psi(h_s)) S_{s-} \, \mu(ds) \right| .
\end{align*}
\] (2.2.14)

Now we divide equation (2.2.14) by adding a null and using the triangle inequality twice:

\[
\begin{align*}
&\sum_{l=1}^{k_n} \int_{T_{l-1}^n}^{T_l^n} h_s \left[ \exp \left( \frac{\theta_{T_l^n} - \theta_{T_{l-1}^n}}{\mu((T_{l-1}^n, T_l^n))} \right) S_{T_l^n-} - \exp(\psi(h_s)) S_{s-} \right] \, \mu(ds) \\
\leq &\sum_{l=1}^{k_n} \int_{T_{l-1}^n}^{T_l^n} \left[ h_s \exp \left( \frac{\theta_{T_l^n} - \theta_{T_{l-1}^n}}{\mu((T_{l-1}^n, T_l^n))} \right) S_{T_l^n-} - h_s \exp(\psi(h_s)) S_{s-} \right] \, \mu(ds) \\
&+ \sum_{l=1}^{k_n} \left| \int_{T_{l-1}^n}^{T_l^n} h_s \exp(\psi(h_s)) S_{T_l^n-} - h_s \exp(\psi(h_s)) S_{s-} \right| \, \mu(ds) \\
\leq &\sum_{l=1}^{k_n} \left| S_{T_l^n-} \right| \left| h_s \right| \exp \left( \frac{\theta_{T_l^n} - \theta_{T_{l-1}^n}}{\mu((T_{l-1}^n, T_l^n))} \right) - \exp(\psi(h_s)) \right| \, \mu(ds) \\
&+ \sum_{l=1}^{k_n} \int_{T_{l-1}^n}^{T_l^n} h_s \exp(\psi(h_s)) \left[ S_{T_l^n-} - S_{s-} \right] \, \mu(ds) \right| .
\end{align*}
\] (2.2.15)

In order to show \(|K_{T_{k-1}^n} - K_{T_{T_{k-1}^n}}^n| \to 0\) as \(n \to 0\), we show that equation (2.2.15) tends to zero as \(n \to \infty\). Both terms in (2.2.15) will be dealt separately.

(i) For showing the first part, define \(l := l(s, n)\) such that \(T_{l-1}^n < s \leq T_l^n\) for \(s \in [0, T]\). We have:

\[
\begin{align*}
&\sum_{l=1}^{k_n} \int_{T_{l-1}^n}^{T_l^n} \left| S_{T_l^n-} \right| \left| h_s \right| \exp \left( \frac{\theta_{T_l^n} - \theta_{T_{l-1}^n}}{\mu((T_{l-1}^n, T_l^n))} \right) - \exp(\psi(h_s)) \right| \, \mu(ds) \\
=: &\sum_{l=1}^{k_n} \int_{T_{l-1}^n}^{T_l^n} \left| F(n, s) \right| \, \mu(ds) \right| \\
= &\int_0^T \left| F(n, s) \right| \, \mu(ds) .
\end{align*}
\] (2.2.16) (2.2.17)

We show that equation (2.2.17) tends to zero as \(n \to \infty\). We split the proof into two parts. In part one we show the pointwise convergence of the integrand, i.e., \(F(n, s) \to 0\) as \(n \to \infty\).
for \( \mu \)-almost all \( s \in [0, T] \). In part two we show that \((F(n, \cdot), n \in \mathbb{N})\) is uniformly \( \mu \)-integrable. Therefore we obtain the \( L^1 \)-convergence, which implies the required convergence in probability.

We assume w.l.o.g. that \( \mu \) is a probability measure (\( \mu \) is finite). Then - as described in the proof of Proposition 2.2.11 - for fixed \( \omega \in \Omega \) the quotient \( \frac{1}{\mu((T_{n-1}^m, T_n^m])} \int_{T_{n-1}^m}^{T_n^m} h_s \mu(ds) \) can be written as \( E_\mu[h | G_n] \), where

\[
G_n = \sigma\left((T_{k-1}^m, T_k^m) : m \leq n, k = 1, \ldots \right) \quad \text{and} \quad \sigma\left( \bigcup_{n \in \mathbb{N}} G_n \right) = \mathcal{B}([0, T]) .
\]

With Proposition 2.2.11 we get

\[
| h_s | \exp\left(\psi(E_\mu[h | G_n])\right) \rightarrow 0
\]
as \( n \to \infty \), because the function \( \exp \circ \psi \) is continuous. Since the semimartingale \( S \) is bounded along each trajectory, \( F(n, s) \to 0 \) for \( \mu \)-a.s. \( s \in [0, T] \).

Now we show the uniform integrability of \((F(n, \cdot), n \in \mathbb{N})\). Equation (2.2.16) can be rewritten:

\[
\sum_{l=1}^{k_n} \int_{T_{l-1}^n}^{T_n^m} F(n, s) \mu(ds) = \sum_{l=1}^{k_n} \int_{T_{l-1}^n}^{T_n^m} \left[ |h_s| \exp\left(\psi(E_\mu[h | G_n])\right) - |h_s| \exp(\psi(h_s)) \right] \mu(ds).
\]

In step 2 we have already seen that \( |h| \exp(\psi(h)) \in \mathcal{L}^1(\mu) \). Moreover \( S \) is bounded along each trajectory. Hence the uniform integrability of the term \((|h| \exp(\psi(E_\mu[h | G_n])))_n \in \mathbb{N}\) is left to be shown. For the proof we use a change of measure argument. By Jensen’s inequality

\[
| h \exp(\psi(E_\mu[h | G_n])) | \leq |h| \exp(\psi(E_\mu[h | G_n]))
\]
it follows that it is sufficient to show that the upper bound is uniformly integrable (p. 131 in [Kl]). Moreover we can assume w.l.o.g. that \( h \geq 0 \). Since the right hand side of inequality (2.2.18) is strictly increasing for \( h \geq 0 \), we assume moreover w.l.o.g. that \( h > 0 \). Let \( \nu \sim \mu \) with the Radon-Nikodym derivative \( \frac{d\nu}{d\mu} = h \). We have by assumption (2.2.11)

\[
E_\nu[\exp(\psi(h))] = E_\mu[h \exp(\psi(h))] < \infty .
\]

Therefore, \( \exp(\psi(h)) \in \mathcal{L}^1(\nu) \) and as in (ii) of the proof of Proposition 2.2.11 already shown
the family of conditional expectations \((E_\nu[\exp(\psi(h))|G_n])_{n\in\mathbb{N}}\) is uniformly integrable w.r.t. measure \(\nu\), i.e., for any \(\epsilon > 0\) there exists a \(\delta(\epsilon) > 0\) such that \(E_\nu[1_A E_\nu[\exp(\psi(h))|G_n]] < \epsilon\) for all \(A \in \mathcal{B}([0, T])\) with \(\nu(A) < \delta\) (see e.g. p. 133 in [Kl]).

For the next calculations (the first two equalities) we have in mind that on the one hand \(\frac{d\mu}{d\nu} = \frac{1}{h}\) and on the other hand \(\frac{d\nu}{d\mu} = E_\mu[h|G_n]\), respectively \(\frac{d\mu}{d\nu} = E_\nu[\frac{1}{h}|G_n]\) (Lemma 2.2.12). Therefore we get for all \(A \in \mathcal{B}([0, T])\) with \(\nu(A) < \delta(\epsilon)\):

\[
E_\mu[1_A h \exp(\psi(E_\mu[h|G_n])]) = E_\nu[1_A \exp(\psi(E_\mu[h|G_n]))]
= E_\nu[1_A \exp(\psi(E_\mu[h|G_n]))].
\] (2.2.20)

By Jensen’s inequality we can find an upper bound for the last equation (2.2.20), because the function \(x \mapsto \frac{1}{x}\) with \(x \in \mathbb{R}_+\) is convex and \(\exp \circ \psi\) is strictly increasing. We get:

\[
E_\nu[1_A \exp(\psi(E_\mu[h|G_n])]) \leq E_\nu[1_A \exp(\psi(E_\nu[h|G_n])]).
\] (2.2.21)

Again we apply Jensen’s inequality on the last equation (2.2.21). We use the assumption, that \(\psi''(x) \geq -(\psi'(x))^2\), because then the function \(C(x) := \exp(\psi(x))\) is convex:

\[
C'(x) = \exp(\psi(x))\psi'(x)
\]
\[
C''(x) = \exp(\psi(x))((\psi'(x))^2 + \psi''(x)) \geq 0 \iff \psi''(x) \geq -(\psi'(x))^2.
\] (2.2.22)

If the function \(\psi\) fulfills condition (2.2.22), the Jensen inequality can be applied:

\[
E_\nu[1_A \exp(\psi(E_\nu[h|G_n])]) \leq E_\nu[1_A E_\nu[\exp(\psi(h))|G_n]].
\] (2.2.23)

As already mentioned \((E_\nu[\exp(\psi(h))|G_n])_{n\in\mathbb{N}}\) is uniformly integrable. Moreover \(A \in \mathcal{B}([0, T])\) with \(\nu(A) < \delta\) such that \(E_\nu[1_A E_\nu[\exp(\psi(h))|G_n]] < \epsilon\). Consequently, we have shown the uniform integrability of our initial sequence \((h|\exp(\psi(E_\mu[h|G_n]))))_{n\in\mathbb{N}}.

All in all we have shown the pointwise convergence and the uniform integrability of \(F(n, s)\), which completes the proof of the first part, that is

\[
\int_0^T |F(n, s)| \mu(ds) \xrightarrow{n \to \infty} 0.
\]

(ii) In the second part of step three we study the other term of equation (2.2.15), that is

\[
\sum_{l=1}^{k_n} \int_{T_{l-1}}^{T_l} h_s \exp(\psi(h_s)) [S_{T_l} - S_{T_{l-1}}] \mu(ds).
\] (2.2.24)
In order to show that (2.2.24) tends to zero as \( n \to \infty \), we split the semimartingale into its two canonical parts \( S^M \) and \( S^B \). We show \( L^2 \)-convergence, respectively \( L^1 \)-convergence. Both concepts imply the required convergence in probability and are supportive concepts in the context of semimartingales. The critical part will be of course the control of the differences between the time points of the semimartingale.

\( S \) is a locally bounded semimartingale based on and adapted to a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\). With Proposition 2.2.3 \( S \) is a special semimartingale and the price process can be decomposed as \( S = S^M + S^B \), where \( S^M_0 = S^B_0 = 0 \), \( S^M \) is a local \( L^2 \)-integrable martingale and \( S^B \) is a locally bounded predictable finite variation process. We study separately the two locally bounded components \( S^M \) and \( S^B \) of the price process \( S \).

W.l.o.g. we can assume that \( S^M \), \( S^B \) and \( \theta \) (the trading strategy is locally bounded - predictable and has càdlàg paths) are bounded, instead of only locally bounded. This assumption can be justified by the aim of our work. We want to show the uniform convergence in probability of the discrete and continuous-time models, i.e., \( \mathbb{P}[\sup_{t \in [0,T]} |K^n_t - K_t| > \delta] \leq \epsilon \). Exemplarily we consider the process \( S^M \). \( S^M \) is locally bounded, i.e., there exists a sequence of stopping times \( (\tau_n)_{n \geq 1} \) with \( \mathbb{P}[\tau_n \to \infty] = 1 \) such that for each \( n \geq 1 \), the stopped process \( (S^M_t)_{\tau_n \wedge T} = S^M_{T \wedge \tau_n} \) is bounded. For any fixed \( \epsilon > 0 \) there exists a \( m_\epsilon \) that for all \( m \geq m_\epsilon \), \( \mathbb{P}[\tau_m \leq T] \leq \epsilon \). We define \( R_n := \tau_n 1_{\{\tau_n \leq T\}} + T 1_{\{\tau_n > T\}} \). Then we have:

\[
\begin{align*}
\mathbb{P}[\sup_{t \in [0,T]} |(S^M_t)^n - S^M_t| > \delta] &\leq \mathbb{P}[\{\sup_{t \in [0,T]} |(S^M_t)^n - S^M_t| > \delta\} \cap \{R_n = T\}] + \mathbb{P}[\{\sup_{t \in [0,T]} |(S^M_t)^n - S^M_t| > \delta\} \cap \{R_n < T\}] \\
&\leq \mathbb{P}[\{\sup_{t \in [0,T]} |(S^M_t)^n - S^M_t| > \delta\} \cap \{R_n = T\}] + \mathbb{P}[R_n < T] \\
&\leq \mathbb{P}[\{\sup_{t \in [0,T]} |(S^M_t)^n - S^M_t| > \delta\} \cap \{R_n = T\}] + \epsilon \\
&= \mathbb{P}[\{\sup_{t \in [0,T]} |(S^M_{T \wedge \tau_n})^n - S^M_{T \wedge \tau_n}| > \delta\}] + \epsilon.
\end{align*}
\]

Therefore it is sufficient to show the convergence of the stopped and therefore bounded process.

Now back to the problem of the convergence in (2.2.24). As already mentioned we use the canonical decomposition of the semimartingale \( S = S^M + S^B \) (compare 2.2.3), which both now w.l.o.g. can be assumed to be bounded. We replace both parts in (2.2.24) and show the result respectively. We work with the following notation: \( S^M_t = \int_0^t 1 \, dS^M_r \),
$S^B_t = \int_0^t 1 \, dS^B_r$, $f^l_0 = \int_{[0, t]}$ and $f^{r_1}_0 = \int_{[r, t]}$.

We start with the local martingale part $S^M$ and show $L^2$-convergence. We have:

$$
E \left[ \left( \sum_{l=1}^{k_n} \int_{T^n_{l-1}}^{T^n_l} h_s \exp(\psi(h_s)) \left[ S^M_{T^n_{l-1}} - S^M_s \right] \mu(ds) \right)^2 \right] = E \left[ \left( \sum_{l=1}^{k_n} \int_{T^n_{l-1}}^{T^n_l} h_s \exp(\psi(h_s)) \left[ \int_0^{T^n_l} 1 \, dS^M_r - \int_0^{s} 1 \, dS^M_r \right] \mu(ds) \right)^2 \right] = E \left[ \left( \sum_{l=1}^{k_n} \int_{T^n_{l-1}}^{T^n_l} h_s \exp(\psi(h_s)) \left[ \int_{s}^{T^n_l} 1 \, dS^M_r \right] \mu(ds) \right)^2 \right] = E \left[ \left( \sum_{l=1}^{k_\rho} \int_{T^n_{l-1}}^{T^n_l} h_s \exp(\psi(h_s)) \left[ \int_{T^n_{l-1}}^{T^n_l} \mathbb{1}_{\{r \geq s\}} 1 \, dS^M_r \right] \mu(ds) \right)^2 \right]. (2.2.25)
$$

Next we apply Fubini's theorem for stochastic integrals (see p. 211 in [Pro]). We define $E_t := E(\omega, t, r) := h_t(\omega) \exp(\psi(h_t(\omega))) \mathbb{1}_{\{r \geq t\}}(\omega)$. Since $h$ is a predictable density, $E(\omega, t, r)$ is $\mathcal{P} \otimes \mathcal{B}([0, T])$ measurable. Moreover we have assumed in Assumption 2.2.5 that $\mu$ is a finite and positive measure on $[0, T]$. Next we have to study whether $(\int_0^T (E_t)^2 \mu(ds))^{1/2} \in L(S^M)$, that is

$$
\left( \int_{T^n_{l-1}}^{T^n_l} (h_s \exp(\psi(h_s)) \mathbb{1}_{\{r \geq s\}})^2 \mu(ds) \right)^{1/2} \in L(S^M).
$$

This is clear, because in (2.2.11) we have assumed that $\int_0^T (|h_s| \exp(\psi(h_s)))^2 \mu(ds) < C$. Therefore all prerequisites are fulfilled and we can apply Fubini's theorem in equation (2.2.26). Moreover we define

$$
G_t(\omega) := \int_0^t h_s(\omega) \exp(\psi(h_s(\omega))) \mu(ds)
$$

and plug in the function $G_t$, where the indicator function constraints the integration area.
We obtain:

\[
E \left[ \left( \sum_{l=1}^{k_n} \int_{T_{l-1}^n}^{T_l^n} (G_r - G_{T_{l-1}^n}) dS_r^M \right)^2 \right] = E \left[ \left( \sum_{l=1}^{k_n} \int_{T_{l-1}^n}^{T_l^n} \left( \Psi_{T_l^n} - \Psi_{T_{l-1}^n} \right) \right)^2 \right]. \tag{2.2.27}
\]

Moreover we can extract the sum out of the expectation value, because the increments of a martingale are uncorrelated. With \( \Psi_{T_l^n} := \int_0^{T_l^n} (G_r - G_{T_{l-1}^n}) dS_r^M \) which is a martingale (integral with bounded integrand with respect to a martingale), we can write for (2.2.27):

\[
E \left[ \max_{l=1, \ldots, k_n} \left( \Psi_{T_l^n} - \Psi_{T_{l-1}^n} \right)^2 \right] \text{ and therefore interpret the integral as increments of martingales.}
\]

Then we use the Itô-isometry formula and finally we estimate the obtained expression. We obtain:

\[
E \left[ \sum_{l=1}^{k_n} \left( \int_{T_{l-1}^n}^{T_l^n} (G_r - G_{T_{l-1}^n}) dS_r^M \right)^2 \right] = E \left[ \sum_{l=1}^{k_n} \int_{T_{l-1}^n}^{T_l^n} (G_r - G_{T_{l-1}^n})^2 d[S_r^M, S_r^M] \right]
\]
\[
\leq E \left[ \max_{l=1, \ldots, k_n} \sup_{r \in [T_{l-1}^n, T_l^n]} (G_r - G_{T_{l-1}^n})^2 \sum_{l=1}^{k_n} (|S_r^M, S_r^M|_{T_l^n}^T - |S_r^M, S_r^M|_{T_{l-1}^n}^T) \right]. \tag{2.2.28}
\]

Now we want to take the limit \( n \to \infty \) in (2.2.28) to receive \( L^2 \)-convergence. For this we use the theorem of dominated convergence twice. At first we consider the inner part of the expectation value. For showing the convergence of

\[
\max_{l=1, \ldots, k_n} \sup_{r \in [T_{l-1}^n, T_l^n]} (G_r - G_{T_{l-1}^n})^2 \xrightarrow{n \to \infty} 0, \tag{2.2.29}
\]

we have to use the theorem of dominated convergence for the first time, because \( G_t := \int_0^t h_s \exp(\psi(h_s)) \mu(ds) \). We need an \( \mu \)-integrable upper bound for the term (2.2.29). With assumption (2.2.11) we get:

\[
\left| \max_{l=1, \ldots, k_n} \sup_{r \in [T_{l-1}^n, T_l^n]} (G_r - G_{T_{l-1}^n})^2 \right| = \left| \max_{l=1, \ldots, k_n} \sup_{r \in [T_{l-1}^n, T_l^n]} \left( \int_{T_{l-1}^n}^{T_l^n} h_s \exp(\psi(h_s)) \mu(ds) \right)^2 \right| \leq \left( \int_0^T |h_s| \exp(\psi(|h_s|)) \mu(ds) \right)^2 \in L^1(\mu).
\]
Therefore we can take the limit within the integral. The integrand is again uniformly integrable and with the decreasing sequence of integration limits we obtain the convergence (compare step 2).

Then we consider the other part of the inner expression of the expected value. We need to establish that the sum is finite. For this we rearrange the sum. For equation (2.2.30) recall that $T_0 = 0$ and $\Delta [S^M, S^M]_{T^n} = [S^M, S^M]_{T^n} - [S^M, S^M]_{T^n-1}$. In equation (2.2.31) we use the fact that $\Delta [S^M, S^M] = (\Delta S^M)^2$. Moreover we have for any $t > 0$ that $\sum_{0<s\leq t} (\Delta S^M_s)^2 \leq [S^M, S^M]_t < \infty$. Therefore the sum in (2.2.32) is bounded below and the whole expression is finite:

\[
\sum_{l=1}^{k_n} ([S^M, S^M]_{T_l^n} - [S^M, S^M]_{T_{l-1}^n})
\]

\[
=[S^M, S^M]_{T_{k_n}} - [S^M, S^M]_0 - \sum_{l=1}^{k_n-1} ([S^M, S^M]_{T_l^n} - [S^M, S^M]_{T_{l-1}^n})
\]

\[
=[S^M, S^M]_T - \sum_{l=1}^{k_n-1} \Delta [S^M, S^M]_{T_l^n}
\]

\[
=[S^M, S^M]_T - \sum_{l=1}^{k_n-1} \Delta S^M_{T_l} \Delta S^M_{T_l}
\]

\[
=[S^M, S^M]_T - \sum_{l=1}^{k_n-1} (\Delta S^M_{T_l})^2.
\]

Now we want to apply the theorem of dominated convergence for the second time, because we have to take the limit within the expectation value. For this we need an \(P\)-integrable upper bound. We consider the inner part of the expected value, where we have in mind, that the series converges. We have

\[
\left| \max_{1 \leq n \leq k_n} \sup_{r \in [T_{n-1}^T, T_n^T]} (G_r - G_{T_{n-1}^T})^2 \sum_{l=1}^{k_n} ([S^M, S^M]_{T_l^n} - [S^M, S^M]_{T_{l-1}^n}) \right| 
\]

\[
\leq \left( \int_0^T h_s \exp(\psi(h_s)) \mu(ds) \right)^2 |\tilde{C}| \leq C^*,
\]

where \(\tilde{C}\) and \(C^*\) are constants. The upper bound in (2.2.33) is bounded and therefore \(P\)-integrable and we can apply the theorem of dominated convergence in equation (2.2.28).
Therefore we get the required $L^2$-convergence of equation (2.2.25), which was:

$$
E \left[ \sum_{l=1}^{k_n} \left( \int_{T_{l-1}^{n}}^{T_{l}^{n}} h_s \exp(\psi(h_s)) \left[ S_{T_l}^M - S_{s-}^M \right] \mu(ds) \right)^2 \right] \rightarrow 0 \text{ as } n \text{ tends to } \infty.
$$

Now we want to examine the other part of the price process, namely the predictable FV process $S^B$. The initial calculations are very similar to the calculations above in the case where we have considered the martingale part of the price process; here we show $L^1$-convergence. We have:

$$
E \left[ \sum_{l=1}^{k_n} \left( \int_{T_{l-1}^{n}}^{T_{l}^{n}} h_s \exp(\psi(h_s)) \left[ S_{T_l}^B - S_{s-}^B \right] \mu(ds) \right) \right] = E \left[ \sum_{l=1}^{k_n} \left( \int_{T_{l-1}^{n}}^{T_{l}^{n}} h_s \exp(\psi(h_s)) \left[ \int_{0}^{T_l} - \int_{0}^{s} \right] ds \right) \mu(ds) \right]
$$

For the same reasons we are again able to change the order of integration by using Fubini’s theorem. After that we put in the already defined function $G_t$. In inequality (2.2.35) we use the fact that any FV process can be written as difference of two increasing processes, i.e., $S^B = S^{B+} - S^{B-}$ and that $S^{B+} = \frac{1}{2}(Var(S^B) + S^B)$, respectively $S^{B-} = \frac{1}{2}(Var(S^B) - S^B)$ (see e.g. Proposition 3.3, p.27 in [JS]). It is clear that $Var(S^B) \geq S^B$. We have

$$
E \left[ \sum_{l=1}^{k_n} \left( \int_{T_{l-1}^{n}}^{T_{l}^{n}} h_s \exp(\psi(h_s)) \mathbb{1}_{\{r \geq s\}} \mu(ds) \right) \right]
$$

$$
= E \left[ \sum_{l=1}^{k_n} \left( \int_{T_{l-1}^{n}}^{T_{l}^{n}} G_r - G_{T_{l-1}^{n}} dS_r^B \right) \right]
$$

$$
\leq E \left[ \max_{l=1,\ldots,k_n} \sup_{r \in [T_{l-1}^{n},T_l^{n}]} |G_r - G_{T_{l-1}^{n}}| Var(S^B)_r \right] \quad \text{ (2.2.35)}
$$

$$
\leq E \left[ \max_{l=1,\ldots,k_n} \sup_{r \in [T_{l-1}^{n},T_l^{n}]} \left( Var(S^B)_T - Var(S^B)_{T_{l-1}^{n}} \right) \right] \quad \text{ (2.2.36)}
$$
Again we want to apply the theorem of dominated convergence to take the limit within the expectation value. The integrability of the upper bound is obvious, because in (2.2.11) we have assumed that in particular $|G_T| \leq \int_0^T |h_s \exp(\psi(h_s))| \mu(ds)$ exists. The sum

$$\sum_{l=1}^{k_n} \left( \text{Var}(S^B)_{T_{kn}} - \text{Var}(S^B)_{T_{kn}^{-1}} \right)$$

$$= \text{Var}(S^B)_{T_{kn}} - \text{Var}(S^B)_{T_{0}} - \sum_{l=1}^{k_{n-1}} \left( \text{Var}(S^B)_{T_{kn}} - \text{Var}(S^B)_{T_{kn}^{-1}} \right)$$

$$= \text{Var}(S^B)_{T_{kn}} - \sum_{l=1}^{k_{n-1}} \Delta \text{Var}(S^B)_{T_{kn}}$$

is bounded, because $S^B$ is a FV process. Therefore the whole expression in (2.2.34) tends to zero, because $|G_r - G_{T_{kn}^{-1}}| = |\int_{T_{kn}^{-1}}^r h_s \exp(\psi(h_s)) \mu(ds)|$ tends to zero for $n \to \infty$.

All in all we have shown the required convergence in probability (as $n \to \infty$) of the expression

$$\sum_{l=1}^{k_n} \left( \int_{T_{kn}^{-1}}^{T_{kn}^n} h_s \exp(\psi(h_s)) \left[ S_{T_{kn}^n} - S_{T_{kn}^{-1}} \right] \mu(ds) \right),$$

because we have shown $L^2$-convergence for the local martingale part $S^M$ and $L^1$-convergence for the FV part $S^B$. 

\[\blacksquare\]
Chapter 3

Absence of Arbitrage

In the second part of this thesis we study arbitrage opportunities in illiquid economies. We will especially consider a modification of the permanent price effect model of Bank and Baum, which was introduced in the second chapter. We will show that there exist no arbitrage opportunities, where we have to assume that the price process is described by a continuous semimartingale.

3.1 Problem Setting

Before we study illiquid economies in detail, we want to motivate the necessity of studying different trading strategies in a more general context. For this we consider a Black/Scholes economy (see [BS73]). We will show that a trader is able to create real wealth at no risk (arbitrage), if she follows a doubling strategy. Even if this result is well known and quite intuitive, we will present it within a formal framework.

If the trader realizes losses in one period she will adjust the stock position with the aim to compensate the accumulated losses and to realize a surplus within the succeeding period. For instance she buys one share of Deutsche Telekom for 20 Euro. The price decreases to 15 Euro in the subsequent week. If she purchased for example five additional shares (so a total of six shares) and the price increased only by one Euro, she would have compensated the losses in the amount of five Euro and would have made a surplus of one Euro. We want to show that it is always possible to make a profit within this economy by following such a trading strategy.

A Black/Scholes economy is characterized by perfect elasticity, no transaction costs and
no dividend payments. The trader has the possibility to invest in a risky asset \( S \), e.g. a stock, or in a riskless bond \( B \). The price processes of the assets are given by:

\[
\begin{align*}
\frac{dB_t}{B_t} &= rt \, dt \\
\frac{dS_t}{S_t} &= \mu S_t \, dt + \sigma S_t \, dW_t
\end{align*}
\]

\( S_0 = s_0 \),

where \( W \) is a Wiener process and \( r \), \( \mu \) and \( \sigma \) are deterministic constants. The solution to equation (3.1.2) is well known and is given by

\[
S_t = s_0 \exp \left( (\mu - \frac{1}{2} \sigma^2) t + \sigma W_t \right).
\]

Without loss of generality\(^1\) we assume \( \frac{1}{2} \sigma^2 < \mu \), \( s_0 = 1 \) and that the time interval is standardized to \([0, 1]\).

Then we have

\[
S_t = e^{(\mu - \frac{1}{2} \sigma^2) t + \sigma W_t}
\]

or equivalently

\[
S_t e^{-(\mu - \frac{1}{2} \sigma^2) t} = e^\sigma W_t.
\]

We divide the time interval into an infinite number of intervals. The trader starts at time 0 with one stock and observes the development of the portfolio until time \( t = \frac{1}{2} \). If the gain of the portfolio value exceeds a certain level, e.g. 1, she liquidates her stock position at exactly this time and stops trading. On the other hand if the loss of her position crosses a fixed lower bound (e.g. -1), we assume that the ”game” stops, i.e., the trader liquidates the whole portfolio and realizes the loss. Then the trader starts trading at time point \( t = \frac{1}{2} \) again with his decision of buying a new position of shares. The trader has to take into consideration that the probability of compensating her losses increases by buying more shares. Now the game starts again. The trader observes her position up to time point \( t = \frac{3}{4} \) (the half of the remaining time interval). She liquidates her position, if the capital gains compensate the accumulated losses and bless her with a profit of 1. On the other side if the debt position of her portfolio gets again negative (doubles), i.e. crosses the border \(-3\)

\(^1\)Some of the succeeding calculations would have to be modified, if \( \mu \leq \frac{1}{2} \sigma^2 \), but we will waive this case, because the variations are not very difficult.
she stops again, realizes her losses and begins with the new decision of a number of shares for the next period at the new price of the stock at the same time $S_{\frac{3}{4}}$. If the portfolio value does not cross any of the boundaries within the examined period, we will assume, that the portfolio value has crossed the lower bound. Therefore we aggravate the situation, nevertheless the trader is able to make a profit.

Now we formalize the described game. The trader observes her portfolio value during the $n$-th period $[t_n, t_{n+1})$, where $t_n := 1 - 2^{-(n-1)}$ with $n \in \mathbb{N}$. The bisection of the remaining time interval is relevant for producing a potentially infinite number of trading points. We divide the procedure into two steps. At first we determine the upper and lower level $\delta_n$ for the evolution of the Brownian motion. Therefore we isolate the Brownian term and consider $X_t$ (see (3.1.4)). Within the second step we determine the capital commitment $K_n$, which is necessary to generate a profit. Here we choose a new perspective; we do not study the trading strategy $(\theta_n)_n$ (the constant number of shares during period $n$) and the asset price process $(S_t)_{t \in [0,1]}$ separately, but the dynamics of the total amount $K_n = \theta_n S_{t_n}$, which the trader invests at the beginning of a period.

We define the sequence of stopping times $(\tau_{(-\delta_n, \delta_n)})_{n \in \mathbb{N}}$:

$$
\tau_{(-\delta_n, \delta_n)} := \inf\{t \in [t_n, t_{n+1}) : \sigma|W_t - W_{t_n}| \geq \delta_n\}
$$

$$
= \inf\{t \in [t_n, t_{n+1}) : |W_t - W_{t_n}| \geq \frac{\delta_n}{\sigma}\}
$$

$$
=: \inf\{t \in [t_n, t_{n+1}) : |W_t - W_{t_n}| \geq \tilde{\delta}_n\} \quad (3.1.5)
$$

$$
= : \tau_{(\tilde{\delta}_n, \tilde{\delta}_n)}.
$$

The event $\{\tau_{(-\delta_n, \delta_n)} < \infty\}$ gives the time point within the $n$-th period $[t_n, t_{n+1})$, where the Brownian motion hits the lower or upper bound $\delta_n$. Analogously we define the stopping time, where the upper border gets crossed:

$$
\tau_{\tilde{\delta}_n} := \inf\{t \in [t_n, t_{n+1}) : W_t - W_{t_n} \geq \tilde{\delta}_n\}.
$$

Our aim is to show, that the Brownian motion hits the upper bound at least once, that is:

$$
P[\exists \ n \in \mathbb{N} : \tau_{(-\delta_n, \delta_n)} < \infty \ \text{and} \ W_{\tau_{(-\delta_n, \delta_n)}} - W_{t_n} = \tilde{\delta}_n] = 1. \quad (3.1.7)
$$

Moreover, for $n \in \mathbb{N}$ we define the event

$$
A_n := \{\tau_{(-\delta_n, \delta_n)} < \infty \ \text{and} \ W_{\tau_{(-\delta_n, \delta_n)}} - W_{t_n} = \tilde{\delta}_n\}
$$

$$
= \{\tau_{(-\delta_n, \delta_n)} < \infty \ \text{and} \ \tau_{(-\delta_n, \delta_n)} = \tau_{\tilde{\delta}_n}\}. \quad (3.1.8)
$$
This is the event, that within the \( n \)-th period the upper bound is violated. Then

\[
N := \inf \{ n \in \mathbb{N} : \tau_{\{\tilde{\delta}_n, \tilde{\delta}_n\}} < \infty \quad \text{and} \quad W_{\tau_{\{\tilde{\delta}_n, \tilde{\delta}_n\}}} - W_{t_n} = \tilde{\delta}_n \} \tag{3.1.9}
\]

is the first period where the upper bound is crossed. Here the game ends. For showing that (3.1.7) is true, we use the lemma of Borel-Cantelli. The sequence of events \((A_n)_{n \in \mathbb{N}}\) is independent. If \(\sum_{n=1}^{\infty} P[A_n] = \infty\), then \(P[\limsup_{n \to \infty} A_n] = 1\), which implies\(^2\) the assertion.

First of all we can simplify the stopping time due to the Markov property of Brownian motion in (3.1.10), a rescaling of the time axis to the time interval \([0,1)\) in (3.1.11) and the scaling property of Brownian motion in (3.1.12), where these equalities are equalities in distribution, noted by \(\mathcal{D}\). We have

\[
\tau_{\{\tilde{\delta}_n, \tilde{\delta}_n\}} = \inf \{ t \in [t_n, t_{n+1}) : |W_t - W_{t_n}| \geq \tilde{\delta}_n \}
\]

\[\mathcal{D} \inf \{ t \in [0, 2^{-n}) : |W_t| \geq \tilde{\delta}_n \} \tag{3.1.10}\]

\[= 2^{-n} \inf \{ t \in [0, 1) : |W_{t2^{-n}}| \geq \tilde{\delta}_n \} \tag{3.1.11}\]

\[\mathcal{D} 2^{-n} \inf \{ t \in [0, 1) : |W_t| \geq \tilde{\delta}_n \sqrt{2^n} \}. \tag{3.1.12}\]

As described above we have to study the probability of the first hitting time of the upper bound. Within the following calculations we use the just obtained results regarding the stopping time. Therefore we are able to deduce

\[
P[A_n] = P[\tau_{\{\tilde{\delta}_n, \tilde{\delta}_n\}} < \infty \quad \text{and} \quad W_{\tau_{\{\tilde{\delta}_n, \tilde{\delta}_n\}}} - W_{t_n} = \tilde{\delta}_n]
\]

\[= P[\tau_{\{\tilde{\delta}_n, \tilde{\delta}_n\}} < \infty \quad \text{and} \quad \tau_{\{\tilde{\delta}_n, \tilde{\delta}_n\}} = \tau_{\tilde{\delta}_n}]
\]

\[= P[t_{n+1} - t_n > \tau_{\{\tilde{\delta}_n, \tilde{\delta}_n\}} - t_n = \tau_{\tilde{\delta}_n} - t_n]
\]

\[= P[2^{-n} > \tau_{\{\tilde{\delta}_n, \tilde{\delta}_n\}} = \tau_{\tilde{\delta}_n}]
\]

\[= P[1 > \tau_{\{\tilde{\delta}_n \sqrt{2^n}, \tilde{\delta}_n \sqrt{2^n}\}} = \tau_{\tilde{\delta}_n \sqrt{2^n}}]. \tag{3.1.13}\]

If the trader has now the possibility to set these bounds (as motivated above), she will be able to influence the probability of this event and therefore for a free lunch without risk.

\(^2\)With Borel-Cantelli the event enters infinitely often with probability one. We are only interested in a single occurrence.
3.1 Problem Setting

We assume that the trader sets the bound at $\tilde{\delta}_n = \sqrt{2-n}$ for each period $n \in \mathbb{N}$ up to the end of the game. As we see the investor has to increase the efforts to keep the boundaries down. In this special and optional situation we obtain from (3.1.13) for any $n \in \mathbb{N}$:

$$\mathbf{P}[A_n] = \mathbf{P}[1 > \tau_{(-1, 1)} = \tau_1] > 0.$$  \hfill (3.1.14)

With the result in (3.1.14) we obtain $\sum_{n=1}^{\infty} \mathbf{P}[A_n] = \infty$ and with the Lemma of Borel-Cantelli we get $\mathbf{P}[\limsup_{n \to \infty} A_n] = 1$ and therefore in particular the assertion (compare (3.1.7)) for this special situation.

The question at hand is now, whether the trader is able to set (e.g. $\tilde{\delta}_n = \sqrt{2-n}$) or influence these bounds. At the beginning of this section we have already noted that the trader will have the possibility to compensate her losses in the Deutsche Telekom stock much easier, if she increases the investment, respectively the number of shares. Therefore the investor has the capability to determine $\delta_n$ within each period by choosing the capital commitment $K_n$, where $K_n$ is the absolute amount in monetary units. How much does the trader have to invest in each period that she will realize a profit almost surely?

First of all we consider the potential losses $(L_n)_{n \in \mathbb{N}}$ and profits $(P_n)_{n \in \mathbb{N}}$ of one period. For all $n < N$ the trader realizes a loss $L_n$:

$$L_n = K_n - K_n e^{-\tilde{\delta}_n} e^{(\mu - \frac{1}{2} \sigma^2)\left((\tau_{(-\tilde{\delta}_n)} - t_n) \land (t_{n+1} - t_n)\right)}.$$  \hfill (3.1.15)

$K_n$ is the initial investment at time point $t_n$. The second expression describes the negative development of the stock position. The Brownian term, which we have studied before, has crossed the lower bound or has not violated the upper bound; in this case we assume the crossing of the lower bound at time point $t_{n+1}$ (the trader has lost). The other term $e^{(\mu - \frac{1}{2} \sigma^2)(\tau_{(-\tilde{\delta}_n)} \land t_{n+1} - t_n)}$ describes the drift part of the price process. This term anticipates the duration of the game. If the process hits the lower bound, the game will end at $\tau_{-\tilde{\delta}_n}$, otherwise we have to take into account the whole period. After some estimations, using at first the assumption $\frac{1}{2} \sigma^2 < \mu$ and then the inequality $e^b \geq 1 + b$ we get:

$$L_n = K_n \left(1 - e^{-\tilde{\delta}_n} e^{(\mu - \frac{1}{2} \sigma^2)(\tau_{(-\tilde{\delta}_n)} \land t_{n+1} - t_n)}\right) \leq K_n \left(1 - e^{-\tilde{\delta}_n}\right) \leq K_n \left(1 - (1 - \tilde{\delta}_n)\right) = K_n \tilde{\delta}_n.$$  \hfill (3.1.16)
Analogously we obtain an inequality for the profit of the trader in period $n = N$:

$$P_n = K_n e^{\tilde{\delta}_n} e^{(\mu - \frac{1}{2} \sigma^2) (\tau(\tilde{\delta}_n) \wedge t_{n+1} - t_n)} - K_n$$

$$= K_n \left( e^{\delta_n} e^{(\mu - \frac{1}{2} \sigma^2) (\tau(\tilde{\delta}_n) \wedge t_{n+1} - t_n)} - 1 \right)$$

$$\geq K_n \left( e^{\tilde{\delta}_n} - 1 \right)$$

$$\geq K_n \tilde{\delta}_n.$$

The aim of the trader is to generate a profit of one Euro, i.e., the additional value of the portfolio in period $n = N$ has to dominate at least the losses of the periods before and one monetary unit. Consequently we get the following condition:

$$1 + \sum_{l=1}^{n-1} K_l \tilde{\delta}_l \leq K_n \tilde{\delta}_n.$$  \hspace{1cm} (3.1.17)

Therefore we are able to determine the necessary amount of $K_n$ for any $n$:

$$K_n = \frac{1 + \sum_{l=1}^{n-1} K_l \tilde{\delta}_l}{\tilde{\delta}_n}.$$

In the optional case above ($\tilde{\delta}_n = \sqrt{2^n}$), where the winning situation was forced by the setting of the bounds, the trader will have to invest $K_n = \frac{1 + \sum_{l=1}^{n-1} K_l \sqrt{2^l}}{\sqrt{2^n}}$. Also in the formal framework we recognize that $\tilde{\delta}_n$ depends on the amount of the capital commitment $K_n$. All in all we have shown that the trader has the possibility to create riskless profits by following the presented doubling trading strategy within this Black/Scholes-economy. We realize, that the trader potentially needs quite huge funds of capital to purchase large numbers of stocks to minimize the border $\tilde{\delta}_n$. Of course this assumption of a potential infinite credit facility is not very realistic.

To obtain efficient and consistent markets some trading strategies have to be excluded. This may be done by using the concept of admissible integrands, which was introduced in the first chapter (see [HP] and Definition 1.2.4).

The question at hand is now, whether the relaxation of the admissibility condition within an illiquid economy leads to arbitrage opportunities. The idea behind this question is the
fact that trading in an illiquid economy is quite expensive, especially when the trader has to double her position in each period. We hope, that it is not necessary to restrict the set of trading strategies to the admissible ones, because the costs of liquidity restrict the trading strategies automatically.

3.2 Admissibility of trading strategies

3.2.1 Heuristics

In this section we introduce a model for an illiquid financial market with a permanent price impact. As described above, the Rogers/Singh-model belongs to the class of market models with a short-term price impact. Our so-called reference model is based on the framework which Bank and Baum developed and which was introduced in Section 2.1. We have seen that it is more advantageous for the large trader to split the liquidation of the portfolio into smaller packages than to liquidate en bloc. The liquidation value of \( \rho \in \mathbb{R} \) shares at time \( t \in [0,T] \) is \( L(\rho, t) \) (see Definition 2.1.3):

\[
L(\rho, t) := \int_0^\rho S(x, t) \, dx.
\]  

(3.2.1)

In Proposition 2.1.4 we have seen that the wealth process can be divided into different components and that the trader has the possibility to avoid all liquidity costs by using trading strategies whose trajectories are continuous and of finite variation. Therefore the dynamics can be simplified to

\[
V^{\text{real}}_t(\theta) = v_0 + \int_0^t L(\theta_{s-}, ds),
\]  

(3.2.2)

where \( \int_0^t L(\theta_{s-}, ds) \) denotes the stochastic integral\(^3\) of \( \theta_- \) with respect to the semimartingale kernel \( L(\vartheta, ds) \) and where all derivatives are taken with respect to \( \vartheta \). Of course all these terms depend on the specific form of the price process. Therefore we make some reasonable assumptions. Hereby we follow the approach of section 3 in [Kü]. We assume that the price process of the large trader is driven by a continuous dividend process. With Assumption 3.8 and Proposition 3.9 in [Kü] the price dynamics can be

\(^3\)In Chapter 2.1 we gave more information about the stochastic integral. Especially in the book of Kunita [K] the exact definitions can be found.
Chapter 3: Absence of Arbitrage

described by

\[ S_t^\vartheta = S(\vartheta, t) = S(0, t) - \int_0^t \vartheta c_s \, dA_s, \]

where \( \vartheta \in \mathbb{R} \), \( c \) is a \( \mathbb{R} \)-valued predictable process and \( A \) is a predictable locally integrable increasing process. The second part of the price process can be deduced from the characteristics of a semimartingale (see Proposition 2.9 in [JS]). Hereby \( c \cdot A \) describes the predictable quadratic variation of the dividend process, i.e., \( \langle S, S \rangle_t = \int_0^t c_s dA_s \).

Then we plug in the deduced price process in equation (3.2.1) and we get:

\[
L(\rho, t) = \int_0^{\rho} S(x, t) \, dx = \int_0^{\rho} \left( S(0, t) - \int_0^t c_s x \, dA_s \right) \, dx = \int_0^{\rho} S(0, t) \, dx - \int_0^{\rho} \int_0^t c_s x \, dA_s \, dx = \rho S(0, t) - \frac{\rho^2}{2} \int_0^t c_s \, dA_s = \rho S(0, t) - \frac{\rho^2}{2} \langle S, S \rangle_t.
\]

We are interested in the realizable value (see equation 3.2.2) of the portfolio. We define:

\[
V_t^{\text{real}}(\theta) = v_0 + \int_0^t L(\theta_s, ds) := v_0 + \int_0^t \theta_s dS(0, s) - \int_0^t \frac{\theta_s^2}{2} c_s dA_s = v_0 + \int_0^t \theta_s dS(0, s) - \int_0^t \frac{\theta_s^2}{2} d\langle S, S \rangle_s.
\]

With \( M_t := \int_0^t \theta_s dS_s \), we have:

\[
V_t^{\text{real}}(\theta) = v_0 + M_t - \frac{1}{2} \langle M, M \rangle_t.
\]

Now we are ready introduce our so called reference model.

**Definition 3.2.1** Let \( \theta \) be an arbitrary trading strategy. The process \( (R_t)_{t \geq 0} \) describes the wealth dynamics of the **reference portfolio**. \( v_0 \) is the initial amount of cash in the
portfolio and \( a \in \mathbb{R}_+ \):

\[
R_t(\theta) = v_0 + \int_0^t \theta_s dS_s - a \int_0^t \theta_s^2 d\langle S, S \rangle_s ,
\]

(3.2.3)

where \( \langle S, S \rangle \) denotes the predictable quadratic variation of \( S \) (see Proposition 1.1.11). For \( M_t := \int_0^t \theta_s dS_s = (\theta \cdot S)_t \) we have:

\[
R_t(\theta) = v_0 + \int_0^t \theta_s dS_s - a \int_0^t \theta_s^2 d\langle S, S \rangle_s
\]

\[
= v_0 + (\theta \cdot S)_t - a \langle \theta \cdot S, \theta \cdot S \rangle_t
\]

\[
= v_0 + M_t - a \langle M, M \rangle_t .
\]

(3.2.4)

In the following we set \( \lim_{t \to \infty} \langle M, M \rangle_t =: \langle M, M \rangle_\infty \).

### 3.2.2 No arbitrage

Here and now we want to study the illiquid economy, which is characterized by the portfolio value process of Definition 3.2.3. We notice that \( \mathcal{M}(S) \) - the set of equivalent martingale measures (see Definition 1.2.7) - is not an empty set and cannot be reduced to a singleton.

We have to divide the examination into two cases. In the first paragraph we study a price process which is described by a continuous semimartingale. Then we consider the general case.

#### Continuous case

**Assumption 3.2.2** The price process \( S \) is assumed to be a continuous semimartingale, based on and adapted to a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F} = (\mathcal{F}_t)_{t \geq 0}, P)\).

We start with a few preparatory lemmata and already known results.

**Lemma 3.2.3** Let \((B_t)_{t \geq 0}\) be a standard Brownian motion and \( a, b \in \mathbb{R}_+ \). Then there are constants \( C := C_{a,b} \) and \( c := c_a \) such that for \( t \) big enough:

\[
P[\sup_{s \geq t} (B_s - as) > b] \leq Ce^{-ct}.
\]

(3.2.5)

**Proof.** We define the following event

\[
A := \{\sup_{s \geq t} (B_s - as) > b\}
\]
and

\[ B := \{ B_t - at \leq \frac{b}{2} - \frac{at}{2} \}. \]

Our aim is to minimize the probability of the event A. We make the following approach:

\[
P[A] = P[A \cap B] + P[A \setminus B]
= P[A|B]P[B] + P[A \cap B^c]
\leq P[A|B] + P[B^c].
\]

First we examine the second term:

\[
P[B^c] = P\left[ \left\{ B_t - at \leq \frac{b}{2} - \frac{at}{2} \right\}^c \right] = P\left[ B_t - at > \frac{b}{2} - \frac{at}{2} \right]
= \frac{1}{\sqrt{2\pi t}} \int_{\frac{b+at}{2}}^{\infty} e^{-\frac{x^2}{2t}} \, dx
\leq \frac{1}{\sqrt{2\pi t}} \frac{2}{b + at} \int_{\frac{b+at}{2}}^{\infty} xe^{-\frac{x^2}{2t}} \, dx
= \frac{2}{\pi} \frac{t}{\sqrt{t} \sqrt{t(b + at)}} \frac{e^{-\frac{(b+at)^2}{2t}}}{\sqrt{t}}
\leq e^{-\frac{(b+at)^2}{2t}}.
\]

The last inequality is valid for t big enough.

Now we define the stopping time

\[
\tau_{y,\mu} := \inf\{s \geq 0 : B_s \geq y + \mu s\}
\]

and consider the expression \( P[A|B] \). We denote by \( (B^x_t)_{t \geq 0} \) a Brownian motion which starts in \( x \in \mathbb{R} \). Then we have

\[
P[A|B] = P\left[ \sup_{s \geq t} (B^0_s - as) > b \left| B^0_t \leq \frac{b + at}{2} \right. \right]
= P\left[ B^0_s > b + as \text{ for some } s > t \left| B^0_t \leq \frac{b + at}{2} \right. \right].
\]

We want to minimize the probability of the event that after time t the standard Brownian motion hits the line \( b + as \) with the condition that up to t the process is under the threshold
Therefore we assume the worst case scenario that \( B_t = \frac{b + at}{2} \) and increase due to the symmetry of Brownian motion the probability that the line gets crossed (inequality (3.2.7)). Because of the strong Markov property of the Brownian motion we are able to shift the process in (3.2.8) and let the Brownian motion now starts in zero, where we have to adjust the y-axis. The next equality (3.2.9) is clear by the definition of the stopping time (see equation (3.2.6)). The solution of this problem is well known and can be found in e.g. p. 197 in [KS].

\[
P[A|B] = P\left[ B_s^0 > b + as \text{ for some } s > t \middle| B_t^0 \leq \frac{b + at}{2} \right]
\leq P\left[ B_{s+t}^{b+at} > b + a(s + t) \text{ for some } s > 0 \right] \tag{3.2.7}
= P\left[ B_s^0 > \frac{b + at}{2} + as \text{ for some } s > 0 \right] \tag{3.2.8}
= P\left[ \tau_{\frac{b+at}{2},a} < \infty \right] \tag{3.2.9}
= e^{-2\frac{b+at}{2}} = e^{-(ab+a^2t)}.
\]

We have shown that for \( t \) big enough:

\[
P\left[ \sup_{s > t} (B_s - as) > b \right] \leq e^{-(ab+a^2t)} + e^{-\frac{b+at}{2t}}
\leq e^{-ab} e^{-\frac{a^2t}{8}} + e^{-\frac{k^2+2a(t+a^2)I^2}{8t}}
\leq e^{-ab} e^{-\frac{a^2t}{8}} + e^{-\frac{k^2}{8t} e^{-\frac{at}{4}}} e^{-\frac{a^2t}{8}}
= \left( e^{-ab} + e^{-\frac{at}{4}} \right) e^{-\frac{a^2t}{8}}.
\]

With \( C_{a,b} := e^{-ab} + e^{-\frac{ab}{8}} \) and \( c_a := \frac{a^2}{8} \) we obtain the assertion. \( \blacksquare \)

**Lemma 3.2.4** Let \((B_t)_{t \geq 0}\) be a standard Brownian motion and \( a \in \mathbb{R}_+ \). Then \((B_t-at)_{t \geq 0}\) is a submartingale.

**Proof.** First note that \((B_t-at)_{t \geq 0}\) is a supermartingale, because for \( t > s \) we have

\[
E[B_t - at | \mathcal{F}_s] = B_s - at = B_s - as - a(t-s) \leq B_s - as.
\]

Therefore \((-B_t + at)_{t \geq 0}\) is a submartingale. The function \( \phi : X \mapsto (X)^+ \) is convex and
monotone increasing and with Jensen’s inequality we get:

\[
\mathbb{E}[(-B_t + at)^+ | \mathcal{F}_s] = \mathbb{E}[\phi(-B_t + at) | \mathcal{F}_s] \\
\geq \phi(\mathbb{E}[(-B_t + at) | \mathcal{F}_s]) \\
\geq \phi(-B_s + as) = (-B_s + as)^+.
\]

Consequently the positive part of \((-B_t + at)_{t\geq 0}\) is also a submartingale. The assertion follows from the fact that \(-(B_t - at)_{t\geq 0} = (B_t - at)_{t\leq 0}\).

\[\blacksquare\]

**Lemma 3.2.5** Let \(M\) be a locally square integrable local martingale, then

\[
\frac{M_t}{\langle M, M \rangle_t} \xrightarrow{t \to \infty} 0 \text{ a.s.} \quad (3.2.10)
\]
on the set \(\{\langle M, M \rangle_\infty = \infty\}\).

*Proof.* We refer to p. 150 in [Lep] and p. 187 in [RY]. \[\blacksquare\]

**Lemma 3.2.6** A continuous local martingale \(M\) converges a.s. as \(t\) goes to infinity, on the set \(\{\langle M, M \rangle_\infty < \infty\}\).

*Proof.* See for instance p. 131 in [RY]. \[\blacksquare\]

One of the key arguments of Theorem 3.2.9 is the embedding of a local martingale into a time change of a Brownian motion (Theorem of Dambis, Dubins-Schwarz).

**Definition 3.2.7** We call **extension** of the filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) another filtered probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) together with a map \(\pi : \tilde{\Omega} \to \Omega\), such that \(\pi^{-1}(\mathcal{F}_t) \subset \tilde{\mathcal{F}}_t\) for each \(t\) and \(\pi(\tilde{\mathbb{P}}) = \mathbb{P}\). A process \(X\) defined on \(\Omega\) may be viewed as defined on \(\tilde{\Omega}\) by setting \(X(\tilde{\omega}) = X(\omega)\) if \(\pi(\tilde{\omega}) = \omega\).

We define the stopping time \(T_t = \inf\{s : \langle M, M \rangle_s > t\}\). For the following Lemma 3.2.8 we consider the set \(A^c := \{\langle M, M \rangle_\infty < \infty\}\). On \(A^c\) the limit \(M_\infty = \lim_{t \to \infty} M_t\) exists (see Lemma 3.2.6). Thus for any \(\omega \in \Omega\) we can define a process \(W\) by

\[
W_t = \begin{cases} 
M_{T_t}, & \text{if } t < \langle M, M \rangle_\infty \\
M_\infty, & \text{if } \infty > t \geq \langle M, M \rangle_\infty.
\end{cases}
\]

Under these assumptions we get:
Lemma 3.2.8 (Dambis, Dubins-Schwarz) There exist an extension \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) of \((\Omega, \mathcal{F}, \mathbb{G} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) and a Brownian motion \(\tilde{\beta}\) on \(\tilde{\Omega}\) independent of \(M\) such that the process
\[
B_t = \begin{cases} M_t, & \text{if } t < \langle M, M \rangle \infty \\ M_{\infty} + \tilde{\beta}_t - \langle M, M \rangle \infty, & \text{if } t \geq \langle M, M \rangle \infty. \end{cases}
\]
is a standard linear Brownian motion. The process \(W\) is an \((\mathcal{F}_t)\)-Brownian motion stopped at \(\langle M, M \rangle \infty\).

Proof. We refer to [RY] p. 181 and 182. ■

Now we are prepared to present the main result of this section:

Theorem 3.2.9 Let \((R_t)_{t \geq 0}\) be the value process of Definition 3.2.1 and \(Q \in \mathcal{M}(S)\). Then
\[
\lim_{t \to \infty} R_t \text{ exists in } [-\infty, \infty) \text{ and } E_Q[\lim_{t \to \infty} R_t] \leq v_0.
\]

Proof. We define the following events \(A := \{(M, M) \infty = \infty\}\) and \(A^c = \{(M, M) \infty < \infty\}\). Recall that \(M\) converges a.s. as \(t\) goes to infinity on \(A^c\) (Lemma 3.2.6). We show:
\[
E_Q[\lim_{t \to \infty} (M - a \langle M, M \rangle_t)] = E_Q[\lim_{t \to \infty} (M - a \langle M, M \rangle_t) 1_A] + E_Q[(M - a \langle M, M \rangle_t) 1_{A^c}] \leq 0,
\]
which is an equivalent formulation of the assertion.

We split the proof into two parts. In the first part we show
\[
M_t - a \langle M, M \rangle_t \xrightarrow{t \to \infty} -\infty \text{ a.s. on the set } A.
\]
If \(P[A] > 0\), we have to exclude that \(E_Q[(M - a \langle M, M \rangle_t) 1_{A^c}] = \infty\).

Therefore we consider in the second part of the proof the event \(A^c\). It is sufficient to show
\[
E_Q[(M - a \langle M, M \rangle_t) 1_{A^c}] < \infty.
\]

(i) We start with an infinite quadratic variation of \(M\). By using Lemma 3.2.5, we obtain
\[
\frac{M_t}{\langle M, M \rangle_t} \xrightarrow{t \to \infty} 0.
\]
a.s. on \( \{ \langle M, M \rangle_\infty = \infty \} \). Therefore for fixed \( a \in \mathbb{R}_+ \) we get \( M_t - a \langle M, M \rangle_t \xrightarrow{t \to \infty} -\infty \) and we are done. In this case we do not use the continuity of the process \( S \), but the fact that \( S \) and therefore \( M \) are local martingales under the equivalent local martingale measure \( Q \in \mathcal{M}(S) \).

(ii) In order to deal with \( \mathbb{E}_Q[ R_\infty 1_{A^c} ] \) we use Lemma 3.2.8 to represent the local martingale \( M \) as time change of a Brownian motion. Let \( Q \in \mathcal{M}(S) \) and recall that \( S \) is a continuous local martingale under \( Q \) and therefore \( M \) is also a continuous local martingale. As in Lemma 3.2.8 we define the stopping time \( T_t = \inf\{ s : \langle M, M \rangle_s > t \} \). We can define (because \( M_\infty \) exists on \( A^c \)) a process \( W \) by

\[
W_t = \begin{cases} 
M_{T_t}, & \text{if } t < \langle M, M \rangle_\infty \\
M_\infty, & \text{if } t \geq \langle M, M \rangle_\infty.
\end{cases}
\]

Hence, we obtain that \( W \) is an \( (\mathcal{F}_{T_t}) \)-Brownian motion stopped at \( \langle M, M \rangle_\infty \). For \( t \) big enough, namely \( t \geq \langle M, M \rangle_\infty \), we have \( M_\infty = W_{\langle M, M \rangle_\infty} \) and we can write:

\[
\mathbb{E}_Q[(M_\infty - a \langle M, M \rangle_\infty) 1_{A^c}] = \mathbb{E}_Q[(W_{\langle M, M \rangle_\infty} - a \langle M, M \rangle_\infty) 1_{A^c}].
\]

For simplification we define \( T := \langle M, M \rangle_\infty \). The next step is to show that \( \{ \langle M, M \rangle_\infty \leq t \} \in \mathcal{F}_{T_t} \), i.e., \( T \) is an \( (\mathcal{F}_{T_t}) \)-stopping time. We have:

\[
\{ \langle M, M \rangle_\infty \leq t \} = \{ T \leq t \} = \{ T_t = \infty \} \in \mathcal{F}_{T_t}.
\]

In this case (on \( A^c \)) the stopping time is finite. For a bounded stopping time \( T \leq \tau \) we can apply the optional sampling theorem in equation (3.2.12) and we are done:

\[
\mathbb{E}_Q[(W_T - aT) 1_{A^c}] = \mathbb{E}_Q[W_T 1_{A^c}] - \mathbb{E}_Q[aT 1_{A^c}]
\]

\[
= \mathbb{E}_Q[W_0 1_{A^c}] - \mathbb{E}_Q[aT 1_{A^c}] \quad (3.2.12)
\]

\[
= -a\mathbb{E}_Q[T 1_{A^c}] \leq 0, \quad (3.2.13)
\]

where the expectation value in the last expression exists, because \( T \) is bounded.

For a finite, but not necessarily bounded, stopping time we consider the positive part of \((W_T - aT) 1_{A^c}\), because it is sufficient to show that \( \mathbb{E}_Q[(M_\infty - a \langle M, M \rangle_\infty)^+ 1_{A^c}] < \infty \).

If \( \mathbb{E}_Q[(W_T - aT)^- 1_{A^c}] = \infty \), we are finished again. Therefore we assume that \( \mathbb{E}_Q[(W_T - aT)^- 1_{A^c}] < \infty \).

With Lemma 3.2.4 we have that \((W_t - at)_{t \geq 0}\) is a supermartingale and obviously \((W_t - at)^+_{t \geq 0}\)
is also a supermartingale. Since \((W_t - at)\) is nonnegative, we can apply the optional sampling theorem for finite (but not necessarily bounded) stopping times. We have:

\[
E_Q[(W_T - aT)^+] \leq E_Q[W_0^+] = 0,
\]
which completes the proof.

\section*{General case}

Within the last paragraph we could show for a continuous semimartingale that the value process \(R_t\) of the form 3.2.3 is a supermartingale under the equivalent martingale measure \(Q\): \(E_Q[\lim_{t \to \infty} R_t] \leq v_0\). The essential part of the proof was the theorem from Dambis/Dubins-Schwarz. The continuous local martingale \(M\) could be represented as time changed Brownian motion \(M_t = B_{\langle M, M \rangle}_t\). Now we want to consider the general case, i.e., the underlying price process \((S_t)_{t \in [0,T]}\) is a general semimartingale based on and adapted to a filtered probability space.

The question at hand is whether we can find another and similar connection between the representations considered here and that we are in the position to give some statements referring to the value process in the general case. We consider a process \(X(t)\) defined by evaluating Brownian motion \((B(u), u \geq 0)\) at a random time given by an increasing, right continuous process \(A(t)\) that is independent of \(B(t)\); in formulas:

\[
X(t) = B(A(t)).
\]  

(3.2.14)

The most general result in our context is the theorem of Monroe (see [Mo]), that all semimartingales have the representation (3.2.14), possibly defined on some adequately extended probability space.

In the continuous case we have seen, that \(A(t)\), the time change, may be recovered as the predictable quadratic variation process \(\langle X, X \rangle_t\). We note that in contrast to this case of time changes by continuous processes \(A(t)\), where we have this identification, the process \(A(t)\) is hidden in the composite process \(X(t)\) and cannot be recovered from it when \(A(t)\) is purely discontinuous. For instance Geman, Madan and Yor have studied this case (see [GMY00] and [GMY02]).
Chapter 4

Conclusion

In this thesis we have discussed many aspects of modelling liquidity. For incomplete markets, the theory of illiquidity is far less advanced than that of other facets of incompleteness. The aim was at first to give an overview of actual different modelling approaches. We distinguished between feedback effect models and short-term price impact models. As an example for the first class we gave a short description of the model of Bank and Baum [BB] in chapter 2. For the illiquid economy of Rogers and Singh [RS], which is a short-term price impact approach, we gave in Theorem 2.2.13 a formal connection between the discrete-time and continuous-time environment, where we have extended the model to semimartingale price dynamics.

The second aspect, which in my opinion is economically quite more important, is the study of arbitrage opportunities in illiquid economies. We noted that some trading strategies, like doubling strategies, have to be excluded for complete markets, because they can lead to arbitrage possibilities. We presented a model which was related to the Bank and Baum model and could show in Theorem 3.2.9 that in this model of an illiquid economy this admissibility condition can be relaxed. In this context we had to assume that the price process is described by a continuous semimartingale. Therefore the canonical next step would be to extend the result to general semimartingale dynamics, but up to now, there is no solution of this problem in view.

We realize that the studied aspects are quite fundamental. The next step should be, e.g., to examine pricing formulas for assets within these illiquid economies. Generally we believe that illiquidity is an underestimated issue of financial markets regarding the risk and the business development. The natural evolution, which has already begun, is the securitiza-
tion of illiquid assets to convertible assets, as we have seen it within the credit derivatives area. The capacity of this market segment cannot be overestimated. Therefore enough problems wait for the researchers’ solutions.
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<tr>
<td>$FV$</td>
<td>Finite variation $(Var(A) &lt; \infty)$</td>
</tr>
<tr>
<td>$\mathcal{H}^2$</td>
<td>Space of semimartingales with finite $\mathcal{H}^2$-norm</td>
</tr>
<tr>
<td>$K_t = (K_t(\theta))_{t \in [0,T]}$</td>
<td>Discounted holdings in the bank account</td>
</tr>
<tr>
<td>$L(S)$</td>
<td>Set of $S$-integrable processes</td>
</tr>
<tr>
<td>$L(\rho,t)$</td>
<td>Liquidation value of $\rho \in \mathbb{R}$ shares at time $t \in [0,T]$</td>
</tr>
<tr>
<td>$\mathcal{M}(S)$</td>
<td>Set of equivalent martingale measures</td>
</tr>
<tr>
<td>$\mathcal{P}$</td>
<td>Predictable $\sigma$-field</td>
</tr>
<tr>
<td>$\langle X, X \rangle$</td>
<td>Predictable quadratic variation, resp. covariation</td>
</tr>
<tr>
<td>$[X, X] = ([X, X]<em>t)</em>{t \in [0,T]}$</td>
<td>Quadratic variation process, resp. covariation</td>
</tr>
<tr>
<td>$(R_t)_{t \geq 0}$</td>
<td>Reference portfolio wealth dynamics</td>
</tr>
<tr>
<td>$S^0 = (S^0_t)_{t \in [0,T]}$</td>
<td>Price process of the bond</td>
</tr>
<tr>
<td>$S = (S_t)_{t \in [0,T]}$</td>
<td>Price process of the stock</td>
</tr>
<tr>
<td>$S^\vartheta = (S^\vartheta_t)_t \in [0,T]$</td>
<td>Price process of the stock, where the large trader holds a stake of $\vartheta$ shares</td>
</tr>
<tr>
<td>$\theta = (\theta_t)_{t \in [0,T]}$</td>
<td>Trading strategy</td>
</tr>
<tr>
<td>$\theta_t$</td>
<td>Number of stocks in the portfolio at time $t$</td>
</tr>
<tr>
<td>$\mathcal{V}$</td>
<td>Set of all real-valued processes $A$ that are adapted, càdlàg, with $A_0 = 0$, and have $FV$</td>
</tr>
<tr>
<td>$(V_t(\theta))_{t \in [0,T]}$</td>
<td>Portfolio value process</td>
</tr>
<tr>
<td>$(V^b_t(\theta))_{t \in [0,T]}$</td>
<td>Portfolio book value process</td>
</tr>
<tr>
<td>$(V^{block}<em>t(\theta))</em>{t \in [0,T]}$</td>
<td>Portfolio block liquidation value process</td>
</tr>
<tr>
<td>$(V^{real}<em>t(\theta))</em>{t \in [0,T]}$</td>
<td>Portfolio realizable value process</td>
</tr>
<tr>
<td>$Var(X)$</td>
<td>Variation process of $X$</td>
</tr>
</tbody>
</table>
Bibliography


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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

_________________________  Mainz, am 22. August 2007